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Relation Between Eshelbyan Mechanics and Topological Derivative Concept

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Synonyms

Eshelby problem; Eshelby tensor; Polarization tensor; Shape sensitivity analysis; Topological derivative

Definitions

The Pólya-Szegö polarization tensor, introduced in 1951, naturally appears on the famous Eshelby problem, also referred to as Eshelby theorem. This problem, formulated by Eshelby in 1957 and 1959, forms the basis to the theory of elasticity involving the determination of effective elastic properties of materials with multiple inhomogeneities (inclusions, pores, defects, cracks, etc.). This important result represents one of the major advances in the continuum mechanics theory of the twentieth century. In this work, a relation between Eshelby Mechanics and the modern concept of topological derivative is discussed. The topological derivative is defined as the first term (correction) of the asymptotic expansion of a given shape functional with respect to a small parameter that measures the size of singular domain perturbations, such as holes, cavities, inclusions, source terms, and cracks. Therefore, the Eshelby problem in general and the polarization tensor in particular play a crucial role on the topological asymptotic analysis, leading to a rich and fascinating theory. On the other hand, the topological derivative can also be defined as the singular limit of the classical shape derivative. The shape gradient can be interpreted as the flux of the Eshelby energy-momentum tensor, introduced by Eshelby in 1975, across to the moving boundary. It means that the polarization tensor – and thus the topological derivative – is related to the Eshelby tensor through the limit passage with respect to the small parameter measuring the size of the singular domain perturbation. Therefore, the Eshelby tensor together with the Eshelby problem can be seen as the main theoretical foundation for the topological derivative concept.

Introduction

In continuum mechanics, the famous problem formulated by Eshelby (1957, 1959), also called as Eshelby theorem, refers to the effect produced by a set of elastic inclusions embedded within an infinite homogeneous elastic body. Analytical

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solutions to these problems were first derived by John D. Eshelby in the 1950s. In particular, it was shown that the associated strain and stress fields are uniform inside the elastic inclusion. In addition, these fields were written in an elegant and compact form in terms of the background solution. The Eshelby problem plays a central role in the theory of elasticity involving the determination of effective elastic properties of materials with multiple inhomogeneities (inclusions, pores, defects, cracks, etc.). It has been considered one of the major advances in the continuum mechanics theory of the twentieth century (Kachanov et al. 2003), representing an important theoretical foundation for many problems in elasticity theory (Leugering et al. 2012). In this work, a relation between Eshelby Mechanics and the relatively new concept of topological derivative is discussed.

The topological derivative, rigorously introduced for the first time by Sokołowski and Zochowski (1999), measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation, such as the insertion of holes, cavities, inclusions, source terms, or cracks. This concept has proved to be useful in the treatment of a wide range of physical and engineering problems such as topology optimization, inverse problems, image processing, multiscale material design, fracture mechanics sensitivity analysis, and damage evolution modeling. For a comprehensive account on the topological derivative concept and its applications, see, for instance, the book by Novotny and Sokołowski (2013). Since the topological derivative represents the sensitivity with respect to singular domain perturbations, the Pólya-Szegö polarization tensor (Pólya and Szegö 1951) plays a crucial role on the topological asymptotic analysis, which naturally appears on the Eshelby theorem. On the other hand, the topological derivative can also be defined as the singular limit of the classical shape derivative as shown in the book by Novotny and Sokołowski (2013). The shape gradient can be interpreted as the flux of the Eshelby energy-momentum tensor, introduced by Eshelby (1975), across to the moving boundary. It means that the polarization tensor – and thus the topological derivative – is related to the Eshelby tensor through the limit passage with respect to the small parameter measuring the size of the singular domain perturbation.

Therefore, the Eshelby tensor together with the Eshelby problem in general and the polarization tensor in particular can be seen as the main theoretical foundation for the topological derivative concept, leading to a rich and fascinating theory. In order to show the applicability of these fundamental results, a simple numerical example in the context of structural topology optimization into three spatial dimensions is presented.

