The Topological Derivative for the Poisson's Problem

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Abstract

The Topological Derivative has been recognized as a powerful tool in obtaining the optimal topology of several problems of engineering interest. More specifically, the Topological Derivative gives the sensitivity of the problem when a small hole is created at each point of the domain under consideration. In this work the Topological Derivative for the Poisson's problem is calculated using two different approaches: the Domain Truncation Method, and a new method based on Shape Sensitivity Analysis concepts. This comparison shows that this novel approach, which we call Topological-Shape Sensitivity Method, has led to a simpler and more general methodology than the former one. To point out the general applicability of this new methodology, the most general set of boundary conditions for the Poisson's problem is considered, namely, Dirichlet, Neumann (both homogeneous and non-homogeneous) and Robin boundary conditions are considered. Finally, a comparative analysis of these two methodologies will also show that the Topological-Shape Sensitivity Method has the advantage that it can be easily generalized to solve other classes of problems.

keywords: Topological Derivative, Shape Sensitivity Analysis, Topological Optimization, Shape Optimization, Asymptotic Expansion.

1 Introduction

In Schumacher[13], Sokolowski & Żochowski[14, 15] and Garreau et al.[4, 5] the so-called Topological Derivative concept was introduced. The Topological Derivative gives the sensitivity of a cost function defined in the domain of definition of a boundary-value problem when a small hole is introduced in the domain. More specifically, the idea is to make a perturbation on the domain Ω by subtracting a ball of radius ϵ, denoted by B_ϵ, centered in a point $\hat{\mathbf{x}} \in \Omega$. This originates a new domain $\Omega_\epsilon = \Omega - \overline{B}_\epsilon$. Therefore, if a cost function j defined in Ω is considered, then the Topological Derivative, here denoted by $D_T j$, can be defined as

$$j(\Omega_\epsilon) = j(\Omega) + f(\epsilon) D_T j + \mathcal{R}(f(\epsilon)). \tag{1}$$

In the expression above, $f(\epsilon)$ is a negative function that depends on the problem under analysis and that monotonically decreases so that $f(\epsilon) \to 0$ when $\epsilon \to 0$. $\mathcal{R}(f(\epsilon))$ contains all higher order terms than $f(\epsilon)$, that is, it satisfies

$$\lim_{\epsilon \to 0} \frac{\mathcal{R}(f(\epsilon))}{f(\epsilon)} = 0.$$

In general, j depends explicitly and implicitly on ϵ. The implicit dependence arises from the solution of the boundary value problem defined in Ω_ϵ. If this problem is elliptic, conditions in the whole boundary of Ω_ϵ must be imposed. Therefore, when $\overline{B}_\epsilon$ is introduced, boundary conditions must also be defined on ∂B_ϵ.

To obtain $D_T j$, an asymptotic analysis of the problem above was carried out by Garreau et al.[4] using the Dirichlet-to-Neumann map. This methodology, called the Domain Truncation Method, can be used for singular problems such as those with Dirichlet boundary conditions imposed on ∂B_ϵ.

In Céa et al.[2] a relation between the Topological Derivative and the classical Shape Derivative is discussed. The authors arrived at the conclusion that these derivatives differ by a factor 2 for the case of homogeneous Neumann boundary conditions imposed on ∂B_ϵ.

In Sokolowski & Żochowski[14] a method to calculate the Topological Derivative via Shape Sensitivity Analysis was presented. Nevertheless, this method provides correct results only for homogeneous Neumann boundary conditions imposed on ∂B_ϵ.

More recently, Novotny et al.[10, 11, 12] formally established the relationship between the Topological Derivative and Shape Sensitivity Analysis. This relationship provides an alternative way to compute the Topological Derivative using Shape Sensitivity Analysis results. Moreover, this new approach, which we will refer as the Topological-Shape Sensitivity Method, can be applied to any cost function and to any type of boundary condition on ∂B_ϵ. Therefore, this method is more general than others found in the literature (namely those in[2, 4, 5, 14, 15]).

In this work, the Topological Derivative is calculated for Poisson's problem using the Domain Truncation Method and the Topological-Shape Sensitivity Method. This will allow us to compare the two approaches. As a consequence, we will show that our new approach is more generic and that it provides a straightforward way to calculate the Topological Derivative. Furthermore, a more general set of boundary conditions on ∂B_ϵ is considered, extending the results obtained by Garreau et al.[4] and Novotny et al.[12].

2 An Elliptic Boundary Value Problem

Let $\Omega_\epsilon \subset \mathbb{R}^2$ be an open bounded domain, whose boundary, denoted by $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon$, is sufficiently smooth. Then, the Poisson's problem with Dirichlet, Neumann or Robin boundary conditions on ∂B_ϵ can be stated as

$$\left\{\begin{array}{l} \text{Find } u_\epsilon \text{ such that} \\ \begin{array}{rcll} -\Delta u_\epsilon & = & b & \text{in} \quad \Omega_\epsilon \\ u_\epsilon & = & \bar{u} & \text{on} \quad \partial\Omega \\ g\left(\alpha,\beta,\gamma\right)\left(u_\epsilon\right) & = & 0 & \text{on} \quad \partial B_\epsilon \end{array} \end{array}\right. , \tag{2}$$

where

$$g\left(\alpha,\beta,\gamma\right)\left(u_\epsilon\right) = \alpha\left(u_\epsilon - h\right) + \beta\left(\frac{\partial u_\epsilon}{\partial n} - h\right) + \gamma\left(u_\epsilon + \frac{\partial u_\epsilon}{\partial n} - h\right) \tag{3}$$

and $\alpha,\beta,\gamma \in \{0,1\}$, with $\alpha+\beta+\gamma = 1$. Using this notation, the three possible types of boundary conditions that can be imposed on ∂B_ϵ are obtained in the following way

$$g(\alpha,\beta,\gamma)(u_\epsilon) = \left\{\begin{array}{lll} u_\epsilon - h, & \text{if } \alpha=1, \beta=0, \gamma=0, & \text{Dirichlet b. c.} \\ \frac{\partial u_\epsilon}{\partial n} - h, & \text{if } \alpha=0, \beta=1, \gamma=0, & \text{Neumann b. c.} \\ u_\epsilon + \frac{\partial u_\epsilon}{\partial n} - h, & \text{if } \alpha=0, \beta=0, \gamma=1, & \text{Robin b. c.} \end{array}\right. .$$

In the following sections the Topological Derivative will be calculated for the set of boundary conditions shown in (2). The derivative will be calculated using the Domain Truncation Method and the Topological-Shape Sensitivity Method. It will be shown that the second approach leads to a simple and constructive formulation.

3 The Domain Truncation Method

The standard weak formulation corresponding to the boundary-value problem (2) is defined in the domain Ω_ϵ. The inconvenience of this kind of approach is that the domain of integration as well as the integrands that appear in the variational equation depend on ϵ. As will be seen, this creates a complication that the Domain Truncation Method (see Garreau et al.[4]) will avoid. This method consists of separating problem (2) into two problems where one of them is solved analytically using separation of variables. This allows the authors to perform an asymptotic analysis of (2) with respect to ϵ.

Let B_R and B_ϵ be open balls of radii R and ϵ, respectively, centered in $\hat{\mathbf{x}} \in \Omega$ with $R > \epsilon$. Define $A = B_R - \overline{B}_\epsilon$ as the open ring of inner radius ϵ and outer radius R. We now define the following problem on A

$$\left\{\begin{array}{rcll} \text{Find } w_\epsilon \text{ such that:} & & & \\ -\Delta w_\epsilon &=& b & \text{in } A, \\ w_\epsilon &=& \psi & \text{on } \partial B_R, \\ g\left(\alpha,\beta,\gamma\right)\left(w_\epsilon\right) &=& 0 & \text{on } \partial B_\epsilon. \end{array}\right. \tag{4}$$

We also define the operator $T^\epsilon : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ such that

$$T^\epsilon(\psi) = \left.\frac{\partial w_\epsilon}{\partial n}\right|_{\partial B_R}.$$

Given the previous definitions, the following problem in $\Omega_R = \Omega - \overline{B}_R$ is stated

$$\left\{\begin{array}{rcll} \text{Find } v_\epsilon \text{ such that:} & & & \\ -\Delta v_\epsilon &=& b & \text{in } \Omega_R, \\ v_\epsilon &=& \bar{u} & \text{on } \partial\Omega, \\ \frac{\partial v_\epsilon}{\partial n} &=& T^\epsilon(\psi) & \text{on } \partial B_R. \end{array}\right. \tag{5}$$

It is straightforward to show the following equivalence theorem:

Theorem 1 *If $\psi = u_\epsilon|_{\partial B_R}$, then*

$$w_\epsilon = u_\epsilon|_A \quad \textit{and} \quad v_\epsilon = u_\epsilon|_{\Omega_R}.$$

Therefore u_ϵ satisfies

$$a_\epsilon(u_\epsilon,\eta) = l\left(\eta\right) \qquad \forall\, \eta \in V, \tag{6}$$

where $V = \{\eta \in H^1(\Omega_R) \mid \eta|_{\partial\Omega} = 0\}$,

$$a_\epsilon(u_\epsilon,\eta) = \int_{\Omega_R} \nabla u_\epsilon \cdot \nabla\eta \, d\Omega_R - \int_{\partial B_R} T^\epsilon\left(u_\epsilon|_{\partial B_R}\right)\eta \, d\partial B_R, \tag{7}$$

and

$$l\left(\eta\right) = \int_{\Omega_R} b\eta \, d\Omega_R. \tag{8}$$

Note that in (6) the domain of integration is independent of ϵ.

