CRACK GROWTH CONTROL BASED ON THE TOPOLOGICAL DERIVATIVE OF THE RICE'S INTEGRAL

M. XAVIER, A.A. NOVOTNY AND J. SOKOŁOWSKI

Abstract. In fracture mechanics, an important question concerns the useful life of mechanical components. Such components are, usually, submitted to the actions of external forces and/or degrading agents which can trigger the crack nucleation and propagation process. In particular, when a mechanical component is already partially cracked, the question is how to extend its remaining useful life. In this work, a simple and efficient methodology aiming to extend the remaining useful life of cracked elastic bodies is proposed. More precisely, we want to find a way to retard or even avoid the triggering of the crack propagation process by nucleating hard and/or soft inclusions far from the crack tip. The main idea consists in minimize a shape functional based on the Rice's integral with respect to the nucleation of inclusions by using the concept of topological derivative. The obtained sensitivity, which corroborates with the famous Eshelby theorem, is used to indicate the regions where the controls have to be inserted. According to the Griffith's energy criterion, this simple procedure allows for increasing the remaining useful life of the cracked body. Finally, some numerical experiments are presented showing the applicability of the proposed methodology.

1. Introduction

Mechanical components submitted to the action of external forces and/or degrading agents may become susceptible to crack nucleation and propagation process, eventually up to their total failure. In particular, when the original component is already partially cracked, one important question arises consisting in how to extend its remaining useful life i.e., how to retard or avoid the crack propagation process. The remaining useful life of a cracked component is closed related to its energy release rate, which is defined as the variation of the strain energy stored in the body with respect to the crack growth. More specifically, based on Griffith's energy criterion [11], the lower is the energy release rate of the cracked component the higher is its useful life. One well-known idea to deal with this problem consists in finding different strategies which allow for reducing the energy release rate of the component, including optimal shape and material distribution, for instance. See the pioneering work [7] and more recent publications [14, 17, 19]. See also related works [15, 23, 24].

In this paper we propose a simple strategy aiming to retard or even avoid the triggering of the crack propagation process by nucleating hard and/or soft inclusions far from the crack tip. Therefore, we do not consider obvious solutions consisting in round the crack tip, for instance. The basic idea consists in minimizing a shape functional defined in terms of the Rice's integral (J-integral [22]) with respect to the nucleation of inclusions by using the topological derivative concept [21]. Actually, the introduction of inclusions at the regions where the topological derivative is negative allows for a decreasing on the values of the associated shape functional. According to the Griffith's energy criterion, this simple strategy allows for an increasing on the remaining useful

Key words and phrases. Rice's integral, Griffith's criterion, Eshelby's tensor, topological derivative, topology optimization.

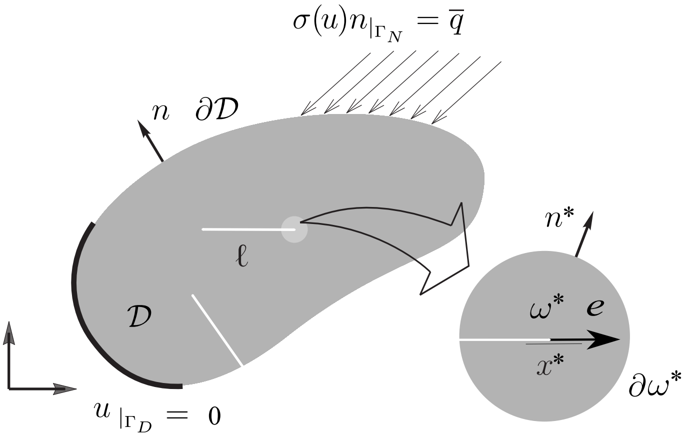


Figure 1. Cracked elastic body.

life of the cracked body. Finally, some numerical experiments are presented, showing the applicability of the proposed methodology. It should be noted however that our approach is developed over a linear elastic model. One well-known limitation of this class of models is that they are not able to distinguish between traction and compression stress states, so that crack closure phenomenon cannot be captured, for example. The extension to the non-linear case by considering inequality type boundary conditions on the crack lips, by following the original ideas introduced in [26], is now under investigation.

More precisely, our procedure for the passive control of crack propagation can be described as follows. We start by evaluating the shape derivative of the elastic energy with respect to the position of the crack tip. In this way, the numerical value of SIF is obtained. We claim however that the precise computations of the SIF at the crack tips is avoided. Actually, in our computational scheme, the SIF is given by the shape derivative with respect to the crack growth or by the Rice's integral. Then, the topological derivative of the associated shape derivative is evaluated by the domain decomposition method. In this way, the influence of singular domain perturbations on SIF is obtained. Finally, the obtained result is used to decide where the controls have to be positioned in order to increase the remaining useful life of the cracked specimen.

The work is organized as follows. The statement of the problem we are dealing with is presented in Section 2. In Section 3, we present the closed formula of the associated topological derivative. Then, some numerical experiments are driven in Section 4 showing the applicability of the proposed methodology. Finally, some concluding remarks are presented in Section 5.

2. Problem statement

In order to introduce the mechanical problem we are dealing with, let us consider an elastic cracked body represented by an open and bounded domain $\mathcal{D} \subset \mathbb{R}^2$, with boundary $\partial\mathcal{D} = \Gamma_N \cup \Gamma_D \cup \ell$, submitted to surface loads on Γ_N and prescribed displacements on Γ_D. The existing cracks, represented by ℓ, are assumed to be straight with length h and direction e, whose tips are denoted by x^*. In addition, the cracks ℓ are free of traction. Finally, a control region $\omega^* \subset \mathcal{D}$ around the crack tip is considered. See sketch in Figure 1.