The Topological Derivative Concept

Let $\mathscr{D} \subset \mathbb{R}^3$ be an open and bounded domain with a Lipschitz boundary $\partial\mathscr{D}$, which is subject to a nonsmooth perturbation confined in a small region $B_\varepsilon(\widehat{x})$ of size ε centered at an arbitrary point $\widehat{x} \in \mathscr{D}$. A characteristic function $x \mapsto \chi(x)$, $x \in \mathbb{R}^3$ is introduced, associated to the unperturbed domain, namely, $\chi = \mathbb{1}_{\mathscr{D}}$, such that

$$|\mathscr{D}| = \int_{\mathbb{R}^3} \chi \, , \tag{1}$$

where $|\mathscr{D}|$ is the Lebesgue measure of $\mathscr{D}$. Then, a characteristic function associated to the topologically perturbed domain of the form $x \mapsto \chi_\varepsilon(\widehat{x}; x)$, $x \in \mathbb{R}^3$, is defined. In the case of a perforation, for example, $\chi_\varepsilon(\widehat{x}) = \mathbb{1}_{\mathscr{D}} - \mathbb{1}_{\overline{B_\varepsilon(\widehat{x})}}$, and the perforated domain is obtained as $\mathscr{D}_\varepsilon(\widehat{x}) = \mathscr{D} \setminus \overline{B_\varepsilon(\widehat{x})}$. Then, it is assumed that a given shape functional $\psi(\chi_\varepsilon(\widehat{x}))$, associated to the topologically perturbed domain, admits the following topological asymptotic expansion

$$\begin{aligned}\psi(\chi_\varepsilon(\widehat{x})) = \psi(\chi) + f(\varepsilon) D_T\psi(\widehat{x}) \\ + \mathscr{R}(f(\varepsilon)) \, ,\end{aligned} \tag{2}$$

where $\psi(\chi)$ is the shape functional associated to the original domain, that is, without perturbation, $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0$ and $\mathscr{R}(f(\varepsilon)) = o(f(\varepsilon))$ is the remainder. The function $\widehat{x} \mapsto D_T\psi(\widehat{x})$ is called the topological derivative of ψ at $\widehat{x}$. Therefore, this derivative can be seen as a first-order correction of $\psi(\chi_\varepsilon(\widehat{x}))$, so that it can be used as a steepest-descent direction in an optimization process like in any method based on the gradient of the cost functional.

Now, the Topological-Shape Sensitivity Method is introduced (Novotny and Sokołowski 2013). It relies on the fact that the topological derivative can be defined as the singular limit of the classical shape derivative. Therefore, this approach can be seen as a generalization of the classical tool in shape optimization, which is summarized through the following proposition:

Proposition 1 *Let $\psi(\chi_\varepsilon(\widehat{x}))$ be the shape functional associated to the topologically perturbed domain, which admits, for ε small enough, the topological asymptotic expansion of the form (2). It is assumed that the remainder $\mathscr{R}(f(\varepsilon))$ in (2) has the additional property $\mathscr{R}'(f(\varepsilon)) \to 0$, when $\varepsilon \to 0$. Then, the topological derivative can be written as*

$$D_T\psi(\widehat{x}) = \lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\frac{d}{d\varepsilon}\psi(\chi_\varepsilon(\widehat{x})) \ , \quad (3)$$

where $\frac{d}{d\varepsilon}\psi(\chi_\varepsilon(\widehat{x}))$ is the (shape) derivative of $\psi(\chi_\varepsilon(\widehat{x}))$ with respect to the small positive parameter ε.

Problem Statement: Elasticity System into Three Spatial Dimensions

Let $\mathscr{D}$ be decomposed into two subdomains, namely, $\omega \subset \mathscr{D}$ and $\mathscr{D}\setminus\overline{\omega}$. The subdomain $\Omega := \mathscr{D}\setminus\overline{\omega}$ represents an elastic and deformable region, whereas ω is filled by a very compliant material. The minimization problem can be defined in the following way

$$\left\{\begin{array}{ll} \underset{\Omega\subset\mathscr{D}}{\text{Minimize}} & -\mathscr{J}_\chi(u) \\ \text{subject to} & |\Omega| \leq M \ , \end{array}\right. \quad (4)$$

where the shape functional $\mathscr{J}_\chi(u)$ represents the total potential energy of the system, $|\Omega|$ is the Lebesgue's measure of Ω, and M represents the required volume at the end of the minimization process. The volume constraint is trivially imposed by using a linear penalization approach. For more elaborate strategies, see, for instance, the work by Campeão et al. (2014). In particular, the constrained optimization problem (4) is replaced by the following unconstrained optimization problem