Now let $dT_R : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ represent a function independent of ϵ which satisfies

$$\lim_{\epsilon\to 0}\left\|\frac{T^\epsilon - T^0 - f(\epsilon)\, dT_R}{f(\epsilon)}\right\|_{\mathcal{L}\left(H^{1/2}(\partial B_R), H^{-1/2}(\partial B_R)\right)} = 0. \tag{9}$$

We now define the following bilinear form for each R

$$\begin{aligned} da_R(\xi,\eta) &:= -\left\langle dT_R(\xi|_{\partial B_R}), \eta|_{\partial B_R}\right\rangle_{\mathcal{L}\left(H^{1/2}(\partial B_R), H^{-1/2}(\partial B_R)\right)} \\ &= -\int_{\partial B_R} dT_R(\xi|_{\partial B_R})\, \eta|_{\partial B_R} \, d\partial B_R, \end{aligned} \tag{10}$$

where $\xi,\eta \in V$. Using (10) the following theorem can be proved

Theorem 2
$$\lim_{\epsilon\to 0}\left\|\frac{a_\epsilon - a_0 - f(\epsilon)\, da_R}{f(\epsilon)}\right\|_{\mathcal{L}(V\times V,\mathbb{R})} = 0.$$

Proof. For all $\xi, \eta \in V$ we have

$$a_\epsilon(\xi,\eta) - a_0(\xi,\eta) - f(\epsilon)\, da_R(\xi,\eta) = -\int_{\partial B_R} (T^\epsilon - T^0 - f(\epsilon) dT_R)(\xi|_{\partial B_R})\, \eta \, d\partial B_R.$$

Dividing the expression above by $f(\epsilon)$, using the Cauchy-Schwarz inequality and the trace theorem we obtain

$$\left|\frac{(a_\epsilon - a_0 - f(\epsilon)\, da_R)\,(\xi,\eta)}{f(\epsilon)}\right| \le C\left\|\frac{T^\epsilon - T^0 - f(\epsilon) dT_R}{f(\epsilon)}\right\|_{\mathcal{L}\left(H^{1/2}(\partial B_R),H^{-1/2}(\partial B_R)\right)} \|\xi\|_V\|\eta\|_V.$$

Using (9) the proof is completed ∎

Remark: *In the previous proof and thereafter, C denotes a positive constant, independent of relevant parameters and possibly different in each case.*

Theorem 3 *Let u_0 be the solution of the Poisson's problem defined in the domain Ω, then the following inequality holds*
$$\|u_\epsilon - u_0\|_V \le C|f(\epsilon)|.$$

Proof. From Theorem 2 we have the following:

$$\begin{aligned} a_\epsilon(u_\epsilon - u_0, u_\epsilon - u_0) &= a_\epsilon(u_\epsilon, u_\epsilon - u_0) - a_\epsilon(u_0, u_\epsilon - u_0) \\ &= a_0(u_0, u_\epsilon - u_0) - a_\epsilon(u_0, u_\epsilon - u_0) \\ &= -f(\epsilon) da_R(u_0, u_\epsilon - u_0) - (a_\epsilon - a_0 - f(\epsilon) da_R)(u_0, u_\epsilon - u_0) \\ &\le C\,|f(\epsilon)|\,\|u_\epsilon - u_0\|_V \end{aligned}$$

where C depends on $\|u_0\|_V$ and $\|da_R\|_{\mathcal{L}(V\times V,\mathbb{R})}$. The final result is obtained by considering the V-Ellipticity of the bilinear form $a_\epsilon(\cdot,\cdot)$ ∎

For the sake of simplicity, the cost function $j\,(\epsilon)$ will be defined in a fixed domain. In other words, $j\,(\epsilon)$ will only depend implicitly on Ω_ϵ through the solution u_ϵ of (2). Thus, the cost function $j\,(\epsilon)$ can be written as $j\,(\epsilon) := J\,(u_\epsilon)$.

To calculate the derivative of $J\,(u_\epsilon)$ with respect to ϵ we will make use of the Lagrangian Method which consists in relaxing the constraint of the problem, in this case the state equation (6), using a Lagrange multiplier λ. We start by defining the Lagrangian as follows

$$\pounds_\epsilon\,(u,\lambda) = J\,(u) + a_\epsilon\,(u,\lambda) - l\,(\lambda) \qquad \forall\, \lambda \in V, \tag{11}$$

Choosing $u = u_\epsilon$, solution of (6), and $\lambda = p_\epsilon \in V$ as the solution of the adjoint equation

$$a_\epsilon\,(\eta, p_\epsilon) = -\left\langle \frac{\partial J\,(u_\epsilon)}{\partial u_\epsilon}, \eta \right\rangle \qquad \forall\, \eta \in V, \tag{12}$$

it follows that the total derivative of the Lagrangian with respect to ϵ coincides with its partial derivative

$$\left.\frac{d}{d\epsilon}\pounds_\epsilon(u,\lambda)\right|_{u=u_\epsilon,\lambda=p_\epsilon} = \left.\frac{\partial}{\partial\epsilon}\pounds_\epsilon(u,\lambda)\right|_{u=u_\epsilon,\lambda=p_\epsilon}.$$

Now, let $\lambda = p_0$ be the solution of the adjoint equation for $\epsilon = 0$, that is

$$a_0(\eta, p_0) = -\left\langle \frac{\partial J\,(u_0)}{\partial u}, \eta \right\rangle \qquad \forall \eta \in V. \tag{13}$$

Then, the following theorem holds:

Theorem 4 *Assuming that $j(\epsilon)$ depends only implicitly on Ω_ϵ through the solution u_ϵ of the boundary value problem (2) and $J \in C^2(H^1(\Omega_R), \mathbb{R})$ then*

$$\lim_{\epsilon \to 0} \frac{j(\epsilon) - j(0) - f(\epsilon)\, da_R(u_\epsilon, p_0)}{f(\epsilon)} = 0.$$

Proof. We start with the following Taylor series expansion of $j(\epsilon)$

$$j(\epsilon) = J(u_0) + \partial_u J(u_0)(u_\epsilon - u_0) + \partial_u^2 J(u_0 + \theta(u_\epsilon - u_0))(u_\epsilon - u_0, u_\epsilon - u_0) + a_\epsilon(u_\epsilon, \lambda) - l(\lambda),$$

$\forall \lambda \in V$ and some $\theta \in (0,1)$. Since $j(0) = J(u_0) + a_0(u_0, \lambda) - l(\lambda)$ and using (11) we have

$$j(\epsilon) = j(0) - a_0(u_0, \lambda) + a_\epsilon(u_\epsilon, \lambda) + \partial_u J(u_0)(u_\epsilon - u_0) + \partial_u^2 J(u_0 + \theta(u_\epsilon - u_0))(u_\epsilon - u_0, u_\epsilon - u_0).$$

Since λ is arbitrary, we can choose $\lambda = p_0$ (see 13) and obtain

$$\begin{aligned} j(\epsilon) &= j(0) - a_0(u_0, p_0) + a_\epsilon(u_\epsilon, p_0) - a_0(u_\epsilon, p_0) + a_0(u_0, p_0) \\ &\quad + \partial_u^2 J(u_0 + \theta(u_\epsilon - u_0))(u_\epsilon - u_0, u_\epsilon - u_0). \end{aligned}$$

