RECONSTRUCTION OF POINTWISE SOURCES IN A TIME-FRACTIONAL DIFFUSION EQUATION

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Abstract. This paper is concerned with an inverse pointwise source problem for the time-fractional diffusion equation in the two-dimensional case. The source term to be identified models the action of a finite number of small particles. Each particle is assumed to be no larger than a single point, characterized by its location and intensity. Both theoretical and numerical aspects are discussed. In the theoretical part, we analyse the well-posedness of the Dirac time-fractional diffusion problem. For the inverse problem, we establish that the unknown point sources can be uniquely identified from local measured data and we derive a local Lipschitz stability result. In the numerical part, we develop a fast and accurate reconstruction approach. The unknown pointwise sources are characterized as solution to an optimization problem minimizing a tracking-type functional. A noniterative reconstruction algorithm is devised, allowing us to determine the number, locations and intensities of the pointwise sources. The efficiency of the proposed approach is confirmed by some numerical examples.

1. Introduction and setting of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with smooth boundary $\Sigma = \partial\Omega$, containing a biological material and let $T > 0$ be a fixed time. We assume that the diffusion phenomena in Ω is governed by the following time-fractional initial boundary value problem

$$\left\{ \begin{array}{rcll} \partial_t^\alpha u - \Delta u &=& F^* & \text{in} \quad \Omega \times (0,T), \\ u &=& u_d & \text{on} \quad \Sigma \times (0,T), \\ u(.,0) &=& 0 & \text{in} \quad \Omega, \end{array} \right. \tag{1.1}$$

where u_d is a given boundary data and F^* is an unknown source term. Moreover, in the model (1.1) the notation ∂_t^α denotes the so-called Caputo (also known as the Djrbashian-Caputo) derivative of order $0 < \alpha < 1$ with respect to the time variable t, defined as (one can see [38] for more details)

$$\partial_t^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u}{\partial \tau}(x,\tau)\,\mathrm{d}\tau, \quad (x,t) \in \Omega \times (0,\,T), \tag{1.2}$$

where Γ denotes the Euler's Gamma function, which is defined on each complex number $z \in \mathbb{C}$ with positive real part (i.e. $\Re\{z\} > 0$), by

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s}\,\mathrm{d}s. \tag{1.3}$$

Diffusion equations with time-fractional derivative have been first introduced in physics by Nigmatullin [53] for describing super slow diffusion process in a porous medium with fractal geometry. During the last few decades, several research studies have shown that fractional diffusion equations are more suitable to model anomalous diffusion processes in which the mean square variance grows faster (in the case of super-diffusion process) or

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slower (in the case of sub-diffusion) than that in a Gaussian process and provide a more accurate fit to experimental data [13, 60]. For example, fractional diffusion equations have been applied to describe relaxation phenomena in a complex viscoelasticity material [25], plasma physics [17, 62, 63], diffusion process in a highly heterogeneous aquifer [3], a non-Markovian diffusion process with memory [49], complex fluid dynamics [26, 27], biological systems [15, 46, 55], kinetic and reaction-diffusion processes [18, 28, 64], dynamics in fractal structure [12, 14], and many other applications.

In this paper, we deal with an inverse source problem related to the time-fractional diffusion equation (1.1). We aim to identify the source term F^* from local measurements of the potential field u. However, this inverse problem is ill-posed in the sense that F^* (in its general form) cannot be uniquely identified from boundary or local internal measurements of u, see for example [36, Section 1.3.1]. To overcome this difficulty, the majority of research works in this issue are focused on the determination of the source term F^* in the following variable separation form

$$F^*(x,t) = f(x)g(t), \quad (x,t) \in \Omega \times (0,\, T).$$

Motivated by significant scientific equations and important industrial applications, many theoretical and numerical approaches have been performed during the last few years for identifying f or g or the two components f and g. The developed studies can be grouped into three main classes:

- *Identification of the space-dependent term*: It consists in identifying the spatial component f of the source term F^* where the temporal component g is assumed to be known. This inverse problem has been studied by many authors. The unique identification of the space-dependent term f from interior observation was proved in [30] using Duhamel's principle and unique continuation principle, which also proposed an iterative threshold algorithm for the reconstruction procedure. The same problem has been investigated by Jiang et al. in [31], where they provided a numerical reconstruction scheme using local input data defined in a small sub-domain $\omega \times (0,T) \subset\subset \Omega \times (0,T)$. In [59], the authors recovered the spatial component f using exact final measurement data and established a stability result. Wang and Ran [65] dealt with a conditional stability and proposed an iteration method to reconstruct f from the final measurement data. Wei and Wang proposed in [69, 70] a modified quasi-boundary value method for identifying the space-dependent term f with the help of final observation data. The same inverse problem has been examined by Wang et al. [66], where they used a simplified Tikhonov regularization method and established some convergence results. Zhang and Xu [74] developed a recovered approach for determining the space-dependent term f from a single point of measure and proposed a numerical procedure for solving the corresponding inverse problem. Rundell and Zhang [57] examined the case where f is the characteristic function of an unknown sub-domain $\omega \subset \Omega$. They developed a Newton-type iterative procedure to reconstruct the location and shape of ω from external boundary measurements, whereas Prakash et al. [56] proposed a non-iterative identification procedure based on the second-order topological gradient to reconstruct ω. The topological derivative method, introduced in the fundamental papers [24, 61] and further developed in the book by Novotny and Sokołowski [54], can be seen as a particular case of the broader class of asymptotic methods fully developed in the books by Ammari et al. [8] and Ammari and Kang [10], for instance. See also related works [48, 52]. The same geometric inverse problem has been considered in [67] in the case where the component g depends also on the space variable x. In this last work, the authors applied the reproducing kernel space method for the reconstruction procedure.

- *Identification of the time-dependent term*: This case concerns the recovery of the temporal component g of the source term F^* assuming that the spatial component f is given. In this context, Sakamoto and Yamamoto [58] established a stability estimate of determining the term g from an observation data taken at one point over $(0, T)$. The same inverse problem has been discussed in [44, 45], where the authors proposed identification approaches for the reconstruction of g from a single monitoring point. Wei et al. [71] recovered g by using the usual initial and boundary data and an additional measurement information at an inner point. While Wei et al. in [68] applied the conjugate gradient method combined with Morozov's discrepancy principle to recovery the time-dependent term from the boundary Cauchy data. Then Li et al. considered in [41] the multi-term case of time-fractional order. In [73], the authors developed an identification method based on the Fourier regularization technique. The same approach was extended in [23] to the case where f depends both on t and x.

- *Identification of the spatial and temporal components*: This case is devoted to the identification of a space-time-dependent source term of the form $F^*(x,t) = f(x)g(t)$ where both f and g are unknown. Kian et al. [35] showed the simultaneous recovery of both temporal and spatial components under suitable assumptions. For completeness, we also mention the developed works in [32, 37] concerning the reconstruction of a source term of the form $F^*(x,t) = f(x',t)R(x,t)$ (with $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^{n-1}$, $n \geq 2$) where f is unknown and R is given. In [32] the authors showed a conditional stability result for the inverse problem using a novel perturbation argument and also proposed an iterative reconstruction algorithm. Then in [37] the authors proved uniqueness and stability results for the inverse source problem of recovering $f(x',t)$ of the sub-diffusion model in a cylindrical domain.

Despite a considerable amount of works done in this topic, several mathematical issues of high interest are still lacking [33]. Especially, for the case where the source terms to be identified are represented by the Dirac delta functions, which are known as point source inverse problems. Actually, there are few studies dealing with such a model inverse problem. For example, one can cite the developed approaches for the classical parabolic case (i.e. $\alpha = 1$) [11, 21, 42, 43]. In [21] El Badia and Ha-Duong established a uniqueness result and proposed an approach for reconstructing pointwise sources with positive intensities and vanishing when the time t becomes greater than a given threshold $T^* < T$. Then in [11], Andrle and El Badia conducted a complete proof of uniqueness when the intensities are not necessarily positive and developed an algebraic identification method. Whereas Ling and Takeuchi in [42] considered the point source identification problems for heat equations from noisy observation data taken at the minimum number of spatially fixed measurement points. In [43] Ling et al. proved that one measurement point is sufficient to identify the number of sources. Also, they showed that three measurement points are sufficient to identify all unknown source locations as well as they developed a numerical reconstruction approach. Other problems of this kind have been studied by several other authors [4, 6, 7, 9, 16, 19, 39, 50, 51].