$$\underset{\Omega\subset\mathscr{D}}{\text{Minimize}}\ \mathscr{F}_\chi(u) = -\mathscr{J}_\chi(u) + m\,|\Omega| \ , \quad (5)$$

where $m > 0$ is a fixed multiplier used to impose the volume constraint of elastic material. This means that the shape functional to be minimized is the strain energy stored into the structure with a volume constraint. In addition, the total potential energy $\mathscr{J}_\chi(u)$ is written as

$$\mathscr{J}_\chi(u) = \frac{1}{2}\int_{\mathscr{D}}\sigma(u)\cdot\nabla u^s - \int_{\Gamma_N}\overline{q}\cdot u \ , \quad (6)$$

where the vector function u is the solution of the following variational problem: Find $u \in \mathscr{U}$, such that

$$\int_{\mathscr{D}}\sigma(u)\cdot\nabla\eta^s = \int_{\Gamma_N}\overline{q}\cdot\eta \quad \forall\eta\in\mathscr{U}_0 \ . \quad (7)$$

Some terms in the above variational equation require explanation. The Cauchy stress tensor is given by

$$\sigma(u) = \rho\mathbb{C}\nabla u^s, \quad \text{with} \quad \nabla u^s = \frac{1}{2}(\nabla u + \nabla u^\top) \ , \quad (8)$$

where the parameter ρ is defined as

$$\rho = \rho(x) := \left\{\begin{array}{ll} 1 & \text{if } x\in\Omega \ , \\ \rho_0 & \text{if } x\in\omega \ , \end{array}\right. \quad (9)$$

with $0 < \rho_0 \ll 1$ used to represent the voids. The constitutive tensor $\mathbb{C}$ is given by

$$\mathbb{C} = \frac{E}{1+\nu}\left(\mathbb{I} + \frac{\nu}{1-2\nu}\mathrm{I}\otimes\mathrm{I}\right) \ , \quad (10)$$

where I and $\mathbb{I}$ are the second- and fourth-order identity tensors, respectively, E the Young modulus, and ν the Poisson ratio, both

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considered constants everywhere. The set $\mathscr{U}$ and the space $\mathscr{U}_0$ are respectively defined as: $\mathscr{U}:=\{\varphi \in H^1(\mathscr{D};\mathbb{R}^3) : \varphi_{|\Gamma_D} = \overline{u}\}$ and $\mathscr{U}_0:=\{\varphi \in H^1(\mathscr{D};\mathbb{R}^3) : \varphi_{|\Gamma_D} = 0\}$. In addition, $\partial\mathscr{D} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ with $\Gamma_D \cap \Gamma_N = \varnothing$, where Γ_D and Γ_N are Dirichlet and Neumann boundaries, respectively. Thus, $\overline{u}$ is a Dirichlet data on Γ_D and $\overline{q}$ is a Neumann data on Γ_N, both assumed to be smooth enough.

The topologically perturbed counterpart of the problem is now introduced. The idea consists in nucleating a spherical inclusion, denoted by $B_\varepsilon(\widehat{x})$, of radius ε and center at the arbitrary point $\widehat{x} \in \mathscr{D}$, such that $\overline{B_\varepsilon(\widehat{x})} \subset \mathscr{D}$. In this case $\chi_\varepsilon(\widehat{x})$ is defined as $\chi_\varepsilon(\widehat{x}) = \mathbb{1}_\mathscr{D} - (1-\gamma)\mathbb{1}_{B_\varepsilon(\widehat{x})}$, where $\gamma = \gamma(x)$ is the contrast on the material properties. From these elements, the following piecewise constant function can be introduced

$$\gamma_\varepsilon = \gamma_\varepsilon(x) := \begin{cases} 1 & \text{if } x \in \mathscr{D} \setminus \overline{B_\varepsilon} , \\ \gamma & \text{if } x \in B_\varepsilon . \end{cases} \quad (11)$$