We rewrite the expression above as follows

$$\begin{aligned} j(\epsilon) - j(0) - f(\epsilon) da_R(u_\epsilon, p_0) &= (a_\epsilon - a_0 - f(\epsilon) da_R)(u_\epsilon, p_0) \\ &\quad + \partial_u^2 J(u_0 + \theta(u_\epsilon - u_0))(u_\epsilon - u_0, u_\epsilon - u_0). \end{aligned}$$

Dividing this expression by $f(\epsilon)$ we obtain

$$\left| \frac{j(\epsilon) - j(0) - f(\epsilon) da_R(u_\epsilon, p_0)}{f(\epsilon)} \right| \leq \left| \frac{(a_\epsilon - a_0 - f(\epsilon) da_R)(u_\epsilon, p_0)}{f(\epsilon)} \right| + C \frac{\|u_\epsilon - u_0\|_V^2}{|f(\epsilon)|}.$$

Taking the limit $\epsilon \to 0$, and using Theorems 2 and 3, the final result is obtained. ∎

Remark: *Since*

$$\lim_{R \to 0} a_0(\xi, \eta) = \int_\Omega \nabla \xi \cdot \nabla \eta \, d\Omega \quad \textit{and} \quad \lim_{R \to 0} \left\langle \frac{\partial J(u_0)}{\partial u}, \eta \right\rangle = \left\langle \frac{\partial J(u)}{\partial u}, \eta \right\rangle$$

the adjoint equation associated to the domain Ω is obtained by taking the limit $R \to 0$ in (13). The solution to this problem will be denoted p.

Remark: *The case $\epsilon = 0$ corresponds to $\Omega_0 = \Omega$. To simplify the notation, u will be used instead of u_0 to denote the solution of (2) when $\epsilon = 0$.*

Given the previous results, the Topological Derivative can be defined as follows.

Definition: *The Topological Derivative at $\hat{\mathbf{x}} \in \Omega$ is defined as follows*

$$\begin{aligned} D_T j &= D_T j(u, p) = \lim_{R \to 0} da_R(u_0, p_0) \\ &= -\lim_{R \to 0} \int_{\partial B_R} dT_R(u_0|_{\partial B_R})\, p_0|_{\partial B_R}\, d\partial B_R, \end{aligned} \tag{14}$$

where u and p are the solutions of the state and adjoint equations, respectively, both defined in the original domain Ω.

Finally, in order to verify (9), it is enough to choose $f(\epsilon)$ and dT_R such that

$$\lim_{\epsilon \to 0} \left\| \frac{(T^\epsilon - T^0)(\psi) - f(\epsilon)\, dT_R(\psi)}{f(\epsilon)} \right\|_{H^{-1/2}(\partial B_R)} = 0 \qquad \forall \psi \in H^{1/2}(\partial B_R). \tag{15}$$

The computation the Topological Derivative via the Domain Truncation Method can be summarized in the following steps:

1. Perform an asymptotic analysis of w_ϵ (solution of problem (4)).

2. Calculate $(T^\epsilon - T^0)(\psi)$ and find the dominant term of its asymptotic development.

3. For each type of boundary condition on ∂B_ϵ:

   - Define the functions $f(\epsilon)$ and $dT_R(\psi)$ that satisfy (15).
   - Using $\psi = u_\epsilon|_{\partial B_R}$ calculate the Topological Derivative given by (14).

In the next subsections we will perform each of the steps above.

3.1 Calculation of the asymptotic development of w_ϵ

The solution w_ϵ of (4) can be written as follows

$$w_\epsilon = \sum_{n=0}^{\infty} \left(\varphi_n(r)\cos n\theta + \hat{\varphi}_n(r)\sin n\theta\right).$$

Imposing the boundary conditions

$$\tilde{\alpha} w_\epsilon + \tilde{\beta}\frac{\partial w_\epsilon}{\partial n} = h \text{ on } \partial B_\epsilon$$

and

$$w_\epsilon = \psi = \sum_{n=0}^{\infty}\left(R^n \psi_n \cos n\theta + R^n \hat{\psi}_n \sin n\theta\right) \text{ on } \partial B_R,$$

where $\tilde{\alpha} = \alpha + \gamma$ and $\tilde{\beta} = \beta + \gamma$, the Fourier coefficients must satisfy

$$\left\{\begin{array}{l} \varphi_0'' + \frac{1}{r}\varphi_0' = -b \\ \varphi_0(R) = \psi_0 \\ \tilde{\alpha}\varphi_0(\epsilon) - \tilde{\beta}\varphi_0'(\epsilon) = h \end{array}\right. , \quad \left\{\begin{array}{l} \varphi_n'' + \frac{1}{r}\varphi_n' - \frac{n^2}{r^2}\varphi_n = 0 \\ \varphi_n(R) = R^n\psi_n \\ \tilde{\alpha}\varphi_n(\epsilon) - \tilde{\beta}\varphi_n'(\epsilon) = 0 \end{array}\right. , \quad \left\{\begin{array}{l} \hat{\varphi}_n'' + \frac{1}{r}\hat{\varphi}_n' - \frac{n^2}{r^2}\hat{\varphi}_n = 0 \\ \hat{\varphi}_n(R) = R^n\hat{\psi}_n \\ \tilde{\alpha}\hat{\varphi}_n(\epsilon) - \tilde{\beta}\hat{\varphi}_n'(\epsilon) = 0 \end{array}\right. .$$

The final expression for w_ϵ is given as follows

$$\begin{aligned} w_\epsilon \quad = \quad & -\frac{b}{4}r^2 + \left(\frac{\tilde{\alpha}\frac{b}{4}(R^2-\epsilon^2) + \tilde{\alpha}\psi_0 - h + \tilde{\beta}\frac{b}{2}\epsilon}{\tilde{\alpha}(\ln R - \ln\epsilon) + \tilde{\beta}/\epsilon}\right)\ln r \\ & + \frac{\ln R\left(h + \tilde{\alpha}\frac{b}{4}\epsilon^2 - \tilde{\beta}\frac{b}{2}\epsilon\right) - (\tilde{\alpha}\ln\epsilon - \tilde{\beta}/\epsilon)(\psi_0 + \frac{b}{4}R^2)}{\tilde{\alpha}(\ln R - \ln\epsilon) + \tilde{\beta}/\epsilon} \\ & + \sum_{n=1}^{\infty}\frac{(\epsilon\tilde{\alpha} + n\tilde{\beta})r^n - \epsilon^{2n}(\epsilon\tilde{\alpha} - n\tilde{\beta})r^{-n}}{(\epsilon\tilde{\alpha} + n\tilde{\beta}) - (\epsilon/R)^{2n}(\epsilon\tilde{\alpha} - n\tilde{\beta})}\left(\psi_n\cos n\theta + \hat{\psi}_n\sin n\theta\right). \end{aligned}$$

3.2 Calculation of $(T^\epsilon - T^0)(\psi)$

$T^\epsilon(\psi)$ is computed as follows

$$\begin{aligned} T^\epsilon(\psi) \quad = \quad & -\left.\frac{\partial w_\epsilon}{\partial r}\right|_{r=R} = \frac{b}{2}R - \frac{1}{R}\left(\frac{\tilde{\alpha}\frac{b}{4}(R^2-\epsilon^2) + \tilde{\alpha}\psi_0 - h + \tilde{\beta}\frac{b}{2}\epsilon}{\tilde{\alpha}(\ln R - \ln\epsilon) + \tilde{\beta}/\epsilon}\right) \\ & - \sum_{n=1}^{\infty} nR^{n-1}\frac{(\epsilon\tilde{\alpha} + n\tilde{\beta}) + (\epsilon/R)^{2n}(\epsilon\tilde{\alpha} - n\tilde{\beta})}{(\epsilon\tilde{\alpha} + n\tilde{\beta}) - (\epsilon/R)^{2n}(\epsilon\tilde{\alpha} - n\tilde{\beta})}\left(\psi_n\cos n\theta + \hat{\psi}_n\sin n\theta\right) \end{aligned}$$

and

$$T^0(\psi) = \frac{b}{2}R - \sum_{n=1}^{\infty} nR^{n-1}\left(\psi_n \cos n\theta + \hat{\psi}_n \sin n\theta\right).$$

Finally,

$$\begin{aligned}(T^\epsilon - T^0)(\psi) &= -\frac{1}{R}\left(\frac{\tilde{\alpha}\frac{b}{4}(R^2-\epsilon^2) + \tilde{\alpha}\psi_0 - h + \tilde{\beta}\frac{b}{2}\epsilon}{\tilde{\alpha}(\ln R - \ln\epsilon) + \tilde{\beta}/\epsilon}\right) \\ &\quad - \sum_{n=1}^{\infty} 2nR^{n-1}\frac{(\epsilon/R)^{2n}(\epsilon\tilde{\alpha} - n\tilde{\beta})\left(\psi_n \cos n\theta + \hat{\psi}_n \sin n\theta\right)}{(\epsilon\tilde{\alpha} + n\tilde{\beta}) - (\epsilon/R)^{2n}(\epsilon\tilde{\alpha} - n\tilde{\beta})}.\end{aligned}$$

3.3 Calculation of $f(\epsilon)$, dT_R and $D_T j(u,p)$.

For each type of boundary condition on ∂B_ϵ, the functions $f(\epsilon)$ and $dT_R(\psi)$ will be constructed according to (15). The Topological Derivative will be calculated using $\psi = u_\epsilon|_{\partial B_R}$.