In contrast to the aforementioned works, in this work we deal with an inverse pointwise source problem for the time-fractional diffusion equation (1.1). More precisely, the source term to be identified F^* represents the action of a finite number $m^* \in \mathbb{N}\backslash\{0\}$ of small particles $\{P_i,\, 1 \leq i \leq m^*\}$ (micro-organisms) located inside the domain Ω. Each particle P_i is assumed to be no larger than a single point, characterized by its location $z_i^* \in \Omega$ and its intensity $\lambda_i^* \in \mathbb{R}\backslash\{0\}$. The source term F^* is defined by the total collection of the local point-sources, which mathematically expressed in terms of the Dirac delta distribution as

$$F^*(x) = \sum_{i=1}^{m^*} \lambda_i^* \delta(x - z_i^*), \tag{1.4}$$

where $\delta(x - z_i^*)$ are used to denote Dirac masses with poles at z_i^*. For simplicity, we suppose that the point-sources locations $z_i^* \in \Omega$, $1 \leq i \leq m^*$ are mutually distinct.

The aim of this paper is to develop an efficient and accurate method for identifying the unknown source term F^* from internal measurements of the potential u taken within an open subdomain $\Omega_0 \subset \Omega\backslash\{z_i^*,\, 1 \leq i \leq m^*\}$. It is an inverse source problem which can be formulated as follows: Given an observation data $u_{obs} \in L^2(0,T; L^2(\Omega_0))$, determine the number m^*, the locations z_i^* and the intensities λ_i^* such that the potential field u coincides with the measure u_{obs} in $\Omega_0 \times (0,T)$, i.e.

$$u = u_{obs} \ \text{ in } \Omega_0 \times (0,T). \tag{1.5}$$

In this work, we will discuss some theoretical and numerical aspects related to the considered inverse source problem. We start our study by examining the well-posedness of the Dirac time-fractional diffusion problem. For the inverse problem, we will prove that the unknown point sources can be uniquely identified from local measured data. Also, we will derive a local Lipschitz stability result.

In the numerical part, a fast and accurate pointwise source reconstruction approach is developed and implemented. The inverse problem is formulated as an optimization one minimizing a tracking-type functional with respect to a set of admissible pointwise sources. The considered objective function measures the misfit between the simulated and measured potentials inside the sub-domain $\Omega_0 \times (0,T)$. A non-iterative reconstruction algorithm is devised, allowing us to determine the number, the locations and the intensities of the unknown pointwise sources. The efficiency of the proposed approach is confirmed by some numerical examples.

Finally, let us briefly describe the organization of the paper. In Section 2, we examine the time-fractional diffusion equation with pointwise source term. We address the existence and uniqueness questions. Section 3 is devoted to the considered inverse source problem. In Section 3.1, we show that the inverse problem admits a unique solution. Section 3.2 is concerned with the stability question. In Section 4, we present the main ideas of the proposed reconstruction method. Then, in Section 5 we devise a noniterative reconstruction procedure, providing the main characteristics (number, locations and intensities) of the unknown pointwise sources. Some numerical examples showing the efficiency of the proposed reconstruction approach are presented in Section 6. The paper ends with some concluding remarks summarized in Section 7.

2. Well-posedness of the direct problem

In this section, we discuss the well-posedness of the Dirac time-fractional diffusion problem. We shall prove an existence and uniqueness results for the direct problem (1.1)-(1.4). We proceed by decomposing the solution to (1.1)-(1.4) into a singular and a regular part. The singular component has a finite number of singularities caused by the point-sources locations. To this end, we introduce the following Sobolev spaces

$$H^{r,s}(\Omega \times (0,T)) = L^2(0,T; H^r(\Omega)) \cap H^s\left(0,T; L^2(\Omega)\right),$$
$$H^{r,s}(\Sigma \times (0,T)) = L^2(0,T; H^r(\Sigma)) \cap H^s\left(0,T; L^2(\Sigma)\right)$$

which are defined for all $r, s \geq 0$ (see e.g. [2]).
In the following theorem, we consider the case where the source term is defined by a single point-source. In the last part of this paragraph, we will generalize the obtained result for the multiple point-sources case.

Theorem 1. *(Single point-source). Let $u_d \in H^{\frac{1}{2},\frac{\alpha}{4}}(\Sigma \times (0,T))$ and $S_{a,\lambda}$ be a single point-source located at $a \in \Omega$ and of intensity $\lambda \in \mathbb{R}\backslash\{0\}$. Then, the following Dirac time-fractional diffusion problem*

$$\begin{cases} \partial_t^\alpha \phi - \Delta\phi &= \lambda\,\delta(\cdot - a) \quad \text{in} \quad \Omega \times (0,T), \\ \phi &= u_d \quad \text{on} \quad \Sigma \times (0,T), \\ \phi(.,0) &= 0 \quad \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

admits a unique solution.

Proof. We start this proof by decomposing the solution u into two parts:

$$\phi = \lambda\,\phi_1 + \phi_0.$$

The term ϕ_1 represents the singular part. Its defined by

$$\phi_1(x) = E(x-a),\ x \in \Omega \tag{2.2}$$

where E is the fundamental solution of the Laplace equation in 2D. It is given by (see, e.g., [10])

$$E(y) = \frac{1}{2\pi}\log|y|, \quad \forall y \in \mathbb{R}^2\backslash\{0\}. \tag{2.3}$$

The second component ϕ_0 is chosen as the solution to the following time-fractional boundary value problem

$$\begin{cases} \partial_t^\alpha \phi_0 - \Delta\phi_0 &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ \phi_0 &= u_d - \lambda\phi_1 \quad \text{on} \quad \Sigma \times (0,T), \\ \phi_0(.,0) &= -\lambda\phi_1 \quad \text{in} \quad \Omega. \end{cases} \tag{2.4}$$

It is important to note here that the function ϕ_1 is smooth on Σ (of class C^∞). Indeed, since Ω is an open set one can determine $r_0 > 0$ such that $|x-a| \geq r_0$, $\forall x \in \Sigma$.
According to the previous decomposition, to prove an existence and uniqueness result for the problem (2.1), it is sufficient to prove that the problem (2.4) has a unique weak solution. To this end, we split problem (2.4) into the following two problems by taking $\phi_0 = \phi_0^1 + \phi_0^2$, where ϕ_0^1 is the solution of

$$\begin{cases} \partial_t^\alpha \phi_0^1 - \Delta\phi_0^1 &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ \phi_0^1 &= u_d - \lambda\phi_1 \quad \text{on} \quad \Sigma \times (0,T), \\ \phi_0^1(.,0) &= 0 \quad \text{in} \quad \Omega, \end{cases} \tag{2.5}$$

and ϕ_0^2 is the solution of

$$\begin{cases} \partial_t^\alpha \phi_0^2 - \Delta\phi_0^2 &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ \phi_0^2 &= 0 \quad \text{on} \quad \Sigma \times (0,T), \\ \phi_0^2(.,0) &= -\lambda\phi_1 \quad \text{in} \quad \Omega. \end{cases} \tag{2.6}$$

The time-fractional diffusion equation (2.5) has been considered in [34, Corollary 3.6]. It is proved that this boundary value problem admits a unique weak solution ϕ_0^1 and satisfying the regularity property:

$$\phi_0^1 \in {}_0H^{1,\frac{\alpha}{2}}(\Omega \times (0,T)),$$

where ${}_0H^{1,\frac{\alpha}{2}}(\Omega \times (0,T))$ is the following subspace of $H^{1,\frac{\alpha}{2}}(\Omega \times (0,T))$:

$${}_0H^{1,\frac{\alpha}{2}}(\Omega \times (0,T)) = \Big\{\theta \in H^{1,\frac{\alpha}{2}}(\Omega \times (0,T));\ \theta(.,0) = 0\Big\}.$$

On the other hand, from [58, Theorem 2.1(i)] Sakamoto and Yamamoto proved that the problem (2.6) has a unique weak solution

$$\phi_0^2 \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega)).$$

Consequently, the time-fractional boundary value problem (2.4) has a unique weak solution. Thus the proof of Theorem 1 is completed. $\square$

It is important to note that the previous result can be easily extended for the case of multiple point-sources. Then, by applying Theorem 1 for each point-source z_i^*, $1 \leq i \leq m^*$, one can deduce the following corollary.