The shape functional associated to the topologically perturbed problem, denoted by $\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon)$, is defined as

$$\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon) = \frac{1}{2}\int_\mathscr{D} \sigma_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon^s - \int_{\Gamma_N} \overline{q} \cdot u_\varepsilon , \quad (12)$$

where the vector function u_ε is the solution to the following variational problem: Find $u_\varepsilon \in \mathscr{U}$, such that

$$\int_\mathscr{D} \sigma_\varepsilon(u_\varepsilon) \cdot \nabla \eta^s = \int_{\Gamma_N} \overline{q} \cdot \eta, \quad \forall \eta \in \mathscr{U}_0 , \quad (13)$$

with the Cauchy stress tensor $\sigma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon \sigma(u_\varepsilon)$, where γ_ε is given by (11). In addition, two transmission conditions on the interface ∂B_ε comes out naturally from the variational formulation (13), namely, $[\![u_\varepsilon]\!] = 0$ and $[\![\sigma_\varepsilon(u_\varepsilon)]\!] n = 0$, where n is the normal unit vector field pointing toward the exterior of the inclusion B_ε. Here, the operator $[\![\varphi]\!]$ is used to denote the jump of the function φ on the interface ∂B_ε, i.e., $[\![\varphi]\!] = \varphi_{|_{\mathscr{D}\setminus\overline{B_\varepsilon}}} - \varphi_{|_{B_\varepsilon}}$ on ∂B_ε.

Topological Sensitivity Analysis

In this section all arguments concerning the existence of the topological derivative as well as the derivation of its closed formula through both methods introduced in the previous section are presented.

Existence of the Topological Derivative

The existence of the topological derivative is ensured by the following result:

Lemma 1 *Let u_ε and u be the solutions of problems (13) and (7), respectively. Then, the following estimate holds true*

$$\|u_\varepsilon - u\|_{H^1(\mathscr{D};\mathbb{R}^3)} \le C\varepsilon^{3/2} , \quad (14)$$

where the constant C is independent of the small parameter ε.

Proof By subtracting (7) from (13) and taking into account the definition for the contrast γ_ε, given by (11), it follows

$$\int_{\mathscr{D}\setminus B_\varepsilon} (\sigma(u_\varepsilon) - \sigma(u)) \cdot \nabla \eta^s + \int_{B_\varepsilon} (\gamma\sigma(u_\varepsilon) - \sigma(u)) \cdot \nabla\eta^s = 0 . \quad (15)$$

After adding and subtracting the term $\int_{B_\varepsilon} \gamma\sigma(u) \cdot \nabla\eta^s$ in the above expression, there is

$$\int_\mathscr{D} \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla\eta^s + (\gamma - 1)\int_{B_\varepsilon} \sigma(u) \cdot \nabla\eta^s = 0 . \quad (16)$$

Now, by taking $\eta = u_\varepsilon - u$ as test function in (16), the following equality is obtained

$$\int_\mathscr{D} \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla^s(u_\varepsilon - u) = \int_{B_\varepsilon} \mathrm{T}(u) \cdot \nabla^s(u_\varepsilon - u) , \quad (17)$$

where the notation $\mathrm{T}(u) = (1-\gamma)\sigma(u)$ has been introduced. From the Cauchy-Schwarz inequality, it follows that

$$\int_{\mathscr{D}} \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla^s(u_\varepsilon - u) \le \|\mathrm{T}(u)\|_{L^2(B_\varepsilon;\mathbb{R}^3)} \, \|\nabla^s(u_\varepsilon - u)\|_{L^2(B_\varepsilon;\mathbb{R}^3)} \le C_1 \varepsilon^{3/2} \, \|u_\varepsilon - u\|_{H^1(\mathscr{D};\mathbb{R}^3)} \ . \quad (18)$$

From the coercivity of the bilinear form on the left-hand side of (18), there is

$$c \, \|u_\varepsilon - u\|^2_{H^1(\mathscr{D};\mathbb{R}^3)} \le \int_{\mathscr{D}} \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla^s(u_\varepsilon - u) \ , \quad (19)$$

which leads to the result with $C = C_1/c$ independent of the small parameter ε. □

Topological Derivative Evaluation

The topological derivative of the shape functional (6) is now evaluated by using the methods previously introduced.

Direct Approach

Following similar derivations as presented in the proof of Lemma 1, the difference between the functionals $\mathscr{J}_\chi(u)$ and $\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon)$, respectively, defined in (6) and (12), can be written as an integral concentrated in the topological perturbations B_ε as follows

$$\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon) - \mathscr{J}_\chi(u) = \int_{B_\varepsilon} \frac{\gamma - 1}{2\gamma} \sigma_\varepsilon(u_\varepsilon) \cdot \nabla u^s \ . \quad (20)$$