The total potential energy of the system is given by

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathcal{D}} \sigma(u)\cdot\nabla u^s - \int_{\Gamma_N} \overline{q}\cdot u \ , \tag{2.1}$$

where $\overline{q}$ represents the set of applied boundary traction on Γ_N. The displacement field u is solution to the following variational problem: Find $u \in \mathcal{U}$ such that

$$\int_{\mathcal{D}} \sigma(u)\cdot\nabla\eta^s = \int_{\Gamma_N} \overline{q}\cdot\eta \ \ \forall\eta\in\mathcal{V} \ , \tag{2.2}$$

where the stress tensor $\sigma(\varphi) = \mathbb{C}\nabla\varphi^s$. We consider isotropic material, so that the elasticity tensor $\mathbb{C}$ can be written as

$$\mathbb{C} = 2\mu\mathbb{I} + \lambda(\mathrm{I}\otimes\mathrm{I}) \ , \tag{2.3}$$

where I and $\mathbb{I}$ are the second and fourth order identity tensors, respectively, and μ and λ are the Lamé's coefficients. In particular, we have

$$\mu = \frac{E}{2(1+\nu)} \ , \ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \ \text{and} \ \lambda^* = \frac{\nu E}{1-\nu^2} \ , \tag{2.4}$$

where λ and λ^* are associated with plane strain and plane stress assumptions, respectively. In addition, E is the Young's modulus and ν the Poisson's ratio. The strain tensor is given by

$$\nabla\varphi^s = \frac{1}{2}(\nabla\varphi + (\nabla\varphi)^\top) \ . \tag{2.5}$$

Finally, the set $\mathcal{U}$ and the space $\mathcal{V}$ are defined as

$$\mathcal{U} := \mathcal{V} := \left\{\varphi\in H^1(\mathcal{D}) : \varphi_{|\Gamma_D} = 0\right\} \ . \tag{2.6}$$

From this scenario, we assume that the crack propagation mechanism may be activated according to some dissipation criterion. Therefore, the idea is to find a way to retard or even avoid the triggering of the crack propagation process by nucleating hard and/or soft inclusions far from the crack tip. According to the Griffith's energy criterion, this simple procedure may increase the remaining useful life of the cracked body. Inspired on these ideas, we proposed to minimize a shape functional based on the Rice's integral with respect to the nucleation of inclusions by using the concept of topological derivative.

2.1. **Rice's integral.** The Rice's integral, denoted here by $\Pi(u)$, is defined as

$$\Pi(u) := -\frac{d}{dh}\mathcal{W}(u) \ , \tag{2.7}$$

where $\mathcal{W}(u)$ is the energy released [22]. By taking the strain energy to compute the energy release rate, i.e., $\mathcal{W}(u) = -\mathcal{J}(u)$, we have

$$\Pi(u) := \frac{d}{dh}\mathcal{J}(u) = e\cdot\int_{\partial\omega^*} \Sigma(u)n^* \ , \tag{2.8}$$

where $\mathcal{J}(u)$ is defined in (2.1), n^* is the outward unit normal vector to $\partial\omega^*$, e is the direction of the crack growth and $\Sigma(u)$ is the Eshelby tensor, defined as

$$\Sigma(u) = \frac{1}{2}(\sigma(u)\cdot\nabla u^s)\mathrm{I} - \nabla u^\top\sigma(u) \ . \tag{2.9}$$

Remark 1. *Note that* $\Sigma(u)$ *in* (2.8)*, defined by* (2.9)*, is the* Eshelby energy-momentum tensor *introduced in* [8]*. This tensor appears in the analysis of defects in elasticity and it plays a central role in the continuum mechanics theory involving inhomogeneities (inclusions, pores, cracks, etc.) in solids* [21]*.*

For the purposes of this work, it is necessary to introduce a representation of $\Pi(u)$ as an integral over the cracked domain. Alternative representations of $\Pi(u)$ can be found in [9, 24, 27], for instance. For a more general expression of $\Pi(u)$ into three spatial dimensions see [10]. So, in order to write $\Pi(u)$ as an integral over the cracked domain, let us first define a shape change velocity field V, with compact support in ω^*, such that

$$V \in C^\infty(\mathcal{D}) : V = e \text{ in } \omega^* \ . \tag{2.10}$$

Then the derivative of $\mathcal{J}(u)$ from (2.1) with respect to the crack length h can be written as

$$\frac{d}{dh}\mathcal{J}(u) = \int_{\mathcal{D}} \Sigma(u) \cdot \nabla V \ . \tag{2.11}$$

Therefore, the following equivalent form for the Rice's integral $\Pi(u)$ holds true

$$\Pi(u) = \int_{\mathcal{D}} \Sigma(u) \cdot \nabla V \ , \tag{2.12}$$

where $\Sigma(u)$ is the Eshelby tensor defined by (2.9). The proof of the equivalence between the different representations of the Rice's integral given by (2.8) and (2.12) can be found in details in [27], for instance.

Remark 2. *Note that, since the distributed shape gradient of the total potential energy is given by the product of* $\Sigma(u)$ *and* ∇V*, it follows that* $\Sigma(u)$ *can be interpreted in terms of the* configurational forces [12] *acting in the elastic body with a small defect inside* [21].

2.2. **Topology optimization problem.** The topology optimization problem is based on Griffith's energy criterion for crack propagation [11]. This criterion can be written in terms of the Rice's integral in the following way:

$$G_s + \Pi(u) \begin{cases} < & 0 \ \text{ the crack is unstable;} \\ = & 0 \ \text{ the crack is in equilibrium;} \\ > & 0 \ \text{ the crack is stable,} \end{cases} \tag{2.13}$$

where $G_s > 0$ is used to denote the Griffith's surface energy.

Since G_s is a positive number and taking into account that $\Pi(u)$ is a negative quantity, the less negative is $\Pi(u)$ the higher is the useful life of mechanical components. Therefore, by avoiding trivial solution which consists in rounding the crack tip, the idea is to maximize $\Pi(u)$ with respect to the nucleation of hard and/or soft inclusions far from the crack tip. Thus, the optimization problem we are dealing with can be formulated as follows:

$$\underset{\Omega \subset \mathcal{D}}{\text{Minimize}} \ \{-\Pi(u)\}, \text{ subject to } (2.2) \ , \tag{2.14}$$

where $\Omega := \mathcal{D} \setminus \omega^*$ and $\Pi(u)$ is the Rice's integral defined through (2.12). Here, the domain Ω, which is free of geometrical singularities produced by the crack tip, is assumed to be smooth, with Lipschitz boundary $\partial\Omega$.