Corollary 2. *(Multiple point-sources case). Let* $u_d \in H^{\frac{1}{2},\frac{\alpha}{4}}(\Sigma \times (0,T))$. *Then, there exists a unique solution* u *to the direct problem* (1.1)-(1.4) *given by*

$$u = u_0 + \sum_{i=1}^{m^*} \lambda_i^* u_i$$

where the singular terms u_i $i = 1, \cdots, m^*$ *are defined with the help of the Laplace fundamental solution*

$$u_i(x) = E(x - z_i^*),\ x \in \Omega \tag{2.7}$$

and u_0 *solves the following system*

$$\left\{\begin{array}{rcll} \partial_t^\alpha u_0 - \Delta u_0 & = & 0 & in \quad \Omega \times (0,T), \\ u_0 & = & u_d - \displaystyle\sum_{i=1}^{m^*} \lambda_i^* u_i & on \quad \Sigma \times (0,T), \\ u_0(.,0) & = & -\displaystyle\sum_{i=1}^{m^*} \lambda_i^* u_i & in \quad \Omega. \end{array}\right. \tag{2.8}$$

Remark 3. *In this section, focused on the theoretical analysis of the considered time-fractional diffusion problem* (1.1)-(1.4)*, several mathematical issues of high interest could not be discussed. The regularity properties of the solution* ϕ_0 *is one of them. As mentioned by Yamamoto* [72, Section 5.1] *there are not complete works on regularity properties within the class of Sobolev spaces* $H^{r,s}(\Omega \times (0,\,T))$. *This question is, up to our knowledge, still an open one which deserves attention.*

3. Mathematical analysis of the inverse problem

This section is concerned with a mathematical analysis of the considered inverse source problem. More precisely, we will present two main theoretical results. In Section 3.1, we discuss the question of identifiability of the inverse problem. Then, Section 3.2 is devoted to show local stability result.

3.1. **Uniqueness.** Here, we aim to prove that the source F^* can be uniquely determined from a local measurement u_{obs} of the potential field u in a sub-domain $\Omega_0 \subset \Omega$. The established result is summarized in the following theorem.

Theorem 4. *Let u_ℓ, with $\ell = 1, 2$, be the solutions of the problems*

$$\begin{cases} \partial_t^\alpha u_\ell - \Delta u_\ell &= F_\ell \quad in \quad \Omega \times (0, T), \\ u_\ell &= u_d \quad on \quad \Sigma \times (0, T), \\ u_\ell(.,0) &= 0 \quad in \quad \Omega, \end{cases} \tag{3.1}$$

where $F_\ell = \sum_{j=1}^{m_\ell} \lambda_j^\ell \delta(x - z_j^\ell)$, and such that

$$u_1 = u_2 \ \ in \ \ \Omega_0 \times (0, T). \tag{3.2}$$

Then,

$$m_1 = m_2 (:= m^*), \ \ \lambda_j^1 = \lambda_j^2, \ \ and \ \ z_j^1 = z_j^2 \quad for\ all\ j = 1, \cdots, m^*. \tag{3.3}$$

Proof. Consider the difference $u_{2,1} = u_2 - u_1$, which is the solution to

$$\begin{cases} \partial_t^\alpha u_{2,1} - \Delta u_{2,1} &= F_2 - F_1 \quad \text{in} \quad \Omega \times (0, T), \\ u_{2,1} &= 0 \quad \text{on} \quad \Sigma \times (0, T), \\ u_{2,1}(.,0) &= 0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.4}$$

According to (3.2), we have

$$u_{2,1} = 0 \quad \text{in} \ \ \Omega_0 \times (0, T). \tag{3.5}$$

In the following, we extend the solution $u_{2,1}$ to the boundary value problem (3.4) by zero outside the interval $(0, T)$. For simplicity, we still denote the extension by $u_{2,1}$. On the other hand, we denote by $\widehat{u}_{2,1}$ the Laplace transform of the solution $u_{2,1}$ with respect to the variable t. Using the following relation issued from the Laplace transform of fractional derivative in the Caputo sense (see, e.g., Kubica, Ryszewska and Yamamoto [40])

$$\widehat{\partial_t^\alpha f(s)} = s^\alpha \widehat{f}(s) - s^{\alpha-1} f(0+), \text{ for each } s \in \mathbb{C} \ \text{ such that } \ \Re\{s\} > 0, \tag{3.6}$$

one can derive that the transformed algebraic equation, satisfied by $\widehat{u}_{2,1}$, is given as

$$\begin{cases} s^a \widehat{u}_{2,1}(x; s) - \Delta \widehat{u}_{2,1}(x; s) &= \widehat{F}_2(x; s) - \widehat{F}_1(x; s), \quad x \in \Omega, \\ \widehat{u}_{2,1}(x; s) &= 0, \quad x \in \Sigma, \end{cases} \tag{3.7}$$

where

$$\widehat{u}_{2,1}(x; s) = \int_0^\infty e^{-st} u_{2,1}(x, t) \mathrm{d}t, \tag{3.8}$$

$$\widehat{F}_\ell(x; s) = \sum_{j=1}^{m} s^{-1} \lambda_j^\ell \delta(x - z_j^\ell), \ \ \ell = 1, 2. \tag{3.9}$$

We set $k = \boldsymbol{i} s^{\frac{\alpha}{2}}$, with $\boldsymbol{i}$ used to denote the imaginary number, i.e., $\boldsymbol{i} = \sqrt{-1}$, then we get

$$\begin{cases} \Delta \widehat{u}_{2,1}(x; s) + k^2 \widehat{u}_{2,1}(x; s) &= \widehat{F}_1(x; s) - \widehat{F}_2(x; s), \quad x \in \Omega, \\ \widehat{u}_{2,1}(x; s) &= 0, \quad x \in \Sigma. \end{cases} \tag{3.10}$$

For $\ell = 1, 2$, let $B_\varepsilon(z_j^\ell)$ be the ball centered at z_j^ℓ with radius $\varepsilon > 0$ small enough such that

$$\overline{B_\varepsilon(z_j^\ell)} \subset \Omega \quad \text{and} \quad \overline{B_\varepsilon(z_j^\ell)} \cap \Omega_0 = \emptyset. \tag{3.11}$$

Observe now that $\widehat{u}_{2,1}$ is a solution to

$$\begin{cases} \Delta\widehat{u}_{2,1}(x;s)+k^2\widehat{u}_{2,1}(x;s) &= 0, \quad x\in\Omega_\varepsilon=\Omega\backslash\bigcup\limits_{\ell,j}B_\varepsilon(z_j^\ell), \\ \widehat{u}_{2,1}(x;s) &= 0, \qquad\qquad x\in\Sigma. \end{cases} \tag{3.12}$$