In order to know the asymptotic behavior of the function u_ε with respect to the small parameter ε, the following ansätz is introduced

$$u_\varepsilon = u + w_\varepsilon + \tilde{u}_\varepsilon \ , \quad (21)$$

where u is the solution of the unperturbed problem (7), w_ε is the solution to an auxiliary exterior problem, and $\tilde{u}_\varepsilon$ is the remainder. In particular, the following auxiliary boundary value problem is considered and formally obtained when $\varepsilon \to 0$: Find $S_\varepsilon(w_\varepsilon)$, such that

$$\begin{cases} \operatorname{div} S_\varepsilon(w_\varepsilon) = 0 & \text{in } \mathbb{R}^3 \ , \\ S_\varepsilon(w_\varepsilon) \to 0 & \text{in } \infty \ , \\ [\![S_\varepsilon(w_\varepsilon)]\!] n = g & \text{on } \partial B_\varepsilon \ , \end{cases} \quad (22)$$

where $S_\varepsilon(w_\varepsilon) = \gamma_\varepsilon \mathbb{C} \nabla w_\varepsilon^s$ and $g = ((\gamma - 1) S(u)(\widehat{x}))n$, which has been obtained from a Taylor series expansion of $\sigma(u(x))$ around the point $\widehat{x}$, with $S(u) = \mathbb{C}\nabla u^s$. The boundary value problem (22) admits an explicit solution. Since the stress $S_\varepsilon(w_\varepsilon)$ is uniform inside the inclusion, the solution of (22) can be written in a following compact form

$$\begin{aligned} S_\varepsilon(w_\varepsilon)_{|B_\varepsilon} &= \mathbb{T}_\gamma S(u)(\widehat{x}) \\ &\Rightarrow \quad \sigma_\varepsilon(w_\varepsilon)_{|B_\varepsilon} \\ &= \rho S_\varepsilon(w_\varepsilon)_{|B_\varepsilon} \\ &= \mathbb{T}_\gamma \sigma(u)(\widehat{x}) \ , \end{aligned} \quad (23)$$

where $\mathbb{T}_\gamma$ is a fourth-order uniform (constant) tensor given by

$$\mathbb{T}_\gamma = \gamma \left((3\beta - 1)\, \mathbb{I} + (\alpha - \beta)\, \mathrm{I} \otimes \mathrm{I} \right) \ , \quad (24)$$

with

$$\alpha = \frac{(1 - \nu)}{3(1 - \nu) - (1 + \nu)(1 - \gamma)} \quad \text{and}$$

$$\beta = \frac{5(1 - \nu)}{15(1 - \nu) - (8 - 10\nu)(1 - \gamma)} \ . \quad (25)$$

Remark 1 As mentioned, the stress tensor field associated with the solution of the exterior problem (22) is uniform inside the inclusion $B_\varepsilon(\widehat{x})$. It means that the stress acting within the inclusion embedded into the whole three-dimensional space $\mathbb{R}^3$ can be written in the compact form (23). Therefore, the above result fits the famous Eshelby problem. This problem, formulated by Eshelby (1957, 1959), represents one of the major advances in the continuum mechanics theory of the twentieth century (Kachanov et al. 2003). It plays a central role in the theory of elasticity involving the determination of effective elastic properties of materials with multiple inhomogeneities. For more details, see the book by Mura (1987), for instance.

Now $\tilde{u}_\varepsilon$ can be constructed in such a way that it compensates for the discrepancies introduced by the higher-order terms in ε as well as by the boundary layer w_ε on the exterior boundary $\partial\mathscr{D}$ and on the interface $\partial\omega$. It means that the remainder $\tilde{u}_\varepsilon$ must be the solution to the following boundary value problem: Find $\tilde{u}_\varepsilon$, such that

$$\begin{cases} \operatorname{div}\sigma_\varepsilon(\tilde{u}_\varepsilon) = 0 & \text{in } \mathscr{D} , \\ \sigma_\varepsilon(\tilde{u}_\varepsilon) = \gamma_\varepsilon \sigma(\tilde{u}_\varepsilon) , & \\ \tilde{u}_\varepsilon = g_1 & \text{on } \Gamma_D , \\ \sigma(\tilde{u}_\varepsilon)n = g_2 & \text{on } \Gamma_N , \\ \left.\begin{array}{r} [\![\tilde{u}_\varepsilon]\!] = 0 \\ [\![\sigma_\varepsilon(\tilde{u}_\varepsilon)]\!]\, n = g_3 \end{array}\right\} & \text{on } \partial\omega , \\ \left.\begin{array}{r} [\![\tilde{u}_\varepsilon]\!] = 0 \\ [\![\sigma_\varepsilon(\tilde{u}_\varepsilon)]\!]\, n = \varepsilon h \end{array}\right\} & \text{on } \partial B_\varepsilon , \end{cases} \tag{26}$$