A natural approach to deal with such a minimization problem consists in apply the topological derivative concept. Therefore, in order to simplify further analysis we introduce the following adjoint state: Find $v \in \mathcal{V}$, such that

$$\int_{\mathcal{D}} \sigma(v) \cdot \nabla \eta^s = \langle D_u \Pi(u), \eta \rangle$$
$$= \int_{\mathcal{D}} \mathrm{tr}(\nabla V) \sigma(u) \cdot \nabla \eta^s - \int_{\mathcal{D}} \sigma(\eta) \cdot (\nabla u \nabla V) - \int_{\mathcal{D}} \sigma(u) \cdot (\nabla \eta \nabla V) \ \forall \, \eta \in \mathcal{V} \, . \quad (2.15)$$

where V is the shape change velocity field defined in (2.10).

3. Topology optimization method

The present methodology is based on the fact that the introduction of an inclusion at the region where the topological derivative is negative allows for a decreasing on the values of the associated shape functional. Therefore, the topological derivative of the shape functional $\{-\Pi(u)\}$, where $\Pi(u)$ is the Rice's integral defined by (2.12), with respect to the nucleation of a small circular inclusion, is obtained. Then, the resulting expression will be used to indicate the regions where the inclusions should be nucleated in order to solve the minimization problem (2.14).

3.1. **The topological derivative concept.** The topological derivative [25] is a scalar field that measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation, such as the insertion of holes, inclusions, source-terms or even cracks. This concept has been successfully applied in many physical and engineering problems [21] including damage and fracture mechanics [3, 27, 28, 29], for instance. In order to introduce these ideas, let us consider the domain $\Omega = \mathcal{D} \setminus \omega^*$ free of cracks, which is now subject to a nonsmooth perturbation confined in a small region $B_\varepsilon(\hat{x})$ of size ε centered at an arbitrary point $\hat{x} \in \Omega$, as sketched in Figure 2. We introduce a characteristic function $x \mapsto \chi(x)$, $x \in \mathbb{R}^2$, associated with the unperturbed domain, namely $\chi = \mathbb{1}_\Omega$, such that:

$$|\Omega| = \int_{\mathbb{R}^2} \chi(x) \, , \quad (3.1)$$

where $|\Omega|$ is the Lebesgue's measure of Ω. Then, we define a characteristic function associated with the topologically perturbed domain of the form $x \mapsto \chi_\varepsilon(\hat{x}; x)$, $x \in \mathbb{R}^2$. In the case of a perforation, for example, $\chi_\varepsilon(\hat{x}) = \mathbb{1}_\Omega - \mathbb{1}_{B_\varepsilon(\hat{x})}$ and the perforated domain is obtained as $\Omega_\varepsilon(\hat{x}) = \Omega \setminus \overline{B_\varepsilon(\hat{x})}$. Then, we assume that a given shape functional $\psi(\chi_\varepsilon(\hat{x}))$, associated to the topologically perturbed domain, admits the following topological asymptotic expansion:

$$\psi(\chi_\varepsilon(\hat{x})) = \psi(\chi) + f(\varepsilon)\mathcal{T}(\hat{x}) + o(f(\varepsilon)) \, , \quad (3.2)$$

where $\psi(\chi)$ is the shape functional associated to the original domain, that is, without perturbation, $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0$ and $o(f(\varepsilon))$ is the remainder. The function $\hat{x} \mapsto \mathcal{T}(\hat{x})$ is called the topological derivative of ψ at $\hat{x}$. Therefore, this derivative can be seen as a first order correction on $\psi(\chi)$ to approximate $\psi(\chi_\varepsilon(\hat{x}))$. In fact, after rearranging (3.2) we have

$$\frac{\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi)}{f(\varepsilon)} = \mathcal{T}(\hat{x}) + \frac{o(f(\varepsilon))}{f(\varepsilon)} \, . \quad (3.3)$$

The limit $\varepsilon \to 0$ in the above expression leads to the general definition for the topological derivative, namely

$$\mathcal{T}(\hat{x}) = \lim_{\varepsilon \to 0} \frac{\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi(x))}{f(\varepsilon)} \ . \tag{3.4}$$

It is worth to mention that the topological derivative is defined by a limit passage when the small parameter governing the size of the topological perturbation goes to zero in (3.4). Therefore, it can also be used as a steepest-descent direction in an optimization process like in any method based on the gradient of the cost functional.

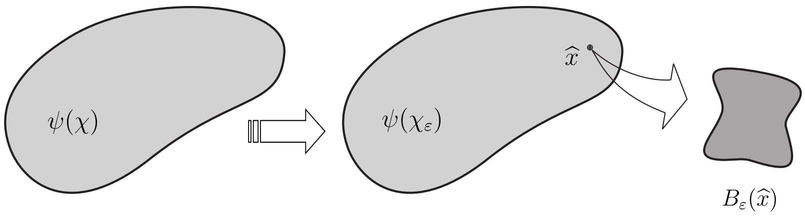


FIGURE 2. The topological derivative concept.

3.2. **Domain decomposition method.** In order to evaluate the topological derivative associated with the minimization problem (2.14), let us first decompose $\mathcal{D}$ into two parts, namely, $\omega^* \subset \mathcal{D}$ and $\Omega := \mathcal{D} \setminus \omega^*$ such that ω^* is the subdomain which contains the singularity produced by the crack tip. In addition, we consider an intact domain ω of the form $\omega := \omega^* \cup \ell$ as sketched in Figure 3. Then, the following boundary value problem is considered: Find w, such that

$$\left\{ \begin{array}{rcll} \operatorname{div}\sigma(w) & = & 0 & \text{in} \quad \omega^* \ , \\ \sigma(w) & = & \mathbb{C}\nabla w^s \ , & \\ \sigma(w)n & = & 0 & \text{on} \quad \ell \ , \\ w & = & \varphi & \text{on} \quad \partial\omega \ . \end{array} \right. \tag{3.5}$$

By using (3.5), we can define the Steklov-Poincaré pseudo-differential boundary operator as follows

$$\mathcal{S} : \varphi \in H^{1/2}(\partial\omega) \mapsto \sigma(w)n^* \in H^{-1/2}(\partial\omega) \ , \tag{3.6}$$

where n^* is the outward normal vector to the boundary $\partial\omega$. Therefore, the following variational problem is considered: Find $u \in \mathcal{U}$, such that

$$\int_\Omega \sigma(u) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u) \cdot \eta = \int_{\Gamma_N} \overline{q} \cdot \eta \quad \forall \eta \in \mathcal{V} \ . \tag{3.7}$$

Note that, by setting $\varphi = (u)_{|\partial\omega}$, we have $w = (u)_{|\omega^*}$. For more details on the domain decomposition method, see [1, 18, 26] for instance.