In other hand, the internal condition (3.5) and relation (3.8) implies

$$\widehat{u}_{2,1}(\cdot;s)=0 \text{ in } \Omega_0. \tag{3.13}$$

Since $\overline{\Omega_0}\subset\Omega_\varepsilon$, we then conclude from the unique continuation principle [29, Theorem 3.3.1] that $\widehat{u}_{2,1}(\cdot;s)$ vanishes in Ω_ε. If we take $\varepsilon\to 0$ this implies that $\widehat{u}_{2,1}(\cdot;s)=0$ in $\Omega\backslash\cup\{z_j^1,z_j^2\}$. Therefore, we can easily extend $\widehat{u}_{2,1}$ outside of Ω by 0; we denote this extension also by $\widehat{u}_{2,1}$, one gets

$$\Delta\widehat{u}_{2,1}(.;s)+k^2\widehat{u}_{2,1}(.;s)=\widehat{F}_1(\cdot;s)-\widehat{F}_2(\cdot;s) \quad \text{in } \mathbb{R}^2. \tag{3.14}$$

We can then obtain its explicit expression by a convolution with the fundamental solution Ψ_s of the Helmholtz equation (in two dimensions) with the wave number $k=\boldsymbol{i}s^{\frac{\alpha}{2}}$

$$\begin{aligned}\widehat{u}_{2,1}(x;s)&=\Psi_s*\widehat{F}_1(x;s)-\Psi_s*\widehat{F}_2(x;s)\\ &=\sum_{j=1}^{m_1}s^{-1}\lambda_j^1\Psi_s\left(x-z_j^1\right)-\sum_{j=1}^{m_2}s^{-1}\lambda_j^2\Psi_s\left(x-z_j^2\right),\end{aligned}$$

where Ψ_s is given by

$$\Psi_s(x)=\frac{1}{2\pi}K_0\Big(s^{\frac{\alpha}{2}}|x|\Big),$$

with K_0 used to denote the modified Bessel function of the second kind [1]. Since $\widehat{u}_{2,1}$ is analytic in the connected domain $\mathbb{R}^2\backslash\cup\{z_j^1,z_j^2\}$ and null outside of Ω, it is null also in $\mathbb{R}^2\backslash\cup\{z_j^1,z_j^2\}$. Consequently,

$$\sum_{j=1}^{m_1}\lambda_j^1\Psi_s\left(x-z_j^1\right)-\sum_{j=1}^{m_2}\lambda_j^2\Psi_s\left(x-z_j^2\right)=0, \text{ for all } x\in\mathbb{R}^2\backslash\cup\{z_j^1,z_j^2\}. \tag{3.15}$$

Now, suppose that

$$\exists\, j_0\in\{1,\ldots,m_2\} \text{ such that } z_{j_0}^2\neq z_k^1, \text{ for all } k\in\{1,\ldots,m_1\}.$$

Thanks to equation (3.15), we have

$$\sum_{j=1}^{m_1}\lambda_j^1\Psi_s\left(x-z_j^1\right)-\sum_{j=1,j\neq j_0}^{m_2}\lambda_j^2\Psi_s\left(x-z_j^2\right)=\lambda_{j_0}^2\Psi_s\left(x-z_{j_0}^2\right). \tag{3.16}$$

For small arguments $0<|z|\ll 1$, we have (see [1])

$$K_0(z)=-\ln\left(\frac{z}{2}\right)-\gamma,$$

where γ is the Euler-Mascheroni constant. Therefore,

$$\left|\lambda_{j_0}^2\Psi_s\left(x-z_{j_0}^2\right)\right|\to\infty \text{ as } x\to z_{j_0}^2, \tag{3.17}$$

and

$$\lim_{x\to z_{j_0}^2}\left|\sum_{j=1}^{m_1}\lambda_j^1\Psi_s\left(x-z_j^1\right)-\sum_{j=1,j\neq j_0}^{m_2}\lambda_j^2\Psi_s\left(x-z_j^2\right)\right|<\infty. \tag{3.18}$$

Now by tending x to $z_{j_0}^2$ in (3.16), one gets

$$\lim_{x\to z_{j_0}^2}\left|\sum_{j=1}^{m_1}\lambda_j^1\Psi_s\left(x-z_j^1\right)-\sum_{j=1,j\neq j_0}^{m_2}\lambda_j^2\Psi_s\left(x-z_j^2\right)\right|=\infty,$$

which is impossible, since this limit is finite from (3.18). Thus, the sets $\left\{z_j^\ell, 1\le j\le m_\ell\right\}$, $(\ell=1,2)$, must be identical. Then, one can write $z_j^1=z_j^2$ after renumbering z_j if necessary and the same argument yields $\lambda_j^1=\lambda_j^2$. Thus the proof of the Theorem is completed. □

3.2. **Local stability.** In this section, based on the Laplace transform properties, we prove a local Lipschitz stability result of our inverse pointwise-source problem. More precisely, we prove, in the same way as in [20], a local Lipschitz stability result derived from the Gâteaux differentiability of the observed data u_{obs} by establishing that its Gâteaux derivative is not null. To this end, let F be a source term defined as

$$F(x)=\sum_{i=1}^{m}\lambda_i\delta(x-z_i). \tag{3.19}$$

Let $\psi=(p_i,a_i)_{1\le i\le m}\in\mathbb{R}\times\mathbb{R}^2$ be arbitrary vectors. For a sufficiently small step $h\neq 0$ such that $z_i+h\,a_i\in\Omega$, $1\le i\le m$, we define the following perturbed source term

$$F^h(x)=\sum_{i=1}^{m}(\lambda_i+h\,p_i)\,\delta(x-(z_i+h\,a_i)). \tag{3.20}$$

Denoting by u^h the solution to the following perturbed problem

$$\left\{\begin{array}{rcll}\partial_t^\alpha u^h-\Delta u^h &=& F^h & \text{in}\quad \Omega\times(0,T),\\ u^h &=& u_d & \text{on}\quad \Sigma\times(0,T),\\ u^h(.,0) &=& 0 & \text{in}\quad \Omega,\end{array}\right. \tag{3.21}$$

and by setting

$$u_{obs}^h=u^h\ \text{ in }\ \Omega_0\times(0,T), \tag{3.22}$$

the following theorem states the obtained stability result.

Theorem 5. *(Local Lipschitz stability). If there exists $i\in\{1,\cdots,m\}$ such that $p_i\neq 0$ or $a_i\neq 0$, then*

$$\lim_{h\to 0}\frac{1}{|h|}\left\|u_{obs}^h-u_{obs}\right\|_{L^2(0,T;L^2(\Omega_0))}\neq 0. \tag{3.23}$$

The proof of this theorem is based on the following local stability of the Helmholtz problem, which is described in the next section.

3.2.1. *Local stability result for the Helmholtz problem.* We suppose that the domain Ω contains m monopolar sources, located at S_i with intensities $\gamma_i\neq 0$, $1\le i\le m$. Furthermore, the points S_i are assumed to be mutually distinct. Then, for a sufficiently small $h\neq 0$, we define the perturbed source f^h as follows

$$f^h=\sum_{i=1}^{m}\gamma_i^h\delta(x-S_i^h), \tag{3.24}$$

where

$$(\gamma_i^h,S_i^h):=(\gamma_i+h\tau_i,S_i+h\,Z_i),\quad i\in\{1,\cdots,m\},$$
$$(\tau_i,Z_i)\in\mathbb{R}\times\mathbb{R}^2\ \text{ for } 1\le i\le m.$$

For a given non-vanishing boundary data $g \in H^{1/2}(\Sigma)$ and a wave number k, let w^h be the solution to the following Helmholtz equation

$$\left\{ \begin{array}{rcll} \Delta w^h + k^2 w^h & = & f^h & \text{in} \quad \Omega, \\ w^h & = & g & \text{on} \quad \Sigma. \end{array} \right. \tag{3.25}$$

Particularly, w^0 represents the solution to (3.25) with a non perturbed source term f^0 (i.e. $h = 0$) defined as

$$f^0 = \sum_{i=1}^{m} \gamma_i \delta(x - S_i). \tag{3.26}$$

Denoting by w^h_{obs} and w^0_{obs} respectively the observed Helmholtz equation solution in the subdomain Ω_0, i.e.

$$w^h_{obs} = w^h \quad \text{in} \quad \Omega_0 \quad \text{and} \quad w^0_{obs} = w^0 \quad \text{in} \quad \Omega_0, \tag{3.27}$$

the following lemma summarizes a local stability result for the Helmholtz problem.