where $g_1 = -w_\varepsilon$, $g_2 = -\sigma(w_\varepsilon)n$, $g_3 = -(1-\rho_0)S(w_\varepsilon)n$ and $h = (1-\gamma)(\nabla\sigma(u(\xi))n)n$, with ξ used to denote an intermediate point between x and $\widehat{x}$. The remainder $\tilde{u}_\varepsilon$, solution of (26), enjoys the asymptotic behavior of the form $\|\tilde{u}_\varepsilon\|_{H^1(\mathscr{D};\mathbb{R}^3)} = O(\varepsilon^3)$. The proof is analogous to the one obtained in the two spatial dimensions case (Novotny and Sokołowski 2013, Ch. 5, pp 155).

From the above results, the variation of the energy shape functionals, given by (20), can be developed in power of ε as follows

$$\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon) - \mathscr{J}_\chi(u) = -\frac{4}{3}\pi\varepsilon^3 \,\frac{1-\gamma}{2\gamma} \left[\gamma\sigma(u)(\widehat{x}) + \mathbb{T}_\gamma\sigma(u)(\widehat{x})\right] \cdot \nabla u^s(\widehat{x}) + o(\varepsilon^3) , \tag{27}$$

By defining the function $f(\varepsilon) = \frac{4}{3}\pi\varepsilon^3$ and after applying the topological derivative concept (2), there is

$$D_T\mathscr{J}_\chi(\widehat{x}) = -\mathbb{P}_\gamma\sigma(u)(\widehat{x}) \cdot \nabla u^s(\widehat{x}) , \tag{28}$$

where $\mathbb{P}_\gamma$ is a fourth-order isotropic tensor obtained from Bonnet and Delgado (2013), namely,

$$\mathbb{P}_\gamma = \frac{1-\gamma}{2}\left[3\beta\mathbb{I} + (\alpha-\beta)\,\mathrm{I}\otimes\mathrm{I}\right] . \tag{29}$$

See also Ammari et al. (2008).

Remark 2 As can be seen from Eq. (27), the polarization tensor comes out naturally from the famous Eshelby problem. In fact, the tensor $\mathbb{T}_\gamma$ given by (24) represents one term contribution to the polarization tensor coming from the solution to the exterior problem (22).

Alternative Approach

In order to evaluate the topological derivative by using the Topological-Shape Sensitivity Method summarized through Proposition 1, the Eshelby energy-momentum tensor (Eshelby 1975) is introduced, namely,

$$\Sigma_\varepsilon = \frac{1}{2}(\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s)\mathrm{I} - \nabla u_\varepsilon^\top\sigma_\varepsilon(u_\varepsilon) , \tag{30}$$

together with a shape change velocity field V over $\mathscr{D}$

$$\mathscr{V} = \left\{V \in C_0^2(\mathscr{D};\mathbb{R}^3) : V_{|\partial\omega} = 0,\ V_{|\partial B_\varepsilon} = n\right\}, \tag{31}$$

representing a uniform expansion of the inclusion B_ε, where $n = (x-\widehat{x})/\varepsilon$, with $x \in \partial B_\varepsilon$. Therefore, from these elements the following result can be stated.

Proposition 2 *Let $\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon)$ be the shape functional defined by (12). Then, the derivative of $\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon)$ with respect to the small parameter ε is given by*

$$\dot{\mathscr{J}}_{\chi_\varepsilon}(u_\varepsilon) = \frac{d}{d\varepsilon}\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon) = \int_{\mathscr{D}} \Sigma_\varepsilon \cdot \nabla V , \tag{32}$$

where V is the shape change velocity field defined through (31) and Σ_ε is the Eshelby energy-momentum tensor given by (30).

Remark 3 Note that Σ_ε in (32) is the energy-momentum tensor introduced by Eshelby (1975). This tensor appears in the analysis of defects in three-dimensional elasticity, and it plays a central role in the continuum mechanics theory involving inhomogeneities (inclusions, pores, cracks, etc.) in solids. Since the distributed shape gradient of the total potential energy is given by the product of Σ_ε and ∇V, it follows that Σ_ε can be interpreted in terms of the configurational forces (Gurtin 2000) acting in the elastic body with a small defect inside.