Let us now introduce the topologically perturbed counterpart of problem (3.7). The idea consists in nucleating a circular inclusion, denoted by $B_\varepsilon(\hat{x})$, of radius ε and center at the arbitrary point $\hat{x} \in \Omega$, such that $\overline{B_\varepsilon(\hat{x})} \subset \Omega$. See sketch in Figure 4. In this case $\chi_\varepsilon(\hat{x})$ is defined as follows:

$$\chi_\varepsilon(\hat{x}) = \mathbb{1}_\Omega - (1-\gamma)\mathbb{1}_{B_\varepsilon(\hat{x})} \ , \tag{3.8}$$

where $\gamma = \gamma(x)$ is the contrast in the material properties. From these elements, we define a piecewise constant function of the form

$$\gamma_\varepsilon = \gamma_\varepsilon(x) := \left\{ \begin{array}{ll} 1 & \text{if } x \in \Omega \setminus \overline{B_\varepsilon(\hat{x})} \ , \\ \gamma & \text{if } x \in B_\varepsilon(\hat{x}) \ . \end{array} \right. \tag{3.9}$$

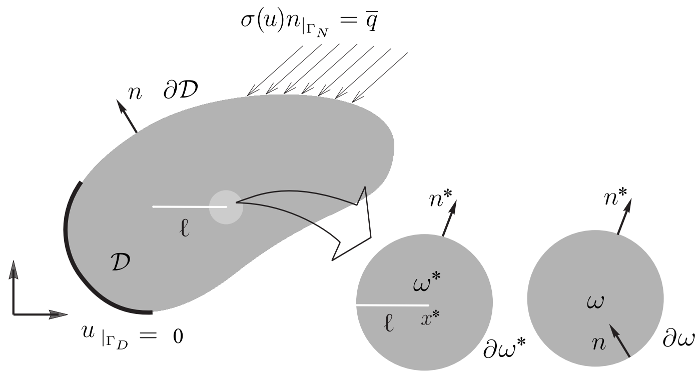


FIGURE 3. Truncated domain.

The variational formulation associated with the topologically perturbed problem is stated as: Find $u_\varepsilon \in \mathcal{U}$, such that

$$\int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon) \cdot \eta = \int_{\Gamma_N} \overline{q} \cdot \eta \quad \forall \eta \in \mathcal{V} \, , \tag{3.10}$$

Note that, by setting $\varphi = (u_\varepsilon)_{|\partial\omega}$, we have $w = (u_\varepsilon)_{|\omega^*}$.

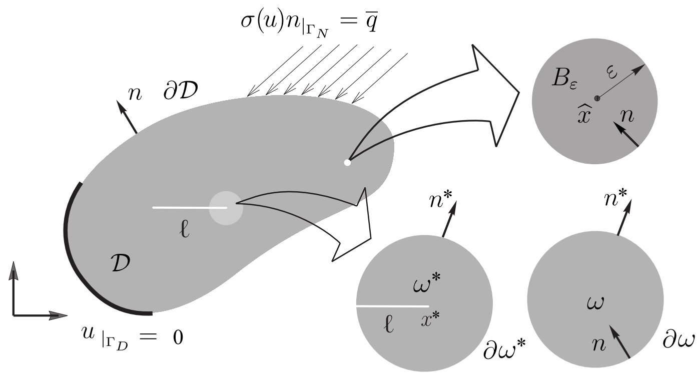


FIGURE 4. Perturbed problem.

3.3. **Existence of the topological derivative.** The existence of the associated topological derivative is ensured by the following result:

Lemma 3. *Let u_ε and u be solutions of problems* (3.10) *and* (3.7)*, respectively. Then, the following estimate holds true:*

$$\|u_\varepsilon - u\|_{H^1(\Omega)} \leq C\varepsilon \, , \tag{3.11}$$

where C is a constant independent of the small parameter ε.

Proof. Let us subtract (3.7) from (3.10). Then, from the definition for the contrast (3.9), we obtain

$$\begin{aligned} 0 &= \int_\Omega (\sigma_\varepsilon(u_\varepsilon) - \sigma(u)) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot \eta \\ &= \int_{\Omega\setminus B_\varepsilon} (\sigma(u_\varepsilon) - \sigma(u)) \cdot \nabla\eta^s + \int_{B_\varepsilon} (\gamma\sigma(u_\varepsilon) - \sigma(u)) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot \eta \, . \end{aligned}$$

After adding and subtracting the term

$$\int_{B_\varepsilon} \gamma\sigma(u) \cdot \nabla\eta^s$$

in the above expression we have

$$\int_\Omega \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla\eta^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot \eta = \int_{B_\varepsilon} (1-\gamma)\sigma(u) \cdot \nabla\eta^s \ . \tag{3.12}$$

By taking $\eta = u_\varepsilon - u$ as test function in (3.12) we obtain the following equality

$$\int_\Omega \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla(u_\varepsilon - u)^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot (u_\varepsilon - u) = \int_{B_\varepsilon} \mathrm{T}(u) \cdot \nabla(u_\varepsilon - u)^s \ , \tag{3.13}$$

where we have introduced the notation

$$\mathrm{T}(u) = (1-\gamma)\sigma(u) \ . \tag{3.14}$$

From the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} \int_{B_\varepsilon} \mathrm{T}(u) \cdot \nabla(u_\varepsilon - u)^s &\leq \|\mathrm{T}(u)\|_{L^2(B_\varepsilon)} \|\nabla(u_\varepsilon - u)\|_{L^2(B_\varepsilon)} \\ &\leq C_0\varepsilon \|\nabla(u_\varepsilon - u)\|_{L^2(B_\varepsilon)} \\ &\leq C_1\varepsilon \|u_\varepsilon - u\|_{H^1(\Omega)} \ . \end{aligned} \tag{3.15}$$