- **Neumann boundary condition** ($\beta = 1, \alpha = \gamma = 0$):

$$\begin{aligned}(T^\epsilon - T^0)(\psi) &= \frac{h}{R}\epsilon - \left[\frac{b}{2R} - \frac{2}{R^2}(\psi_1 \cos\theta + \hat{\psi}_1 \sin\theta)\right]\epsilon^2 \\ &\quad -2\frac{(\epsilon/R)^4}{1+(\epsilon/R)^2}\left(\psi_1 \cos\theta + \hat{\psi}_1 \sin\theta\right) \\ &\quad + \sum_{n=2}^{\infty} 2nR^{n-1}\frac{(\epsilon/R)^{2n}}{1+(\epsilon/R)^{2n}}\left(\psi_n \cos n\theta + \hat{\psi}_n \sin n\theta\right).\end{aligned}$$

  If $h \neq 0$, the choices

$$f(\epsilon) = -2\pi\epsilon \quad \text{and} \quad dT_R(\psi) = -\frac{h}{2\pi R}.$$

  assure that (15) holds. Thus, the Topological Derivative is given by

$$D_T j = -\lim_{R\to 0}\int_{\partial B_R}\left(-\frac{h}{2\pi R}\right)p_0 \, d\partial B_R = h\,p(\hat{\mathbf{x}}). \tag{16}$$

  For $h = 0$, equation (15) is satisfied by taking

$$f(\epsilon) = -\pi\epsilon^2 \quad \text{and} \quad dT_R(\psi) = \frac{b}{2\pi R} - \frac{2}{\pi R^2}(\psi_1 \cos\theta + \hat{\psi}_1 \sin\theta).$$

  Using the divergence theorem, the final expression for the topological derivative is obtained

$$\begin{aligned}D_T j &= \lim_{R\to 0}\left\{-\int_{\partial B_R}\left[\frac{b}{2\pi R} - \frac{2}{\pi R^2}(\psi_1 \cos\theta + \hat{\psi}_1 \sin\theta)\right]p_0 \, d\partial B_R\right\} \\ &= 2\lim_{R\to 0}\left(\frac{\psi_1}{\pi R^2}\int_{\partial B_R} p_0 \cos\theta \, d\partial B_R + \frac{\hat{\psi}_1}{\pi R^2}\int_{\partial B_R} p_0 \sin\theta \, d\partial B_R\right) - bp(\hat{\mathbf{x}}) \\ &= 2\lim_{R\to 0}\frac{1}{\pi^2 R^4}\left(\int_{B_R}\frac{\partial u}{\partial x}\,dB_R \int_{B_R}\frac{\partial p_0}{\partial x}\,dB_R\right. \\ &\qquad \left. + \int_{B_R}\frac{\partial u}{\partial y}\,dB_R \int_{B_R}\frac{\partial p_0}{\partial y}\,dB_R\right) - bp(\hat{\mathbf{x}}) \\ &= 2\nabla u(\hat{\mathbf{x}})\cdot\nabla p(\hat{\mathbf{x}}) - bp(\hat{\mathbf{x}}).\end{aligned} \tag{17}$$

- **Robin boundary condition** ($\gamma = 1, \alpha = \beta = 0$):

$$
\begin{aligned}
(T^\epsilon - T^0)(\psi) &= -\frac{1}{R}\left(\frac{\frac{b}{4}(R^2-\epsilon^2)+\psi_0 - h + \frac{b}{2}\epsilon}{(\ln R - \ln\epsilon)+1/\epsilon}\right) \\
&\quad -\sum_{n=1}^{\infty} 2nR^{n-1}\frac{(\epsilon/R)^{2n}(\epsilon-n)\left(\psi_n\cos n\theta + \hat{\psi}_n \sin n\theta\right)}{(\epsilon+n)-(\epsilon/R)^{2n}(\epsilon-n)}.
\end{aligned}
$$

In this case, equation (15) holds by choosing

$$
f(\epsilon) = -2\pi\epsilon \qquad \text{and} \qquad dT_R(\psi) = \frac{bR}{8\pi} + \frac{\psi_0 - h}{2\pi R}.
$$

Thus, the Topological Derivative becomes

$$
\begin{aligned}
D_T j &= \lim_{R\to 0}\left(-\int_{\partial B_R}\left(\frac{bR}{8\pi}+\frac{\psi_0-h}{2\pi R}\right) p_0\, d\partial B_R\right) \\
&= -\lim_{R\to 0}(\psi_0 - h)\left(\frac{1}{2\pi R}\int_{\partial B_R} p_0\, d\partial B_R\right) \\
&= -\left(u(\hat{\mathbf{x}}) - h\right)\, p(\hat{\mathbf{x}}). \qquad (18)
\end{aligned}
$$

- **Dirichlet boundary condition** ($\alpha = 1, \beta = \gamma = 0$):

$$
\begin{aligned}
(T^\epsilon - T^0)(\psi) &= -\frac{1}{R}\left(\frac{\frac{b}{4}(R^2-\epsilon^2)+\psi_0 - h}{(\ln R - \ln\epsilon)}\right) \\
&\quad -2\frac{(\epsilon/R)^2}{1-(\epsilon/R)^2}\left(\psi_1\cos\theta + \hat{\psi}_1\sin\theta\right) \\
&\quad -\sum_{n=2}^{\infty} 2nR^{n-1}\frac{(\epsilon/R)^{2n}}{1-(\epsilon/R)^{2n}}\left(\psi_n\cos n\theta + \hat{\psi}_n\sin n\theta\right).
\end{aligned}
$$

In this case we have to investigate two possible situations: $h = h^* := \frac{b}{4}(R^2-\epsilon^2)+\psi_0$ and $h \neq h^*$. For $h = h^*$ equation (15) is satisfied using

$$
f(\epsilon) = -\pi\epsilon^2 \quad \text{and} \quad dT_R(\psi) = \frac{2}{\pi R^2}(\psi_1\cos\theta + \hat{\psi}_1\sin\theta).
$$

In this case, the Topological Derivative is given by

$$
\begin{aligned}
D_T j &= -\lim_{R\to 0}\int_{\partial B_R}\frac{2}{\pi R^2}(\psi_1\cos\theta + \hat{\psi}_1\sin\theta)p_0\, d\partial B_R \\
&= -2\nabla u(\hat{\mathbf{x}})\cdot\nabla p(\hat{\mathbf{x}}). \qquad (19)
\end{aligned}
$$

To satisfy (15) for $h \neq h^*$ we need

$$
f(\epsilon) = \frac{2\pi}{\ln\epsilon} \quad \text{and} \quad dT_R(\psi) = \left(\frac{bR}{8\pi}+\frac{\psi_0-h}{2\pi R}\right).
$$

Finally, the Topological Derivative is given by

$$
D_T j = -\lim_{R\to 0}\int_{\partial B_R}\left(\frac{bR}{8\pi}+\frac{\psi_0-h}{2\pi R}\right)p_0\, d\partial B_R = -(u(\hat{\mathbf{x}})-h)p(\hat{\mathbf{x}}). \qquad (20)
$$

Remark : *The exceptional case $h = h^*$ appears in the Saint-Venant theory of torsion of elastic shafts. Indeed,*

$$\int_{\partial B_\epsilon} \frac{\partial w_\epsilon}{\partial n} d\partial B_\epsilon = b\pi\epsilon^2 \quad \Leftrightarrow \quad \frac{b}{4}\left(R^2 - \epsilon^2\right) + \psi_0 - h = 0. \tag{21}$$

This equivalence is obtained from the asymptotic development of w_ϵ, computing the normal derivative of w_ϵ on ∂B_ϵ, and calculating the integral over ∂B_ϵ.