Lemma 6. *Assume that k^2 is not an eigenvalue of $(-\Delta)$ with Dirichlet condition on the boundary Σ. If there exists $i \in \{1, \cdots, m\}$ such that $\tau_i \neq 0$ or $Z_i \neq 0$, then we have*

$$\lim_{h \to 0} \frac{1}{|h|} \| w^h_{obs} - w^0_{obs} \|_{L^2(\Omega_0)} \neq 0. \tag{3.28}$$

Proof. This stability result can be proved by adapting the same analysis presented in [22, Lemma 2]. □

3.2.2. *Proof of Theorem 5.* We start our analysis by extending the observation data u_{obs} by zero outside the interval $[0, T]$. For each $s \in \mathbb{C}$ such that $\Re\{s\} > 0$, we consider the following time-integral quantity

$$\widehat{u_{obs}}(x; s) := \int_0^\infty e^{-s\,t}\, u_{obs}(x, t)\, \mathrm{d}t, \text{ for } x \in \Omega_0. \tag{3.29}$$

Setting,

$$\widehat{u^h}(x; s) = \int_0^\infty e^{-st} u^h(x, t) \mathrm{d}t, \tag{3.30}$$

$$\widehat{u^h_{obs}}(x; s) = \int_0^\infty e^{-st} u^h_{obs}(x, t) \mathrm{d}t, \tag{3.31}$$

$$\widehat{F^h}(x; s) = \sum_{i=1}^{m} (\frac{\lambda_i}{s} + h\, \frac{p_i}{s}) \delta(x - (z_i + h\, a_i)), \tag{3.32}$$

$$\widehat{u_d}(x; s) = \int_0^\infty e^{-st} u_d(x, t) \mathrm{d}t. \tag{3.33}$$

By applying the Laplace transform in (3.21) and making use the formula (3.6), one can check that $\widehat{u^h}$ solves the following transformed algebraic equation

$$\left\{ \begin{array}{rcll} s^a \widehat{u^h}(x; s) - \Delta \widehat{u^h}(x; s) & = & \widehat{F^h}(x; s), & x \in \Omega, \\ \widehat{u^h}(x; s) & = & \widehat{u_d}(x; s), & x \in \Sigma, \\ \widehat{u^h}(x; s) & = & \widehat{u^h_{obs}}(x; s), & x \in \Omega_0. \end{array} \right. \tag{3.34}$$

Let $k = \boldsymbol{i}\, s^{\frac{\alpha}{2}}$, then function $\widehat{u^h}$ can be characterized as the solution of the following Helmholtz type equation

$$\left\{\begin{array}{rcll} \Delta\widehat{u^h}(x;s) + k^2\widehat{u^h}(x;s) & = & \widetilde{F^h}(x;s), & x \in \Omega, \\ \widehat{u^h}(x;s) & = & \widehat{u_d}(x;s), & x \in \Sigma, \\ \widehat{u^h}(x;s) & = & \widehat{u^h_{obs}}(x;s), & x \in \Omega_0, \end{array}\right. \tag{3.35}$$

where

$$\widetilde{F^h}(x;s) := -\widehat{F^h}(x;s) = \sum_{i=1}^{m} (\widetilde{\lambda_i(s)} + h\,\widetilde{p_i(s)})\delta(x - (z_i + h\,a_i)),$$

with

$$\widetilde{\lambda_i(s)} = \frac{-\lambda_i}{s} \quad \text{and} \quad \widetilde{p_i(s)} = \frac{-p_i}{s}.$$

In addition, in the particular case when $h = 0$, function $\widehat{u^0}$, defined as

$$\widehat{u^0}(x;s) = \int_0^\infty e^{-st} u^0(x,t)\mathrm{d}t, \tag{3.36}$$

satisfies the following unperturbed problem

$$\left\{\begin{array}{rcll} \Delta\widehat{u^0}(x;s) + k^2\widehat{u^0}(x;s) & = & \widetilde{F^0}(x;s), & x \in \Omega, \\ \widehat{u^0}(x;s) & = & \widehat{u_d}(x;s), & x \in \Sigma, \\ \widehat{u^0}(x;s) & = & \widehat{u_{obs}}(x;s), & x \in \Omega_0, \end{array}\right. \tag{3.37}$$

where the source term $\widetilde{F^0}(.;s)$ represents the action of m points sources characterized by their locations z_i and intensities $\widetilde{\lambda_i(s)}$; i.e.

$$\widetilde{F^0}(x;s) = \sum_{i=1}^{m} \widetilde{\lambda_i(s)}\delta(x - z_i).$$

Moreover, the source term $x \mapsto \widetilde{F^h}(x;s)$ can be viewed as a linear perturbation of the term $x \mapsto \widetilde{F^0}(x;s)$ in the direction $(\widetilde{p_i(s)}, a_i)_{1\le i\le m}$ with a small step $h \neq 0$ such that $z_i + h\,a_i \in \Omega$, for all $1 \le i \le m$.

As one can observe here, the considered inverse source problem for the time-fractional diffusion equation (1.1)-(1.5) is reformulated as an inverse source problem for the Helmholtz equation with a wave number $k = \boldsymbol{i}\, s^{\alpha/2}$.
Consequently, by Theorem 6 one can deduce the following local stability result

$$\lim_{h\to 0} \frac{\|\widehat{u^h_{obs}} - \widehat{u_{obs}}\|_{L^2(\Omega_0)}}{|h|} \neq 0. \tag{3.38}$$

It is important to point out that the applicability of Theorem 6 follows from the considered assumption on the directions $(\widetilde{p_i(s)}, a_i)_{1\le i\le m}$ and the fact that $k^2 = -s^\alpha$ is not an eigenvalue of $(-\Delta)$ with Dirichlet condition on Σ.

Finally, using Cauchy-Lipschitz inequality one can derive

$$\|\widehat{u^h_{obs}} - \widehat{u_{obs}}\|_{L^2(\Omega_0)} \le \frac{1}{\sqrt{2\,\Re\{s\}}} \|u^h_{obs} - u_{obs}\|_{L^2(0,T;L^2(\Omega_0))}, \tag{3.39}$$

which implies the desired result.

4. Reconstruction method

In this section, we aim to identify the number of point sources m^*, their locations z_i^* and their intensities λ_i^* from local measurements. To achieve this task, we transform our inverse problem into an optimization one. To this end, we start our analysis by characterizing the unknown source term F^* as the solution to a constrained optimization problem, minimizing a least-squares type functional on the following set of admissible source terms

$$\mathcal{C}_\delta(\Omega) = \left\{ F:\ \Omega \to \mathbb{R};\ F(x) = \sum_{i=1}^{n} \lambda_i \delta(x - z_i) \right\}, \tag{4.1}$$

where n is a non-negative integer, λ_i are non-null scalar quantities, and $z_i \in \Omega \backslash \overline{\Omega_0}$, $1 \le i \le n$. Furthermore, the points z_i are assumed to be mutually distinct.