The shape derivative of the functional (12) can also be expressed in an alternative form as shown below:

Proposition 3 *Let $\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon)$ be the shape functional defined by (12). Then, the derivative of $\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon)$ with respect to the small parameter ε is given by*

$$\dot{\mathscr{J}}_{\chi_\varepsilon}(u_\varepsilon) = \int_{\partial\mathscr{D}} \Sigma_\varepsilon n \cdot V - \int_{\partial B_\varepsilon} [\![\Sigma_\varepsilon]\!] n \cdot V - \int_{\partial\omega} [\![\Sigma_\varepsilon]\!] n \cdot V \, , \qquad (33)$$

with V defined through (31) and Σ_ε given by (30).

Corollary 1 *By making use of the divergence theorem at the right-hand side of (32), it follows*

$$\dot{\mathscr{J}}_{\chi_\varepsilon}(u_\varepsilon) = \int_{\partial\mathscr{D}} \Sigma_\varepsilon n \cdot V - \int_{\partial B_\varepsilon} [\![\Sigma_\varepsilon]\!] n \cdot V - \int_{\partial\omega} [\![\Sigma_\varepsilon]\!] n \cdot V - \int_{\mathscr{D}} \operatorname{div}\Sigma_\varepsilon \cdot V. \qquad (34)$$

Since Eqs. (34) and (33) remain valid for all velocity fields V, the last term of the above equation must satisfy

$$\int_{\mathscr{D}} \operatorname{div}\Sigma_\varepsilon \cdot V = 0, \quad \forall V \in \mathscr{V} \;\Rightarrow\; \operatorname{div}\Sigma_\varepsilon = 0 \, . \qquad (35)$$

Remark 4 Note that $\operatorname{div}\Sigma_\varepsilon = F$ in $\mathscr{D}$, with $F = 0$ in this particular case, can be referred to as the balance of configurational forces or simply configurational balance in configurational mechanics theory (Gurtin 2000).

Corollary 2 *Since Σ_ε is a free-divergence tensor field, and in view of the velocity field defined through (31), namely, $V = 0$ on $\partial\omega$, $V = 0$ on $\partial\mathscr{D}$, and $V = n$ on ∂B_ε, the above result (34) reduces to*

$$\frac{d}{d\varepsilon}\psi(\chi_\varepsilon) = \dot{\mathscr{J}}_{\chi_\varepsilon}(u_\varepsilon) = -\int_{\partial B_\varepsilon} [\![\Sigma_\varepsilon]\!] n \cdot n \, . \qquad (36)$$

Remark 5 As can be seen in (36), the shape gradient of the functional $\mathscr{J}_{\chi_\varepsilon}(u_\varepsilon)$ takes the form of a boundary integral concentrated on the moving boundary ∂B_ε, depending on the normal component of the velocity field V. This latter result fits into the so-called Hadamard structure theorem of shape optimization, proved by Sokołowski and Zolésio (1992) and Delfour and Zolésio (2001), for instance.

By taking into account the ansätz (21) together with the explicit solution (23), the integral in (36) yields

$$\int_{\partial B_\varepsilon} [\![\Sigma_\varepsilon]\!] n \cdot n = 4\pi\varepsilon^2 \mathbb{P}_\gamma \sigma(u)(\widehat{x}) \cdot \nabla u^s(\widehat{x}) + o(\varepsilon^2), \qquad (37)$$

with the polarization tensor $\mathbb{P}_\gamma$ given by (29). Then, by applying the result of Proposition 1 in the above expression, it follows

$$D_T \mathscr{J}_\chi(\widehat{x}) = -\lim_{\varepsilon\to 0} \frac{1}{f'(\varepsilon)} (4\pi\varepsilon^2 \mathbb{P}_\gamma \sigma(u)(\widehat{x}) \cdot \nabla u^s(\widehat{x}) + o(\varepsilon^2)), \qquad (38)$$