In this section, the Topological Derivative of a cost functional that does not depend explicitly on the domain Ω, with a constraint given by a Poisson problem, was calculated using the Domain Truncation Method. All possible boundary conditions on the hole ∂B_ϵ were considered. The Topological Derivative for the case of Neumann boundary conditions on ∂B_ϵ is given by (16) for the case $h \neq 0$, and (17) otherwise. For Robin boundary conditions, the topological derivative is given by (18), and finally, for Dirichlet boundary conditions on ∂B_ϵ, the Topological Derivative is given by (19) if $h = h^\star$ and (20) otherwise. These results extend the ones obtained by Garreau et al.[4].

4 The Topological-Shape Sensitivity Method

In the work of Novotny et al.[12] the Topological Derivative of the total potential energy functional was calculated using a new approach, which we called Topological-Shape Sensitivity Method. Since that functional depends explicitly on the domain, the results from [12] can not be directly compared to the results in the previous section of this work. Therefore, and for the sake of completeness, the main results from [12] will be reproduced here with the intent of using them to calculate the Topological Derivative of the functional being treated in this work.

The main idea behind the Topological-Shape Sensitivity Method is to start from a problem in which the hole $\overline{B}_\epsilon$ already exists. The ball $\overline{B}_\epsilon$ is submitted to a small perturbation $\delta\epsilon$, originating a new domain $\Omega_{\epsilon+\delta\epsilon} = \Omega - \overline{B}_{\epsilon+\delta\epsilon}$ with boundary $\partial\Omega_{\epsilon+\delta\epsilon} = \partial\Omega \cup \partial B_{\epsilon+\delta\epsilon}$. Then, the Topological Derivative is redefined as follows

$$D_T^* j := \lim_{\epsilon\to 0}\left\{\lim_{\delta\epsilon\to 0}\frac{j\left(\Omega_{\epsilon+\delta\epsilon}\right) - j(\Omega_\epsilon)}{f\left(\epsilon+\delta\epsilon\right) - f\left(\epsilon\right)}\right\}. \tag{22}$$

It is still necessary to show that both definitions for the Topological Derivative are equivalent, which is asserted in the following theorem.

Theorem 5 *Definitions (14) and (22) for the Topological Derivative are equivalent, i.e.*

$$D_T^* j = D_T j.$$

Proof. Using (1) we have that

$$j(\Omega_\epsilon) = j(\Omega) + f(\epsilon) D_T j + \mathcal{R}\left(f(\epsilon)\right) \tag{23}$$

and

$$j(\Omega_{\epsilon+\delta\epsilon}) = j(\Omega) + f(\epsilon+\delta\epsilon) D_T j + \mathcal{R}\left(f(\epsilon+\delta\epsilon)\right). \tag{24}$$

After subtracting (23) from (24), and dividing the result by $f\left(\epsilon+\delta\epsilon\right) - f\left(\epsilon\right)$, the following is obtained

$$\frac{j(\Omega_{\epsilon+\delta\epsilon}) - j(\Omega_\epsilon)}{f(\epsilon+\delta\epsilon) - f(\epsilon)} = D_T j + \frac{\mathcal{R}\left(f\left(\epsilon+\delta\epsilon\right)\right) - \mathcal{R}\left(f\left(\epsilon\right)\right)}{f\left(\epsilon+\delta\epsilon\right) - f\left(\epsilon\right)}.$$

Taking the limit as shown in (22) of the expression above gives

$$\begin{aligned} D_T^* j &= \lim_{\epsilon\to 0}\left\{\lim_{\delta\epsilon\to 0}\frac{j\left(\Omega_{\epsilon+\delta\epsilon}\right) - j(\Omega_\epsilon)}{f\left(\epsilon+\delta\epsilon\right) - f\left(\epsilon\right)}\right\} \\ &= D_T j + \lim_{\epsilon\to 0}\left\{\lim_{\delta\epsilon\to 0}\frac{\mathcal{R}\left(f\left(\epsilon+\delta\epsilon\right)\right) - \mathcal{R}\left(f\left(\epsilon\right)\right)}{f\left(\epsilon+\delta\epsilon\right) - f\left(\epsilon\right)}\right\}. \end{aligned} \tag{25}$$

Applying L'Hopital's rule, we conclude that the last term of (25) is zero ▮

Using classical concepts in Shape Sensitivity Analysis (see for instance Murat & Simon[9], Section IV-4.2, pp. IV.23, and Zolésio[18], Theorem 1.1, pp. 1090), the perturbation $\Omega_\epsilon \to \Omega_{\epsilon+\delta\epsilon}$ can be represented by a smooth and invertible mapping $\chi(\cdot,\tau) : \mathbb{R}^2 \to \mathbb{R}^2$, which depends on the parameter τ, as follows

$$\chi(\cdot,\tau) : \mathbf{x} \mapsto \mathbf{x}_\tau \qquad \forall \mathbf{x} \in \Omega_\epsilon .$$

We now define the perturbed domain Ω_τ as well as its boundary $\partial\Omega_\tau$ as

$$\begin{aligned} \Omega_\tau &:= \left\{ \mathbf{x}_\tau \in \mathbb{R}^2 \mid \exists\, \mathbf{x} \in \Omega_\epsilon,\ \mathbf{x}_\tau = \chi(\mathbf{x},\tau),\ \mathbf{x}_\tau|_{\tau=0} = \mathbf{x},\ \Omega_\tau|_{\tau=0} = \Omega_\epsilon \right\}, \\ \partial\Omega_\tau &:= \left\{ \mathbf{x}_\tau \in \mathbb{R}^2 \mid \exists\, \mathbf{x} \in \partial\Omega_\epsilon,\ \mathbf{x}_\tau = \chi(\mathbf{x},\tau),\ \mathbf{x}_\tau|_{\tau=0} = \mathbf{x},\ \partial\Omega_\tau|_{\tau=0} = \partial\Omega_\epsilon \right\}. \end{aligned}$$

For sufficiently small τ, the mapping $\chi(\cdot,\tau)$ can be written as

$$\mathbf{x}_\tau = \chi(\mathbf{x},\tau) = \mathbf{x} + \tau \mathbf{V}(\mathbf{x}), \tag{26}$$

where $\mathbf{V}(\mathbf{x})$ is the *velocity of change of form* (see for instance Zolésio[18]).

Considering that only the ball $\overline{B}_\epsilon$ is submitted to the perturbation $\delta\epsilon$, we identify $\Omega_{\epsilon+\delta\epsilon} = \Omega_\tau$ and $\partial\Omega_{\epsilon+\delta\epsilon} = \partial\Omega_\tau$. We choose the following for the velocity on the boundary $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon$

$$\begin{cases} \mathbf{V} = V_n \mathbf{n} & \text{on } \partial B_\epsilon \\ \mathbf{V} = 0 & \text{on } \partial\Omega \end{cases}, \tag{27}$$

where V_n is a negative constant ($V_n < 0$).

With the previous definition for $\mathbf{V}$, the following holds on the boundary ∂B_ϵ

$$\mathbf{x}_\tau = \mathbf{x} + \tau V_n \mathbf{n} \quad \forall \mathbf{x} \in \partial B_\epsilon,$$

so it is possible to associate the perturbation $\delta\epsilon$ with the parameter τ in the following way

$$\delta\epsilon = \|\tau V_n \mathbf{n}\| = \tau |V_n| . \tag{28}$$

It is now possible to relate the Topological Derivative and Shape Sensitivity Analysis with the following theorem.

Theorem 6 *The Topological Derivative can be calculated as follows*

$$D_T^* j = \lim_{\epsilon\to 0} \frac{1}{f'(\epsilon)|V_n|} \left.\frac{dj(\Omega_\tau)}{d\tau}\right|_{\tau=0},$$

where $f(\epsilon)$ is a function chosen such that $0 < |D_T^* j| < \infty$.