In this setting, the unknown source term F^* is characterized as the solution to

$$\underset{F \in \mathcal{C}_\delta(\Omega)}{\text{Minimize}}\, \mathcal{K}(F), \text{ subject to (4.4)}, \tag{4.2}$$

where $\mathcal{K}$ is a cost function defined on each trial source term $F \in C_\delta(\Omega)$ by

$$\mathcal{K}(F) := \int_0^T \Big(\int_{\Omega_0} \Big| u_F - u_{obs} \Big|^2 \mathrm{d}x \Big) \mathrm{d}t, \tag{4.3}$$

where $u_F : \Omega \times (0,T) \to \mathbb{R}$ is the associated potential, solution to the following time-fractional diffusion problem

$$\left\{ \begin{array}{rcll} \partial_t^\alpha u_F - \Delta u_F & = & F & \text{in} \quad \Omega \times (0,T), \\ u_F & = & u_d & \text{on} \quad \Sigma \times (0,T), \\ u_F(.,0) & = & 0 & \text{in} \quad \Omega. \end{array} \right. \tag{4.4}$$

As one can remark here, the function $\mathcal{K}$ measures the discrepancy between the computed and observed potential in the sub-domain Ω_0.

Now, we present the well-posedness of the problem (4.2). More precisely, we prove that the solution of (4.2) is "equivalent" to the solution of the considered inverse source problem. This is the subject of the next proposition. We begin firstly by introducing the following concept of data compatibility:

Definition 7. *An observation data $u_{obs} \in L^2(0,T; L^2(\Omega_0))$ is said to be compatible if the considered inverse problem admits at least one solution.*

Proposition 8. *Let $u_{obs} \in L^2(0,T; L^2(\Omega_0))$ be a given compatible data in the sense of Definition 7. Hence $F \in \mathcal{C}_\delta(\Omega)$ is a solution of the inverse problem if and only if it is a solution of optimization problem* (4.2).

Proof. Let $F \in \mathcal{C}_\delta(\Omega)$ be the solution of the considered inverse problem. Then $u_F = u_{obs}$ in $\Omega_0 \times (0,T)$. Therefore F is a minimum of $\mathcal{K}$ with $\mathcal{K}(F) = 0$. Let now $F_1 \in \mathcal{C}_\delta(\Omega)$ be the solution of (1.1)-(1.5) and $F_2 \in \mathcal{C}_\delta(\Omega)$ another solution of (4.2) such that $\mathcal{K}(F_2) = 0$. Hence $u_{F_1} = u_{obs} = u_{F_2}$ in $\Omega_0 \times (0,T)$. Thanks to the identifiabilty Theorem 4, we obtain $F_1 = F_2$ which is the solution of inverse problem. □

To solve the minimization problem, we will develop an reconstruction approach based on the sensitivity analysis of the misfit functional (4.3) with respect to the set of admissible solutions (4.1). The main ideas of the proposed reconstruction process are described in the next section.

5. Sensitivity analysis

In this section, the sensitivity of the cost functional (4.3) with respect to the source F is derived. The basic idea consists in introduce a number m of pointwise sources with intensities λ_i, concentrated at the arbitrary points $z_i \in \Omega \setminus \overline{\Omega_0}$, for $i = 1, \cdots, m$.

More precisely, the perturbed counterpart of the source term F can be defined as

$$F_\delta(x) = F(x) + \sum_{i=1}^{m} \lambda_i \delta(x - z_i). \tag{5.1}$$

Therefore, the cost functional associated with the perturbed source term F_δ is written as

$$\mathcal{K}(F_\delta) := \int_0^T \Big(\int_{\Omega_0} \Big| u_{F_\delta} - u_{obs} \Big|^2 \mathrm{d}x \Big) \mathrm{d}t, \tag{5.2}$$

where $u_{F_\delta} : \Omega \times (0,T) \to \mathbb{R}$ is solution to the following time-fractional diffusion problem

$$\left\{ \begin{array}{rcll} \partial_t^\alpha u_{F_\delta} - \Delta u_{F_\delta} & = & F_\delta & \text{in} \quad \Omega \times (0,T), \\ u_{F_\delta} & = & u_d & \text{on} \quad \Sigma \times (0,T), \\ u_{F_\delta}(.,0) & = & \varphi & \text{in} \quad \Omega. \end{array} \right. \tag{5.3}$$

From the above time-fractional boundary value problem and by taking into account that the source is time-independent, the solution u_{F_δ} can be decomposed as

$$u_{F_\delta}(x,t) = u_F(x,t) + \sum_{i=1}^{m} \lambda_i h_i(x), \tag{5.4}$$

where h_i are the solutions of the following auxiliary boundary value problems for $i = 1, \cdots, m$: Find h_i, such that

$$\left\{ \begin{array}{rcll} -\Delta h_i & = & \delta(\cdot - z_i) & \text{in} \quad \Omega, \\ h_i & = & 0 & \text{on} \quad \Sigma. \end{array} \right. \tag{5.5}$$

Finally, the boundary value problem (5.3) is complemented with the following initial condition $\varphi(x) = \sum_{i=1}^m \lambda_i h_i(x)$. From these elements, we have

$$\mathcal{K}(F_\delta) - \mathcal{K}(F) = 2 \sum_{i=1}^{m} \lambda_i \int_{\Omega_0} h_i \left(\int_0^T (u_F - u_{obs}) \mathrm{d}t \right) \mathrm{d}x + T \sum_{i,j=1}^{m} \lambda_i \lambda_j \int_{\Omega_0} h_i h_j \, \mathrm{d}x. \tag{5.6}$$

Now, we want to find a better approximation to the target F^* than the initial guess F based on the derived sensitivity analysis (5.6). Therefore, let us introduce the following quantity

$$\Psi(\beta, \zeta) = 2\beta \cdot d(\zeta) + H(\zeta)\beta \cdot \beta, \tag{5.7}$$

where vectors $\zeta = (z_1, \cdots, z_m)$ and $\beta = (\lambda_1, \cdots, \lambda_m)$. The vector d and the matrix H have entries

$$d(\zeta) = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix} \quad \text{and} \quad H(\zeta) = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1m} \\ H_{21} & H_{22} & \cdots & H_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m1} & H_{m2} & \cdots & H_{mm} \end{pmatrix}, \tag{5.8}$$

where

$$d_i = \int_{\Omega_0} h_i \left(\int_0^T (u_F - u_{obs}) \mathrm{d}t \right) \mathrm{d}x \quad \text{and} \quad H_{ij} = T \int_{\Omega_0} h_i h_j \, \mathrm{d}x. \tag{5.9}$$

Given the function (5.7), its minimum is trivially found when:

$$\langle D_\beta \Psi(\beta,\zeta), \delta\beta\rangle = 0, \quad \forall \delta\beta \in \mathbb{R}^m. \tag{5.10}$$

Since H_{ij} is symmetric positive definite, the minimization of the function $\Psi(\beta,\zeta)$ with respect to β leads to the global minimum. In particular,

$$2(H(\zeta)\beta + d(\zeta)) \cdot \delta\beta = 0, \quad \forall\, \delta\beta \in \mathbb{R}^m \quad \Rightarrow \quad H(\zeta)\beta = -d(\zeta), \tag{5.11}$$

provided that $H = H^\top$. Therefore,

$$\beta = \beta(\zeta) = -H(\zeta)^{-1} d(\zeta), \tag{5.12}$$

such that the quantity β, solving (5.12), becomes a function of the locations ζ. After replacing the solution of (5.12) into $\Psi(\beta,\zeta)$, defined by (5.7), the optimal locations $\zeta^\star$ can be obtained from a combinatorial search over the domain Ω. These locations are the solutions to the following minimization problem:

$$\zeta^\star = \underset{\zeta\subset Z}{\operatorname{argmin}} \left\{ \Psi(\beta(\zeta),\zeta) = \beta(\zeta)\cdot d(\zeta) \right\} , \tag{5.13}$$

where Z is the set of admissible locations of the unknown sources. Since this step is bottleneck of the proposed approach, we refer to [47, 5] for more sophisticated strategies based on meta-heuristic and multigrid schemes for solving the minimization problem (5.13). Then, the optimal sources are characterized by the pair $\zeta^\star$ and $\beta^\star = \beta(\zeta^\star)$ of locations and intensities, respectively.