By coercivity of the bilinear form on the left-hand side of (3.15) we have

$$c\|u_\varepsilon - u\|^2_{H^1(\Omega)} \leq \int_\Omega \sigma_\varepsilon(u_\varepsilon - u) \cdot \nabla(u_\varepsilon - u)^s + \int_{\partial\omega} \mathcal{S}(u_\varepsilon - u) \cdot (u_\varepsilon - u) \ , \tag{3.16}$$

which leads to the result with $C = C_1/c$ independent of the small parameter ε. □

3.4. **Topological derivative evaluation.** From the decomposition of the hold all domain $\mathcal{D}$ into Ω and ω^*, the singularity produced by the crack tip is absorbed by the auxiliary problem (3.5) defined over the cracked subdomain ω^*. Consequently, the remaining subdomain Ω becomes smooth, which allows us to evaluate the associated topological derivative by using known results from the literature. In fact, since the topological perturbation is nucleated far from the control region ω^*, namely $\overline{B_\varepsilon(\hat{x})} \subset \Omega$, and taking into account the definition of the shape change velocity field V from (2.10), the Rice's integral (2.12) becomes concentrated over the fixed domain ω^*. In this particular case, the final expression for the topological derivative can be adapted from [5]. For the general case associated with singular domain perturbations, which is much more complicated from the mathematical point view, see for instance [20, 26]. See also [4] for the complete topological asymptotic expansion of solutions governed by the elasticity system.

Theorem 4. *The topological derivative of the shape functional* $\{-\Pi(u)\}$*, where* $\Pi(u)$ *is given by* (2.12)*, with respect to the nucleation of a small circular inclusion endowed with contrast* γ*, can be written in terms of the solutions to the direct* (3.7) *and adjoint* (2.15) *problems, namely:*

$$\mathcal{T}(x) = \mathbb{P}_\gamma \sigma(u)(x) \cdot \nabla v^s(x), \quad \forall x \in \Omega \ , \tag{3.17}$$

where the polarization tensor $\mathbb{P}_\gamma$ *is given by a fourth order isotropic tensor as follows*

$$\mathbb{P}_\gamma = -\frac{1-\gamma}{1+\beta\gamma}\left((1+\beta)\mathbb{I} + \frac{1}{2}(\alpha-\beta)\frac{1-\gamma}{1+\alpha\gamma}\mathrm{I}\otimes\mathrm{I}\right) \ , \tag{3.18}$$

with the coefficients α *and* β *defined as*

$$\alpha = \frac{\mu+\lambda}{\mu} \quad \textit{and} \quad \beta = \frac{3\mu+\lambda}{\mu+\lambda} \; . \tag{3.19}$$

Corollary 5. *The following limit cases for the contrast parameter* γ *can be formally obtained from Theorem 4, whose rigorous mathematical justification can be found in* [2], *for instance:*
Case 1. *Contrast parameter going to zero* ($\gamma \to 0$),

$$\mathcal{T}_0(x) = \mathbb{P}_0 \sigma(u)(x) \cdot \nabla v^s(x) \; , \tag{3.20}$$

where the polarization tensor $\mathbb{P}_0$ *is given by*

$$\mathbb{P}_0 = -\frac{4\mu+2\lambda}{\mu+\lambda}\left(\mathbb{I} - \frac{\mu-\lambda}{4\mu}\mathrm{I}\otimes\mathrm{I}\right) . \tag{3.21}$$

Case 2. *Contrast parameter going to infinity* ($\gamma \to \infty$),

$$\mathcal{T}_\infty(x) = \mathbb{P}_\infty \sigma(u)(x) \cdot \nabla v^s(x) \; , \tag{3.22}$$

with the polarization tensor $\mathbb{P}_\infty$ *given by*

$$\mathbb{P}_\infty = \frac{4\mu+2\lambda}{3\mu+\lambda}\left(\mathbb{I} + \frac{\mu-\lambda}{4(\mu+\lambda)}\mathrm{I}\otimes\mathrm{I}\right) . \tag{3.23}$$

4. Numerical experiments

Now, some numerical experiments are presented in order to verify the applicability of the proposed methodology. In each one, the topological derivative will be evaluated to detect the regions where the inclusions should be nucleated taking into account the two limit cases presented in Section 3 through Corollary 5. In addition, the elasticity system is solved by using the Finite Element Method with linear triangular elements only.

4.1. **Cracked bar.**

In this first example, the hold all domain $\mathcal{D}$ consists of a rectangle with dimensions $(120 \times 240)\ mm^2$ which contains an initial crack ℓ with length h and growth direction e located at the left side on the bottom face. It is assumed the plane stress state and a distributed load $\bar{q} = (0, 10^2)\ N/mm^2$ is applied on the top side of the cracked bar. The dashed lines are used to represent symmetry conditions. In addition, the control region ω^* is given by a semicircle centered at the crack tip with radius $r^* = 5\ mm$. See Figure 5. All parameters used in this example are summarized in Table 1.

Table 1. Cracked bar. Parameters.