Proof. Using definition (22) we have

$$\begin{aligned} D_T^* j &= \lim_{\epsilon\to 0} \left\{ \lim_{\delta\epsilon\to 0} \frac{j(\Omega_{\epsilon+\delta\epsilon}) - j(\Omega_\epsilon)}{\frac{f(\epsilon+\delta\epsilon)-f(\epsilon)}{\delta\epsilon}\delta\epsilon} \right\} = \lim_{\epsilon\to 0} \frac{1}{f'(\epsilon)} \lim_{\delta\epsilon\to 0} \frac{j(\Omega_{\epsilon+\delta\epsilon}) - j(\Omega_\epsilon)}{\delta\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{1}{f'(\epsilon)} \lim_{\tau\to 0} \frac{j(\Omega_\tau) - j(\Omega_\tau|_{\tau=0})}{\tau |V_n|} = \lim_{\epsilon\to 0} \frac{1}{f'(\epsilon)|V_n|} \left.\frac{dj(\Omega_\tau)}{d\tau}\right|_{\tau=0} \quad ▮ \end{aligned}$$

Corollary 7 *From Theorem 5 and 6, the following holds*

$$D_T j = D_T^* j = \lim_{\epsilon\to 0} \frac{1}{f'(\epsilon)|V_n|} \left.\frac{dj(\Omega_\tau)}{d\tau}\right|_{\tau=0}.$$

Theorem 6 provides a more general definition of the Topological Derivative than the one given in Section 3 (equation (14)). In fact, using this new definition, it is possible to consider cost functions that depend explicitly on the domain, as shown in [12]. This is not the case with the definition provided by (14). Moreover, theorem 6 provides a constructive formula for the Topological Derivative based on the shape derivative of the cost functional. Consequently, it is possible to use classical results in Shape Sensitivity Analysis (Céa[1], Haug et al.[6], Haug & Céa[7], Murat & Simon[9], Sokolowski & Zolésio[16], Zolésio[18]) to calculate the Topological Derivative. For this reason, we call this new method for computing the Topological Derivative, Topological-Shape Sensitivity Method.

4.1 Calculation of the Topological Derivative

The corresponding variational form of (2) is given as follows

$$\left\{ \begin{array}{l} \text{Find } u_\epsilon \in U_\epsilon \text{, such that} \\ \hat{a}_\epsilon \left(u_\epsilon, \eta_\epsilon\right) = l_\epsilon \left(\eta_\epsilon\right) \qquad \forall\, \eta_\epsilon \in V_\epsilon \end{array} \right. , \tag{29}$$

where

$$\hat{a}_\epsilon \left(u_\epsilon, \eta_\epsilon\right) = \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla \eta_\epsilon \, d\Omega_\epsilon + \gamma \int_{\partial B_\epsilon} u_\epsilon \eta_\epsilon \, d\partial B_\epsilon, \tag{30}$$

$$l_\epsilon \left(\eta_\epsilon\right) = \int_{\Omega_\epsilon} b \eta_\epsilon \, d\Omega_\epsilon + (\beta + \gamma) \int_{\partial B_\epsilon} h \eta_\epsilon \, d\partial B_\epsilon. \tag{31}$$

The set of admissible functions U_ϵ and the set of admissible variations V_ϵ are given by

$$U_\epsilon := \left\{ u_\epsilon \in H^1\left(\Omega_\epsilon\right) \mid u_\epsilon = \bar{u} \text{ on } \partial\Omega \quad \text{and} \quad \alpha(u_\epsilon - h) = 0 \text{ on } \partial B_\epsilon \right\}$$

and

$$V_\epsilon := \left\{ \eta_\epsilon \in H^1\left(\Omega_\epsilon\right) \mid \eta_\epsilon = 0 \text{ on } \partial\Omega \quad \text{and} \quad \alpha\eta_\epsilon = 0 \text{ on } \partial B_\epsilon \right\},$$

where $\alpha \in \{0,1\}$. The notation above should be interpreted as follows: when $\alpha = 1$, $u_\epsilon = h$ and $\eta_\epsilon = 0$ on ∂B_ϵ, and when $\alpha = 0$, u_ϵ and η_ϵ are not specified on ∂B_ϵ.

The boundary value problem (29) written in the reference configuration, must also be satisfied for all perturbations τ, which can be written in the perturbed configuration Ω_τ as follows

$$\left\{ \begin{array}{l} \text{Find } u_\tau \in U_\tau \text{, such that} \\ \hat{a}_\tau \left(u_\tau, \eta_\tau\right) = l_\tau \left(\eta_\tau\right) \ \forall \eta_\tau \in V_\tau \end{array} \right. , \tag{32}$$

where $\hat{a}_\tau \left(u_\tau, \eta_\tau\right)$ and $l_\tau \left(\eta_\tau\right)$ are given by

$$\hat{a}_\tau \left(u_\tau, \eta_\tau\right) = \int_{\Omega_\tau} \nabla_\tau u_\tau \cdot \nabla_\tau \eta_\tau \, d\Omega_\tau + \gamma \int_{\partial B_{\epsilon_\tau}} u_\tau \eta_\tau \, d\partial B_{\epsilon_\tau}, \tag{33}$$

$$l_\tau \left(\eta_\tau\right) = \int_{\Omega_\tau} b \eta_\tau \, d\Omega_\tau + (\beta + \gamma) \int_{\partial B_{\epsilon_\tau}} h \eta_\tau \, d\partial B_{\epsilon_\tau}, \tag{34}$$

and $\epsilon_\tau = \epsilon + \tau \left|V_n\right|$ and $\nabla_\tau(\cdot)$ denotes $\nabla_\tau(\cdot) := \frac{\partial}{\partial \mathbf{x}_\tau}(\cdot)$.

To calculate the Topological Derivative using Theorem 6 it is necessary to calculate the sensitivity of the cost function $j\left(\Omega_\tau\right) := J_\tau\left(u_\tau\right)$ as follows

$$\left.\frac{dj\left(\Omega_\tau\right)}{d\tau}\right|_{\tau=0} = \left.\frac{d}{d\tau} J_\tau\left(u_\tau\right)\right|_{\tau=0} = \lim_{\tau \to 0} \frac{J_\tau\left(u_\tau\right) - J_0\left(u_0\right)}{\tau}. \tag{35}$$

This derivative can be evaluated using the Lagrangian Method: let u_τ and p_τ be the solutions of the state and adjoint equations respectively, then

$$\begin{aligned}\frac{d}{d\tau}J_\tau\left(u_\tau\right) &= \frac{\partial}{\partial\tau}\pounds_\tau\left(u_\tau,p_\tau\right)\\ &= \frac{\partial}{\partial\tau}J_\tau\left(u_\tau\right)+\frac{\partial}{\partial\tau}\hat{a}_\tau\left(u_\tau,p_\tau\right)-\frac{\partial}{\partial\tau}l_\tau\left(p_\tau\right). \end{aligned}\tag{36}$$

If the cost function is defined in a fixed domain, then $J_\tau\left(u_\tau\right)=J\left(u_\tau\right)$. As a consequence, the partial derivative with respect to τ is zero, and (36) yields

$$\frac{d}{d\tau}J\left(u_\tau\right)=\frac{\partial}{\partial\tau}\pounds_\tau\left(u_\tau,p_\tau\right)=\frac{\partial}{\partial\tau}\hat{a}_\tau\left(u_\tau,p_\tau\right)-\frac{\partial}{\partial\tau}l_\tau\left(p_\tau\right).\tag{37}$$

It is important to note that the methodology proposed here is not limited to a cost function that depends only implicitly on Ω_τ. Indeed, a more general cost function was considered in [12].

The derivatives in the referential configuration $\Omega_\tau|_{\tau=0}=\Omega_\epsilon$ are obtained using Reynolds' transport theorem as follows

$$\begin{aligned}\left.\frac{\partial}{\partial\tau}\hat{a}_\tau\left(u_\tau,p_\tau\right)\right|_{\tau=0} &= \int_{\Omega_\epsilon}\left[\left.\frac{\partial}{\partial\tau}\left(\nabla_\tau u_\tau\cdot\nabla_\tau p_\tau\right)\right|_{\tau=0}+\nabla u_\epsilon\cdot\nabla p_\epsilon\,\mathrm{d}iv\mathbf{V}\right]d\Omega_\epsilon\\ &\quad+\gamma\int_{\partial B_\epsilon}u_\epsilon p_\epsilon\,\mathrm{d}iv_\Gamma\mathbf{V}\,d\partial B_\epsilon\\ &= -\int_{\Omega_\epsilon}\left[\left(\nabla\mathbf{V}^T+\nabla\mathbf{V}\right)\nabla u_\epsilon\cdot\nabla p_\epsilon-\nabla u_\epsilon\cdot\nabla p_\epsilon\,\mathrm{d}iv\mathbf{V}\right]\,d\Omega_\epsilon\\ &\quad+\gamma\int_{\partial B_\epsilon}u_\epsilon p_\epsilon\,\mathrm{d}iv_\Gamma\mathbf{V}\,d\partial B_\epsilon, \end{aligned}\tag{38}$$

where $\mathrm{d}iv_\Gamma\mathbf{V}=\left(\mathbf{I}-n\otimes n\right)\cdot\nabla\mathbf{V}$ and

$$\left.\frac{\partial}{\partial\tau}l_\tau\left(p_\tau\right)\right|_{\tau=0}=\int_{\Omega_\epsilon}bp_\epsilon\,\mathrm{d}iv\mathbf{V}\,d\Omega_\epsilon+\left(\beta+\gamma\right)\int_{\partial B_\epsilon}hp_\epsilon\,\mathrm{d}iv_\Gamma\mathbf{V}\,d\partial B_\epsilon.\tag{39}$$