To summarize, we have introduced a second order reconstruction algorithm which is able to find the optimal intensities $\beta^\star$ of the hidden pointwise sources and their locations $\zeta^\star$ for a given number m of trial sources. The inputs to the algorithm are:

- the vector d and the matrix H;
- the $M = \text{card}(Z)$ points at which the system (5.12) is solved;
- the number m of pointwise sources to be reconstructed.

The algorithm returns the optimal intensities $\beta^\star$ and locations $\zeta^\star$ for a given number of trial sources m. The above procedure is written in pseudo-code format as shown in Algorithm 1. In the algorithm, Π maps the vector of nodal indices $\mathcal{I} = (i_1, i_2, \ldots, i_m)$ to the corresponding vector of nodal coordinates ζ.

Finally, let us point out some interesting features of the Algorithm 1: (a) when the number m^* of target pointwise sources is unknown, the algorithm can be started based on the assumption that there exists $m > m^*$ pointwise sources and then we should find a number $(m - m^*)$ of trial sources with negligible intensities; (b) if the centers of the target sources ζ^* do not belong to the set of admissible locations Z, the algorithm returns a vector $\zeta^\star$ of optimal locations which is the closest to the true one ζ^*; and (c) since a combinatorial search over all the M-points of the set Z has to be performed, this exhaustive search becomes rapidly infeasible for $M \approx 2m$, as m increases. In the ensuing numerical examples, we set $m \ll M$, with m small, so that Algorithm 1 runs in a few seconds for all examples. For a more detailed discussion on the complexity of Algorithm 1, we refer to [47].

6. Numerical results

In this section, some numerical experiments are presented showing different features of the proposed reconstruction algorithm. We consider that the domain $\Omega = (0,1)\times(0,1)$. The boundary data $u_d = 0$ in all cases. The problems are discretized by using standard Finite Element Method in space and Finite Difference Method in time following the same procedure as described in [57]. In particular, the domain Ω is discretized with three-node

Algorithm 1: Second Order Reconstruction Algorithm

input : d, H, m, M;
output: the optimal solution $\Psi^\star$, $\beta^\star$, $\zeta^\star$;

Initialization: $\Psi^\star \leftarrow \infty$; $\beta^\star \leftarrow 0$; $\zeta^\star \leftarrow 0$;
for $i_1 \leftarrow 1$ **to** M **do**
  for $i_2 \leftarrow i_1 + 1$ **to** M **do**
    $\vdots$
    for $i_m \leftarrow i_{m-1} + 1$ **to** M **do**

$$d \leftarrow \begin{bmatrix} d_{(i_1)} \\ d_{(i_2)} \\ \vdots \\ d_{(i_m)} \end{bmatrix}; \quad H \leftarrow \begin{bmatrix} H_{(i_1,i_1)} & H_{(i_1,i_2)} & \cdots & H_{(i_1,i_m)} \\ H_{(i_2,i_1)} & H_{(i_2,i_2)} & \cdots & H_{(i_2,i_m)} \\ \vdots & \vdots & \ddots & \vdots \\ H_{(i_m,i_1)} & H_{(i_m,i_2)} & \cdots & H_{(i_m,i_m)} \end{bmatrix};$$

      $\mathcal{I} \leftarrow (i_1, i_2, \ldots, i_m)$; $\zeta \leftarrow \Pi(\mathcal{I})$; $\beta \leftarrow -H^{-1}d$; $\Psi \leftarrow \beta \cdot d$;
      if $\Psi < \Psi^\star$ **then**
        $\zeta^\star \leftarrow \zeta$; $\beta^\star \leftarrow \beta$; $\Psi^\star \leftarrow \Psi$;
      end if
    end for
  end for
end for
return $\Psi^\star$, $\beta^\star$, $\zeta^\star$;

finite elements. The mesh is generated from a grid of size 10×10, where each resulting square is divided into four identical triangles. The resulting vertices, excepting the ones on the boundary of the domain Ω, are used as the set of admissible locations of the unknown sources Z, leading to 181 trial points. Finally, each triangle is divided into 4 more triangles in such a way that the initial pattern is preserved. This procedure is repeated five times, leading to 409600 triangles. The time T is set as $T = 1$, which is divided into 100 uniform increments.

Since we are considering synthetic data, in order to alleviate the so-called *inverse crime*, eventually the target u_{obs} is corrupted with White Gaussian Noise (WGN). Therefore, noisy data in our context can be interpreted as uncertainties in the measurements.

In the figures to be presented, the black circles represent the sources, where their radii and centers correspond to the intensities and locations of the pointwise sources, respectively. Finally, the observable domain Ω_0 is represented in gray scale.

6.1. **Example 1.** In this first example, we consider the reconstruction of a single pointwise source of intensity $\lambda_1^* = 0.1$, as shown in Figure 1. The obtained results for $\alpha = 0.2, 0.4, 0.6, 0.8$ are presented in Figure 2, where the reconstructions are almost exact in all cases even for a very small observable domain Ω_0. More precisely, the reconstructed centres are exact, whereas the associated intensities are slightly underestimated. In addition, there are a small difference in the obtained intensities for different values of α, which are reported in Table 1. Since the obtained results are almost the same independently of α, from now on we fix $\alpha = 0.5$ in all examples.

6.2. **Example 2.** In this example we deal with the reconstruction of two simultaneous pointwise sources of different intensities, according to Figure 3. The obtained results for varying configurations for the observable domain Ω_0 are presented Figure 4. We observe

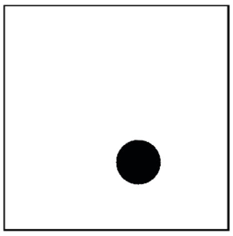

FIGURE 1. Example 1: Target to be reconstructed.

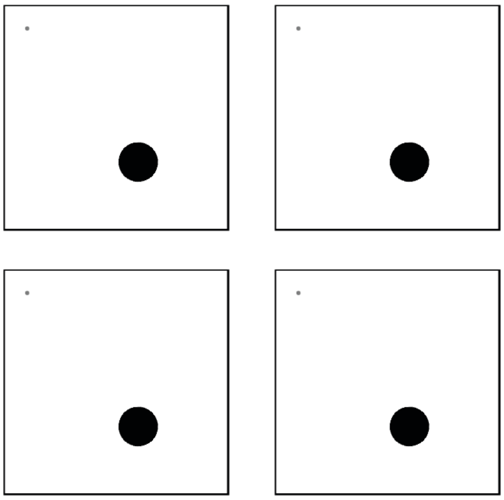

FIGURE 2. Example 1: Obtained reconstructions for varying values of α: 0.2 (top-left), 0.4 (top-right), 0.6 (bottom-left) and 0.8 (bottom-right). The small gray circles represent the observable domains Ω_0.

TABLE 1. Example 1: Obtained results for varying values of α.

α	centre	intensity
0.2	$(0.6, 0.3)$	0.08928
0.4	$(0.6, 0.3)$	0.08879
0.6	$(0.6, 0.3)$	0.08857
0.8	$(0.6, 0.3)$	0.08869

that the algorithm fails for Ω_0 given by a small circle and by two small circles, as reported in Figure 4, top-left and top-right, respectively. On the other hand, the algorithm is able to reconstruct the target after rotate the two small circles in 90° as well as by considering four small circles forming Ω_0, as can be seen in Figure 4, bottom-left and bottom-right, respectively. Therefore, not only the size of Ω_0 is important in the reconstruction process, but also its spatial distribution.

Now, we consider again Ω_0 given by four small circles. However, the measurement u_{obs} is corrupted with varying levels of White Gaussian Noise (WGN). The obtained results are presented in Figure 5 for 20% (left) and 40% (right) of WGN. We observe that for 20% of WGN the reconstruction is quite good. In contrast, the algorithm fails for 40% of WGN. This example shows that the proposed approach is very resilient with respect to noisy data, since 20% of WGN can be considered as a very high level of noise.