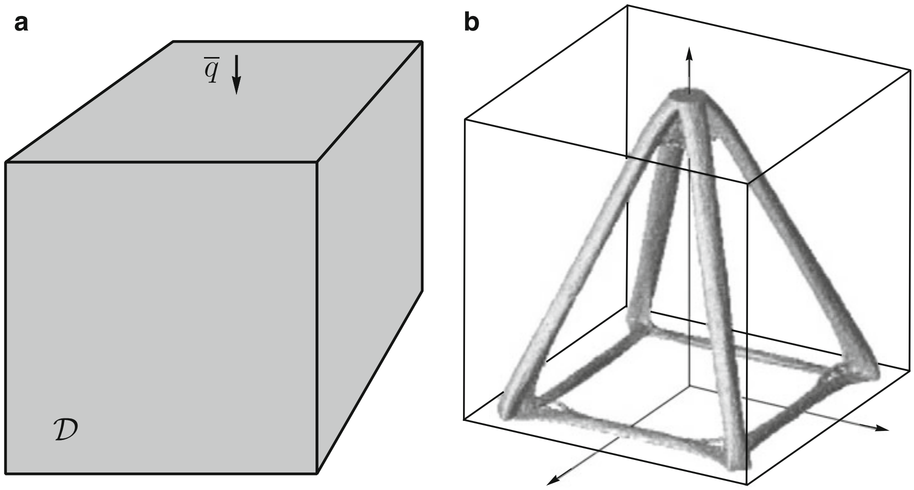
Now, by choosing $f(\varepsilon) = \frac{4}{3}\pi\varepsilon^3$ as before, the main term in the above expansion can be extracted, leading to the final formula for the topological derivative, namely,

$$D_T \mathscr{J}_\chi(\widehat{x}) = -\mathbb{P}_\gamma \sigma(u)(\widehat{x}) \cdot \nabla u^s(\widehat{x}) \, , \qquad (39)$$

which corroborates with the one previously derived (28).

A Numerical Example

In order to illustrate the applicability of the obtained results, a numerical example in the context of structural topology optimization into three



Relation Between Eshelbyan Mechanics and Topological Derivative Concept, Fig. 1 Numerical example. (**a**) Geometry and boundary conditions. (**b**) Final result

spatial dimensions is presented. The topology is identified by the elastic material distribution, and the complaint material is used to represent the voids. The topological derivative is evaluated, and an inclusion is nucleated at the regions where it is negative. This procedure is repeated until the topological derivative becomes positive everywhere. For more elaborated topology design algorithm, see Amstutz and Andrä (2006).

The topological derivative of the shape functional $\mathscr{F}_\chi(u)$, defined by (5), yields

$$D_T \mathscr{F}_\chi(u) = -D_T \mathscr{J}_\chi(x) + m D_T |\Omega|(x) , \tag{40}$$

where $D_T \mathscr{J}_\chi(x)$ is given by (39) and $D_T |\Omega|(x)$ is trivially obtained as

$$D_T |\Omega|(x) = \begin{cases} -1, \text{ if } x \in \Omega , \\ +1, \text{ if } x \in \omega . \end{cases} \tag{41}$$

In addition, the displacement vector field u is evaluated by solving problem (7) numerically.

The hold all domain $\mathscr{D}$ consists of a simply supported cube subject to a vertical load $\bar{q}$ applied on its top, as shown in Fig. 1a. For more details concerning the problem setting, see (Novotny and Sokołowski 2013, Ch. 8, pp 213). The parameter m is chosen in such way that the required final volume $M = 0.02|\mathscr{D}|$ is attained. The cube is discretized by using four-node tetrahedron finite elements, and 5% of material is removed at each iteration. The obtained result is shown in Fig. 1b.

Conclusion

In this work a connection between Eshelby Mechanics and the topological derivative concept has been investigated. It is well-known that the Pólya-Szegö polarization tensor, which naturally appears on the Eshelby theorem, plays a crucial role on the topological asymptotic analysis. In addition, the topological derivative can also be defined as the singular limit of the classical shape derivative. The associated shape gradient is written in terms of the flux of the Eshelby energy-momentum tensor across to the moving boundary. It means that the polarization tensor is related to the Eshelby tensor through the limit passage with respect to the small parameter measuring the size of the singular domain perturbation. Therefore, the Eshelby tensor together with the Eshelby problem in general and the polarization tensor in particular can be seen as the main ingredients for the topological derivative concept. Finally, a simple numerical example has been presented showing the applicability of the obtained results.

Cross-References

- ▶ Continuum Mechanics Basics
- ▶ Generalized Continua
- ▶ Numerical Continuum Mechanics
- ▶ Optimization
- ▶ Tensor Calculus
- ▶ Variational Principles

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