Substituting (38) and (39) in (37) and rearranging terms gives

$$\begin{aligned}\left.\frac{dJ}{d\tau}\right|_{\tau=0} &= -\int_{\Omega_\epsilon}\mathbf{\Sigma}\cdot\nabla\mathbf{V}\,d\Omega_\epsilon-\beta\int_{\partial B_\epsilon}hp_\epsilon\,\mathrm{d}iv_\Gamma\mathbf{V}\,d\partial B_\epsilon\\ &\quad+\gamma\int_{\partial B_\epsilon}p_\epsilon\left(u_\epsilon-h\right)\,\mathrm{d}iv_\Gamma\,\mathbf{V}\,d\partial B_\epsilon, \end{aligned}\tag{40}$$

where

$$\mathbf{\Sigma}=\left(\nabla u_\epsilon\otimes\nabla p_\epsilon\right)+\left(\nabla p_\epsilon\otimes\nabla u_\epsilon\right)+\left(bp_\epsilon-\nabla u_\epsilon\cdot\nabla p_\epsilon\right)\mathbf{I}$$

can be interpreted as a generalization of the Energy-Momentum Tensor of Eshelby (see, for instance, Eshelby[3] and Taroco et al.[17]). Since the state and adjoint equations are satisfied, $\mathrm{d}iv\mathbf{\Sigma}=\mathbf{0}$, and the following tensorial relation holds

$$\mathbf{\Sigma}\cdot\nabla\mathbf{V}=\mathrm{d}iv\left(\mathbf{\Sigma}^T\mathbf{V}\right).\tag{41}$$

Using the previous relation, (40) becomes an integral only defined on the boundary $\partial\Omega_\epsilon$,

$$\begin{aligned}\left.\frac{dJ}{d\tau}\right|_{\tau=0} &= -\int_{\partial\Omega_\epsilon}\mathbf{\Sigma}\mathbf{n}\cdot\mathbf{V}\,d\partial\Omega_\epsilon-\beta\int_{\partial B_\epsilon}hp_\epsilon\,\mathrm{d}iv_\Gamma\mathbf{V}\,d\partial B_\epsilon\\ &\quad+\gamma\int_{\partial B_\epsilon}p_\epsilon\left(u_\epsilon-h\right)\,\mathrm{d}iv_\Gamma\mathbf{V}\,d\partial B_\epsilon. \end{aligned}$$

Using definition (27) and $\epsilon_\tau = \epsilon + \tau |V_n|$, it is easy to show that $div_\Gamma \mathbf{V} = \frac{1}{\epsilon} |V_n|$. Therefore,

$$\left.\frac{dJ}{d\tau}\right|_{\tau=0} = -V_n \int_{\partial B_\epsilon} \left\{ \mathbf{\Sigma n} \cdot \mathbf{n} + \frac{sign\,(V_n)}{\epsilon} \left[\beta h p_\epsilon - \gamma p_\epsilon \left(u_\epsilon - h\right)\right] \right\} d\partial B_\epsilon. \tag{42}$$

Substituting (42) in the result of Theorem 6 and from $sign\,(V_n) = -1$, one has that

$$D_T j = \lim_{\epsilon \to 0} \frac{1}{f'(\epsilon)} \int_{\partial B_\epsilon} \left\{ \mathbf{\Sigma n} \cdot \mathbf{n} - \frac{1}{\epsilon} \left[\beta h p_\epsilon - \gamma p_\epsilon \left(u_\epsilon - h\right)\right] \right\} d\partial B_\epsilon. \tag{43}$$

Using

$$\mathbf{\Sigma n} \cdot \mathbf{n} = 2 \left(\frac{\partial u_\epsilon}{\partial n} \frac{\partial p_\epsilon}{\partial n} \right) + b p_\epsilon - \nabla u_\epsilon \cdot \nabla p_\epsilon,$$

the expression (43) can be written as follows

$$D_T j = \lim_{\epsilon \to 0} \frac{1}{f'(\epsilon)} \int_{\partial B_\epsilon} \left\{ \frac{\partial u_\epsilon}{\partial n} \frac{\partial p_\epsilon}{\partial n} - \frac{\partial u_\epsilon}{\partial t} \frac{\partial p_\epsilon}{\partial t} + b p_\epsilon - \frac{1}{\epsilon} \left[\beta h p_\epsilon - \gamma p_\epsilon \left(u_\epsilon - h\right)\right] \right\} d\partial B_\epsilon. \tag{44}$$

The final expression of the Topological Derivative using the Topological-Shape Sensitivity Method is obtained by evaluating the limit above. Without performing the limiting process, it is still possible to draw some conclusions about the behavior of the different terms. For example, for the case $b = 0$, the following can be inferred:

- For $\beta = 1$ and $\alpha = \gamma = 0$, the solution u_ϵ is regular. Thus, it is expected that the dominant term is given by hp for $h \neq 0$, and $\nabla u \cdot \nabla p$ for $h = 0$.
- In the same way, for $\gamma = 1$ and $\alpha = \beta = 0$ the dominant term is given by $-p\,(u - h)$.
- For $\alpha = 1$, $\beta = \gamma = 0$ and $h = h^*$ (note that the solution is regular in this case), the term $-\nabla u \cdot \nabla p$ is the dominant one. On the other hand, for $h \neq h^*$ (for instance, $h = 0$) the solution is singular and its normal derivative must be related to the solution u_ϵ through a singular function depending on ϵ. Thus, in this last case, the dominant term is given by $-up$.

From the previous analysis, it was possible to recover the results obtained with the Truncation Method (equations (16), (17),(18),(19),(20)) without performing an asymptotic expansion of the solution u_ϵ. In fact, the computation of the Topological Derivative using the Topological-Shape Sensitivity Method can be performed using an adequate numerical procedure. As a consequence, this new methodology can be applied to a more general class of problems.

4.2 Calculation of the limit when $\epsilon \to 0$

The solution of (4) with $\psi = u_\epsilon|_{\partial B_R}$ at ∂B_ϵ is given by

$$\begin{aligned} u_\epsilon|_{\partial B_\epsilon} &= -\frac{b}{4}\epsilon^2 + \ln\epsilon \left(\frac{\tilde{\alpha}\frac{b}{4}(R^2 - \epsilon^2) + \tilde{\alpha}\psi_0 - \tilde{h} + \tilde{\beta}\frac{b}{2}\epsilon}{\tilde{\alpha}(\ln R - \ln\epsilon) + \tilde{\beta}/\epsilon} \right) \\ &\quad + \frac{\ln R \left(\tilde{h} + \tilde{\alpha}\frac{b}{4}\epsilon^2 - \tilde{\beta}\frac{b}{2}\epsilon\right) - \left(\tilde{\alpha}\ln\epsilon - \tilde{\beta}/\epsilon\right)\left(\psi_0 + \frac{b}{4}R^2\right)}{\tilde{\alpha}(\ln R - \ln\epsilon) + \tilde{\beta}/\epsilon} \\ &\quad + \sum_{n=1}^{\infty} \frac{2n\tilde{\beta}\epsilon^n \left(\psi_n \cos n\theta + \hat{\psi}_n \sin n\theta\right)}{\left(\epsilon\tilde{\alpha} + n\tilde{\beta}\right) - (\epsilon/R)^{2n}\left(\epsilon\tilde{\alpha} - n\tilde{\beta}\right)} \end{aligned} \tag{45}$$

and the normal and tangential derivatives of u_ϵ on ∂B_ϵ are given respectively by

$$\begin{aligned} \left.\frac{\partial u_\epsilon}{\partial n}\right|_{\partial B_\epsilon} &= \frac{b}{2}\epsilon - \frac{1}{\epsilon}\left(\frac{\tilde{\alpha}\frac{b}{4}(R^2-\epsilon^2)+\tilde{\alpha}\psi_0-\tilde{h}+\tilde{\beta}\frac{b}{2}\epsilon}{\tilde{\alpha}(\ln R-\ln\epsilon)+\tilde{\beta}/\epsilon}\right) \\ & -\sum_{n=1}^{\infty}\frac{2n\tilde{\alpha}\epsilon^n\left(\psi_n\cos n\theta+\hat{\psi}_n\sin n\theta\right)}{\left(\epsilon\tilde{\alpha}+n\tilde{\beta}\right)-(\epsilon/R)^{2n}\left(\epsilon\tilde{\alpha}-n\tilde{\beta}\right)}, \end{aligned} \tag{46}$$

$$\left.\frac{\partial u_\epsilon}{\partial t}\right|_{\partial B_\epsilon} = \sum_{n=1}^{\infty}\frac{2n^2\tilde{\beta}\epsilon^{n-1}\left(\hat{\psi}_n\cos n\theta-\psi_n\sin n\theta\right)}{\left(\epsilon\tilde{\alpha}+n\tilde{\beta}\right)-(\epsilon/R)^{2n}\left(\epsilon\tilde{\alpha}-n\tilde{\beta}\right)}. \tag{47}$$