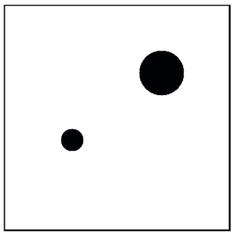

FIGURE 3. Example 2: Target to be reconstructed.

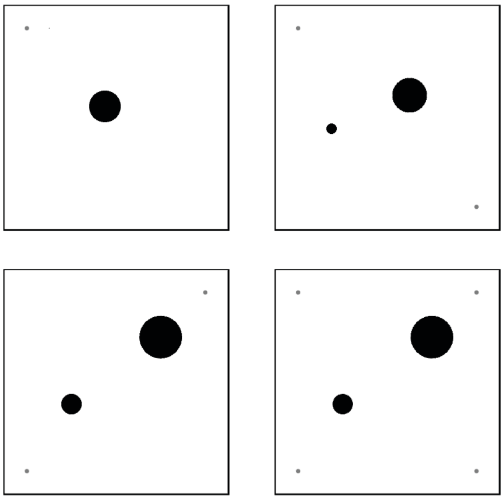

FIGURE 4. Example 2: Obtained reconstructions for varying observable domains Ω_0 in gray.

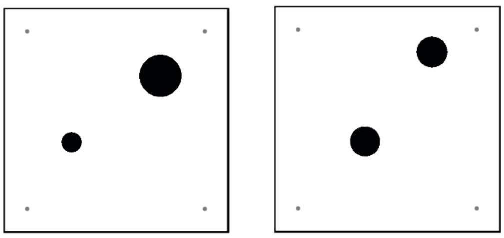

FIGURE 5. Example 2: Obtained results for 20% (left) and 40% (right) of WGN. The small gray circles represent the observable domain Ω_0.

6.3. **Example 3.** In this example we consider again the reconstruction of two simultaneous pointwise sources, but one with intensity ten times smaller than the other one, as shown in Figure 6. The obtained results considering two configurations for the observable domain Ω_0 are presented in Figure 7. We observe that the algorithm fails for Ω_0 given by four small circles as reported in Figure 7, left. In particular, the high intensity source has been found, whereas the low intensity source got lost, as expected. On the other hand, the algorithm is able to reconstruct the target after considering five small circles forming Ω_0, as can be seen in Figure 4, right. This result corroborates with what we have observed in the previous example.

6.4. **Example 4.** In this example we consider the reconstruction of three simultaneous pointwise sources of varying intensities, as shown in Figure 8. The obtained results for

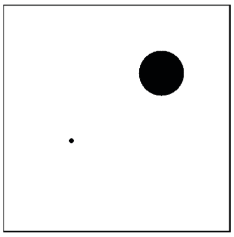

FIGURE 6. Example 3: Target to be reconstructed.

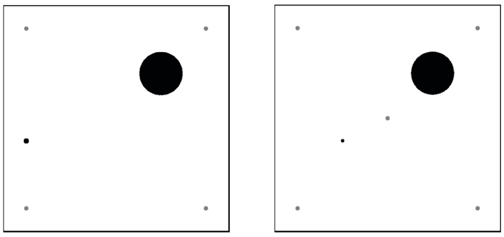

FIGURE 7. Example 3: Obtained reconstructions for varying observable domains Ω_0 in gray.

two different configurations for the observable domain Ω_0 are presented in Figure 9. We observe that the algorithm fails for Ω_0 given by four small circles, according to Figure 9, left. In contrast, the algorithm is able to reconstruct the target for Ω_0 given by five small circles, as reported in Figure 9, right. Therefore, the more complex is the target to be reconstructed, the more information is needed, as expected.

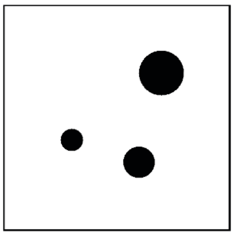

FIGURE 8. Example 4: Target to be reconstructed.

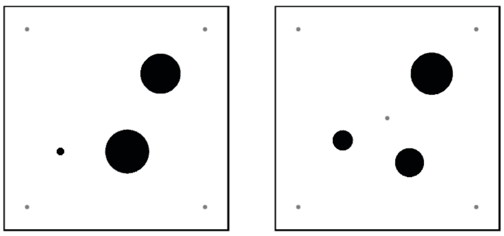

FIGURE 9. Example 4: Obtained reconstructions for varying observable domains Ω_0 in gray.

6.5. **Example 5.** In this example we consider the reconstruction of four simultaneous pointwise sources of varying intensities, as shown in Figure 10, left. The obtained result for Ω_0 given by five small circles is presented in Figure 10, right.

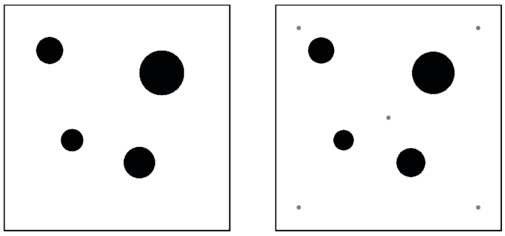

FIGURE 10. Example 5: Target to be reconstructed (left) and obtained reconstruction (right). The small gray circles represent the observable domain Ω_0.

6.6. **Example 6.** Finally, in this last example we consider the reconstruction of three simultaneous pointwise sources of the same intensities, as shown in Figure 11, left. The number of trial sources is set as $m = 4$ in the reconstruction algorithm. The obtained result for Ω_0 given by one big circle containing two of the hidden sources is presented in Figure 11, right. Note that in this case, the assumption $\zeta^* \not\subset \Omega_0$ is violated. Nevertheless, the reconstruction corroborates with the target. Actually, the algorithm returns the correct locations and intensities of the three hidden sources, in addition to one more source of negligible intensity, which is pointed by the red arrow in Figure 11, right.

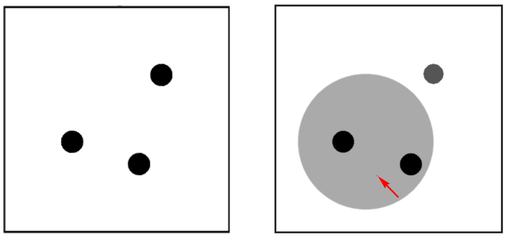

FIGURE 11. Example 6: Target to be reconstructed (left) and obtained reconstruction (right). The big gray circle represents the observable domain Ω_0. Note a fourth tinny ball representing the additional $(m - m^*)$ trial source of negligible intensity, which is pointed by the red arrow.

7. Concluding remarks

In this work, we have considered an inverse pointwise source problem for the time-fractional diffusion equation. We have discussed both theoretical and numerical aspects. In the theoretical part, we have analyzed the well-posedness of the Dirac time-fractional diffusion problem. For the inverse problem, we have established that the unknown point sources can be uniquely identified from local measured data. Also, we have derived a local Lipschitz stability result. The full stability (Lipschitz or logarithmic) problem is, however, up to our knowledge, still an open question which deserves attention.

In the numerical part, we have developed a fast and accurate reconstruction approach. The unknown pointwise sources are characterized as solution to an optimization problem minimizing a tracking-type functional measuring the misfit between the simulated and measured potentials inside the sub-domain $\Omega_0 \times (0, T)$. A noniterative and free of initial guess reconstruction algorithm is devised, allowing us to determine the number, the locations and the intensities of the hidden pointwise sources. The efficiency and accuracy of the proposed approach are confirmed by some numerical examples. Particularly, the influence of the noisy measurement of varying levels has been examined. Simulation results demonstrate that the approach is robust with respect to noise. In addition, the reconstruction is nearly exact when the noise level is low.

As for the crucial regularization issue, the least-squares or energy-like misfit functions have repeatedly been noticed to be self-regularizing, which means that no additional regularization is needed to stabilize the reconstruction process (see, for instance [47, 5]). The sensitivity analysis approach using a least-squares functional that we have been presenting here may thus be seen as a regularization technique. Though numerically once again proved in the present work, this feature still lacks mathematical proof in the specific case we are studying in the present paper.

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