Parameter	Value
E	$2.1 \times 10^5\ N/mm^2$
ν	0.3
e	$(1,0)$
h	$10\ mm$

The finite element mesh used to solve the elasticity system is obtained after the following procedure. First, an initial grid with 120×240 squares is generated. Then, each resulting square is subdivided into four identical triangular elements. After then,

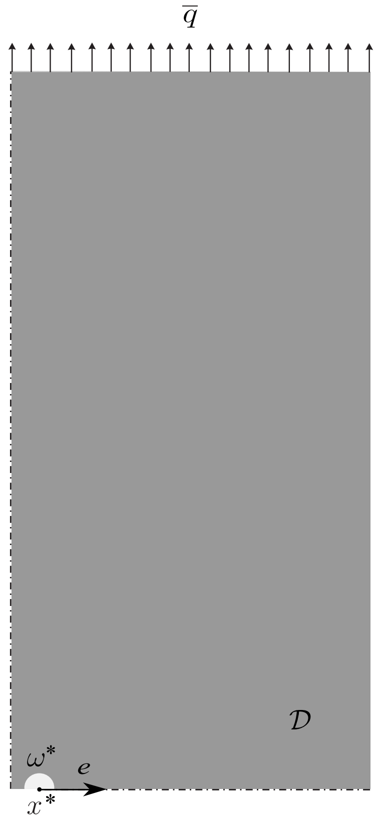


FIGURE 5. Cracked bar. Geometry and boundary conditions.

in order to improve the quality of the solution, a sequence of refinements is applied to the elements into a semicircle $\omega^i \subset \omega^*$ with radius r_i defined as

$$r_i = r^*/i \ , \quad i = 1, 2, ..., n \ , \tag{4.1}$$

where r^* is the radius of the semicircle ω^*. In this process, each triangular element is subdivided into four identical triangular elements again.

The theoretical value of the Rice's integral for this example borrowed from [13] is $\Pi(u) = -1.50616$. The obtained values taking into account the sequence of refinements applied to the elements into the region ω^i with $n = 10$ in (4.1) are presented in Figure 6. These values were normalized according to the theoretical value. Note that for $n = 10$ the result can be considered as quite satisfactory.

The obtained topological derivatives associated with the limit Cases 1 and 2 in Corollary 5, respectively given by (3.20) and (3.22), are presented in Figures 7(a) and 7(b). Therefore, according to the proposed methodology, soft and/or hard inclusions should be nucleated within the region where $\mathcal{T}_0(x)$ and/or $\mathcal{T}_\infty(x)$ are negative, respectively. In order to verify the effects caused by the nucleation of such inclusions, four cases are considered. In the first one, denoted by Case A, no inclusions are nucleated. In Case B, a hard inclusion is nucleated in front of the crack tip at the point (22.5,0) since $\mathcal{T}_\infty(x) < 0$. In Case C, a soft inclusion is nucleated just above the crack tip at the point (10,12.5) since $\mathcal{T}_0(x) < 0$. Finally in Case D, both Cases B and C are combined. In all situations the radius of the inclusion is $r = 5\ mm$. See Figure 8 for details, where white/black circles represent soft/hard inclusions. The obtained results are presented in Table 2, which are also presented in Figure 9 after normalization with respect to the value of $-\Pi(u)$ obtained with $n = 10$ in (4.1).

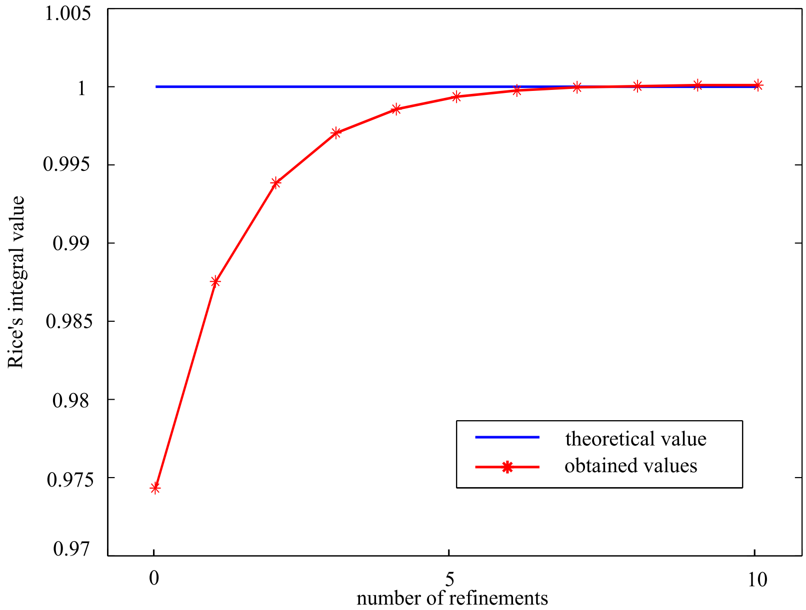
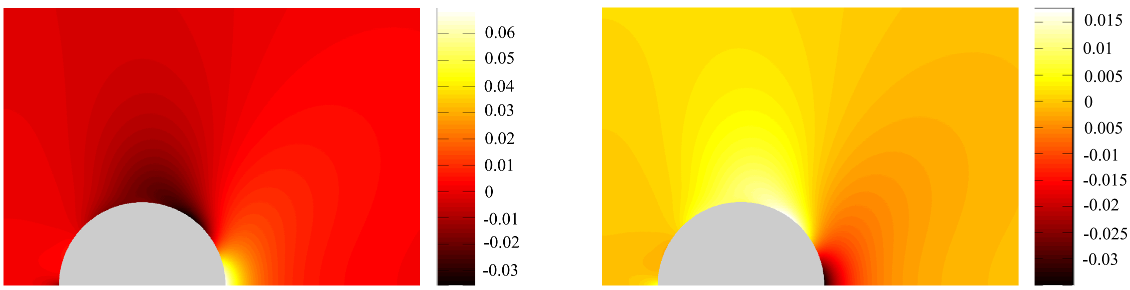


FIGURE 6. Cracked bar. Convergence of the values of Rice's integral.



(a) Case 1: $\mathcal{T}_0(x)$ (b) Case 2: $\mathcal{T}_\infty(x)$

FIGURE 7. Cracked bar. Topological derivatives.

TABLE 2. Cracked bar. Obtained results.

Cases	A	B	C	D
$-\Pi(u)$	1.5063	1.3192	1.2336	1.1516

As expected, the values of the Rice's integral become less negative after introducing the topology changes according to the signal of the topological derivative. In Case D, for example, a gain of approximately 20% is observed.

4.2. **Bittencourt's experiment.** This next example, called Bittencourt's experiments [6], has been proposed by Ingraffea et. al. in 1990 [16]. The geometry of interest is shown in Figure 10. A concentrated force $\bar{q} = -(0, 10^4)\ lbf$ is applied at the middle point of the top face. In addition, the control region ω^* is given by a circle centered at the crack tip with radius $r^* = 0.5\ in$. Finally, it is assumed the plane strain state. The parameters used in this example are shown in Table 3.