With these results, it is now possible to evaluate the limit in (44) to obtain the final expression for the Topological Derivative for different types of boundary conditions on ∂B_ϵ

- **Neumann boundary condition** ($\beta = 1, \alpha = \gamma = 0$): In this case, (44) is given by

$$D_T j = \lim_{\epsilon\to 0}\frac{1}{f'(\epsilon)}\int_{\partial B_\epsilon}\left(\frac{\partial u_\epsilon}{\partial n}\frac{\partial p_\epsilon}{\partial n}-\frac{\partial u_\epsilon}{\partial t}\frac{\partial p_\epsilon}{\partial t}+bp_\epsilon-\frac{1}{\epsilon}hp_\epsilon\right)\, d\partial B_\epsilon. \tag{48}$$

  Considering the case $h \neq 0$ and substituting (45), (46) and (47) in (48) we require that $f'(\epsilon) = -2\pi \Rightarrow f(\epsilon) = -2\pi\epsilon$. Therefore,

$$D_T j = hp(\hat{\mathbf{x}}). \tag{49}$$

  The case $h = 0$ requires that $f'(\epsilon) = -2\pi\epsilon \Rightarrow f(\epsilon) = -\pi\epsilon^2$ and the Topological Derivative is given by

$$D_T j = 2\nabla u(\hat{\mathbf{x}})\cdot\nabla p(\hat{\mathbf{x}}) - bp(\hat{\mathbf{x}}). \tag{50}$$

- **Robin boundary condition** ($\gamma = 1, \alpha = \beta = 0$): In this case, we have

$$D_T j = \lim_{\epsilon\to 0}\frac{1}{f'(\epsilon)}\int_{\partial B_\epsilon}\left(\frac{\partial u_\epsilon}{\partial n}\frac{\partial p_\epsilon}{\partial n}-\frac{\partial u_\epsilon}{\partial t}\frac{\partial p_\epsilon}{\partial t}+bp_\epsilon+\frac{1}{\epsilon}p_\epsilon(u_\epsilon-h)\right)\, d\partial B_\epsilon. \tag{51}$$

  Substituting (45), (46) and (47) in (51) requires that $f'(\epsilon) = -2\pi \Rightarrow f(\epsilon) = -2\pi\epsilon$ which finally gives

$$D_T j = -(u(\hat{\mathbf{x}}) - h)\, p(\hat{\mathbf{x}}). \tag{52}$$

- **Dirichlet boundary condition** ($\alpha = 1, \beta = \gamma = 0$): Finally, the case of a Dirichlet boundary condition on the hole ∂B_ϵ gives

$$D_T j = \lim_{\epsilon\to 0}\frac{1}{f'(\epsilon)}\int_{\partial B_\epsilon}\left(\frac{\partial u_\epsilon}{\partial n}\frac{\partial p_\epsilon}{\partial n}-\frac{\partial u_\epsilon}{\partial t}\frac{\partial p_\epsilon}{\partial t}+bp_\epsilon\right)\, d\partial B_\epsilon. \tag{53}$$

  Taking into account that $\frac{\partial u_\epsilon}{\partial t}\big|_{\partial B_\epsilon} = 0$ and $p_\epsilon = 0$ on ∂B_ϵ (since $p_\epsilon \in V_\epsilon$), equation (53) reduces to

$$D_T j = \lim_{\epsilon\to 0}\frac{1}{f'(\epsilon)}\int_{\partial B_\epsilon}\left(\frac{\partial u_\epsilon}{\partial n}\frac{\partial p_\epsilon}{\partial n}\right)\, d\partial B_\epsilon. \tag{54}$$

  Further, if $h \neq h^*$, from (46) we conclude that the solution u_ϵ is singular. Therefore

$$f'(\epsilon) = -\frac{2\pi}{\epsilon\ln(\epsilon)^2} \Rightarrow f(\epsilon) = \frac{2\pi}{\ln(\epsilon)}.$$

  Using this last result and substituting (46) in (54) we finally obtain

$$D_T j = -(u(\hat{\mathbf{x}}) - h)\, p(\hat{\mathbf{x}}). \tag{55}$$

Table 1: Topological Derivatives for the Poisson's problem in $2D$ domains.

Boundary Conditions	$f(\epsilon)$	$D_T j$
$\beta = 1, \alpha = \gamma = 0$ and $h \neq 0$	$-2\pi\epsilon$	hp
$\beta = 1, \alpha = \gamma = 0$ and $h = 0$	$-\pi\epsilon^2$	$2\nabla u \cdot \nabla p - bp$
$\gamma = 1, \alpha = \beta = 0$	$-2\pi\epsilon$	$-(u-h)\,p$
$\alpha = 1, \beta = \gamma = 0$ and $h = h^*$	$-\pi\epsilon^2$	$-2\nabla u \cdot \nabla p$
$\alpha = 1, \beta = \gamma = 0$ and $h \neq h^*$	$\frac{2\pi}{\log(\epsilon)}$	$-(u-h)\,p$

Remark: *For the exceptional case $h = h^*$ (see equation 21), the expression of the Topological Derivative given by (54) also holds. Hence, from (46) we have $f'(\epsilon) = -2\pi\epsilon \Rightarrow f(\epsilon) = -\pi\epsilon^2$ and the Topological Derivative is given by*

$$D_T j = -2\nabla u(\hat{\mathbf{x}}) \cdot \nabla p(\hat{\mathbf{x}}) . \tag{56}$$

In summary, the Topological Derivative for the Poisson problem with a cost function that depends implicitly on Ω is given by (49) or (50) for the Neumann boundary condition, by (52) for the case of Robin boundary conditions and by (56) or (55) for Dirichlet boundary conditions on ∂B_ϵ. Obviously, these results which were obtained with the Topological-Shape Sensitivity Method are exactly the same as those computed with the Domain Truncation Method as a result of Theorem 5. However, we believe the new method introduced in this work provides a simpler and more constructive mean to calculate the Topological Derivative.

5 Summary of Results

The final expression of the Topological Derivative for a Poisson problem with different types of boundary conditions on ∂B_ϵ are shown in Table 1. The cost function depends implicitly on the domain and u and p are the direct and adjoint solutions of the problem associated to the original domain Ω.

6 Conclusions

The Topological Derivative was calculated for the Poisson problem using the Domain Truncation Method and a new method called Topological-Shape Sensitivity Method. A general set of boundary conditions on the holes was considered: Dirichlet, Neumann (both homogeneous and non-homogeneous), Robin and the exceptional case associated to the Saint-Venant theory of torsion of an elastic shaft.

It was shown that the new methodology leads to a more general and constructive technique for the calculation of the Topological Derivative.

The Domain Truncation Method is limited to cost functions that depend implicitly on Ω through the solution u (see Theorem 4). It appears that this technique can not be easily extended to treat more general cost functions. In particular, the functional given by the total potential energy cannot be treated using this method. On the other hand, the Topological-Shape Sensitivity Method can be used for any cost function and boundary conditions. In fact, in Novotny et al.[12] the total potential energy functional was adopted and the Topological Derivative was obtained in a straightforward manner.

It is also important to mention that the Topological-Shape Sensitivity Method provides a constructive expression for the Topological Derivative and therefore, it makes it possible to consider problems where the limit in (44) can not be obtained analytically. In these cases, numerical techniques can be used to evaluate the limit. We are currently investigating this type of approach.

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