The obtained topological derivatives associated with the limit Cases 1 and 2 in Corollary 5, respectively given by (3.20) and (3.22), are presented in Figures 11(a) and 11(b).

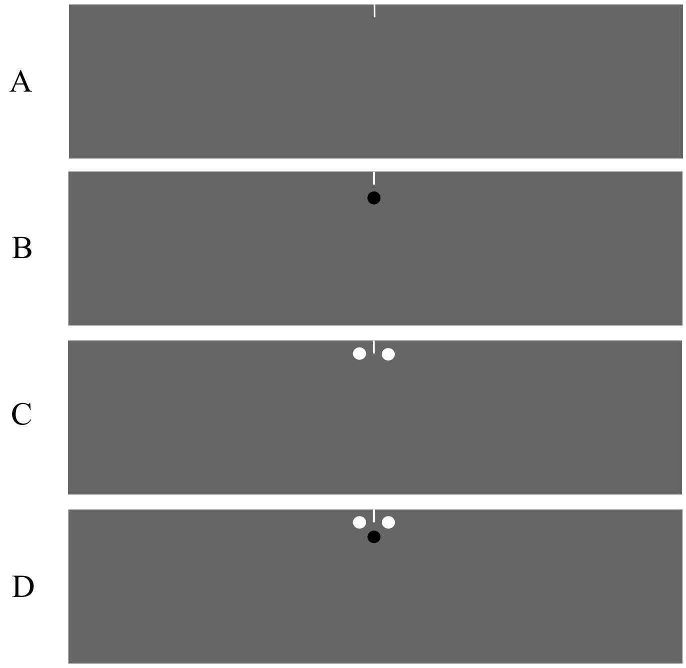


FIGURE 8. Cracked bar. Considered cases.

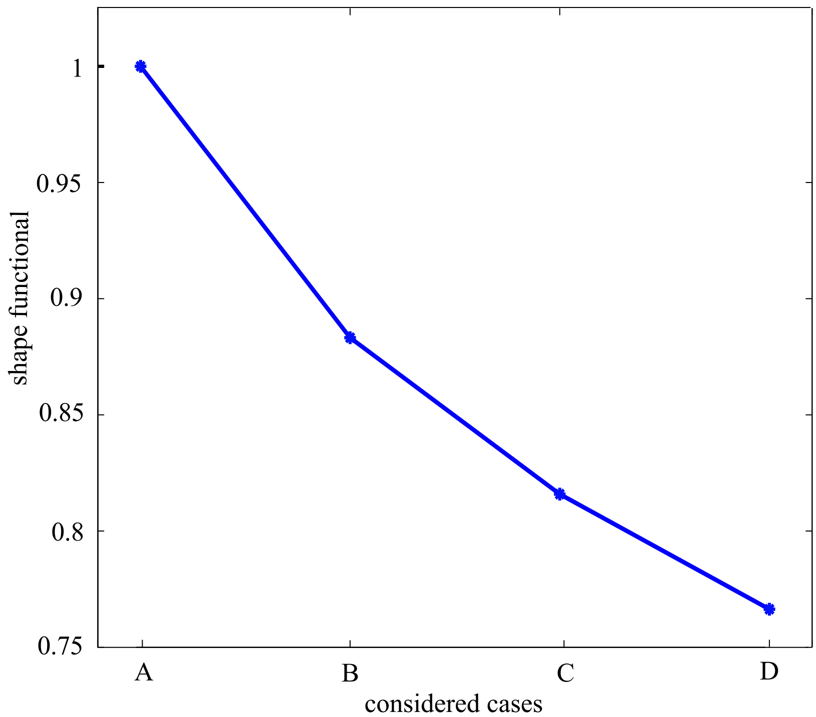


FIGURE 9. Cracked bar. Obtained results.

Note that the final result is quite similar to the previous one: a hard inclusion should be nucleated in front of the crack tip and two soft inclusions at the left and right sides of the crack. However, a small inclination of the negative regions, due to the position of the crack with respect to the applied force $\bar{q}$, should be noted.

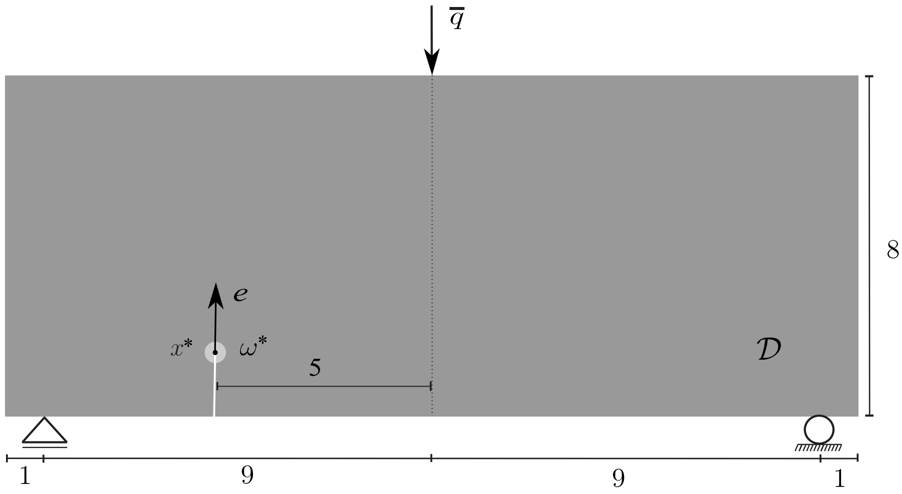
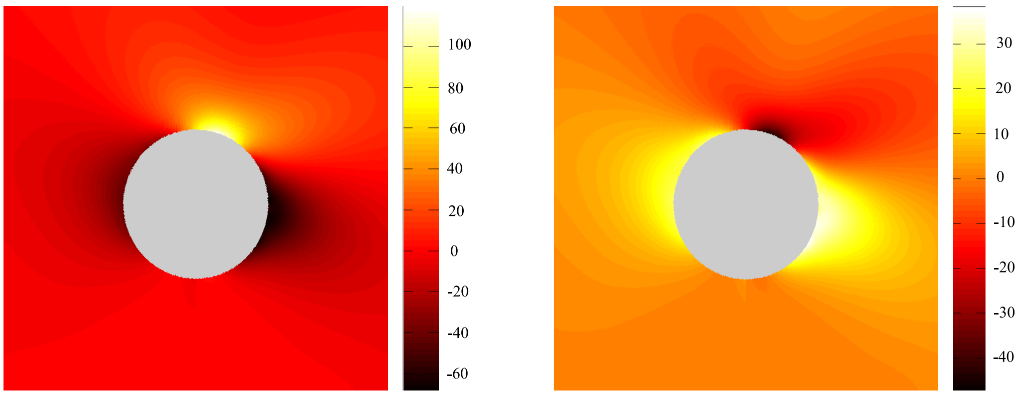


FIGURE 10. Bittencourt's experiment. Geometry and boundary conditions.

TABLE 3. Bittencourt's experiment. Parameters.

Parameter	Value
E	4.5×10^5 psi
ν	0.35
e	$(0, 1)$
h	1.5 in



(a) Case 1: $\mathcal{T}_0(x)$ (b) Case 2: $\mathcal{T}_\infty(x)$

FIGURE 11. Bittencourt's experiment. Topological derivatives.

4.3. **Cracked L-Beam.** In this last example, the hold all domain $\mathcal{D}$ represents a cracked L-beam. The lengths of the horizontal and vertical branches of the L-bracket are 1 m measured along their centre lines. Both have identical width of 0.5 m. The beam is clamped on its top face and submitted to a concentrated load $\bar{q} = -(0, 10^2)$ N as indicated in Figure 12. It is assumed the plane strain state. In addition, the structure contains an initial crack with length h which forms an angle of 45^o with respect to the horizontal plane. Finally, the control region ω^* is given by a circle centered at the crack tip with radius $r^* = 0.025$ m. The used parameters are summarized in Table 4.

The obtained topological derivatives are presented in Figures 13(a) and 13(b). Note

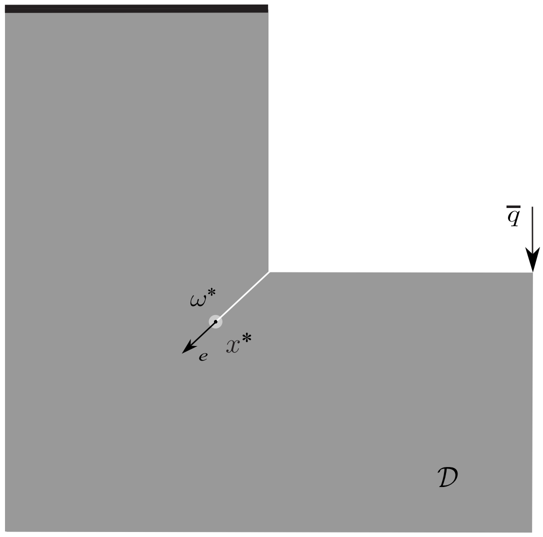
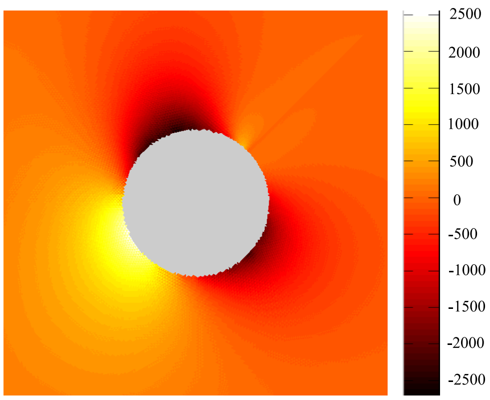


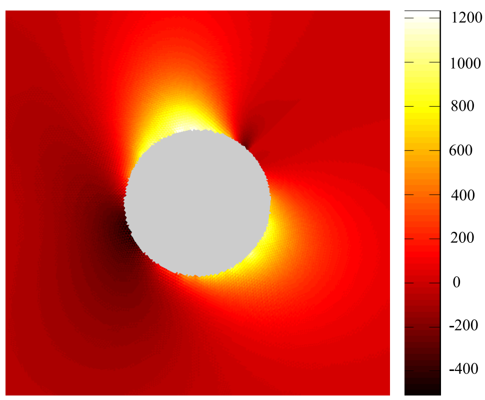
FIGURE 12. Cracked L-beam. Geometry and boundary conditions.

TABLE 4. Cracked L-beam. Parameters.

Parameter	Value
E	$4.5 \times 10^5\ N/m^2$
ν	0.35
e	$(-1, -1)$
h	$0.14\ m$



(a) Case 1: $\mathcal{T}_0(x)$



(b) Case 2: $\mathcal{T}_\infty(x)$

FIGURE 13. Cracked L-Beam. Topological derivatives.

that the results are similar to the previous one, as expected.

5. Conclusions

In this paper, a simple and efficient methodology aiming to extend the remaining useful life of cracked elastic bodies has been proposed. The central idea consists in minimizing a shape functional based on the Rice's integral by nucleating hard and/or soft inclusions far from the crack tip according to the signal of the associated topological derivative. In particular, the sensibility of the shape functional with respect to the nucleation of a small circular inclusion, with different material property from the background, has been obtained. Then, the resulting expression is used to indicate the regions where the inclusions should be nucleated in order to solve the minimization problem. According to Griffith energy criterion, this procedure allows for a decreasing on the values of the shape functional, which increases the remaining useful life of cracked bodies. In fact, numerical experiments confirmed that a gain of 20% in the remaining useful life of the mechanical component can be obtained by following the proposed methodology. It is important to stress that our approach is based on a linear elastic model. Since this class of models do not distinguish between traction and compression stress states, then crack closure phenomenon cannot be captured. Therefore, the extension to the non-linear case by considering inequality type boundary conditions on the crack lips [18] is now under investigation, which shall lead to more realistic results.

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