A NON-ITERATIVE RECONSTRUCTION METHOD FOR AN INVERSE POTENTIAL PROBLEM MODELED BY A MODIFIED HELMHOLTZ EQUATION

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Abstract. This paper deals with an inverse potential problem posed in two dimensional space whose forward problem is governed by a modified Helmholtz equation. The inverse problem consists in the reconstruction of a set of anomalies embedded into a geometrical domain from partial measurements of the associated potential. Since the inverse problem, we are dealing with, is written in the form of an ill-posed boundary value problem, the idea is to rewrite it as a topology optimization problem. In particular, a shape functional is defined to measure the misfit of the solution obtained from the model and the data taken from the partial measurements. This shape functional is minimized with respect to a set of ball-shaped anomalies by using the concept of topological derivatives. It means that the shape functional is expanded asymptotically and then truncated up to the desired order term. The resulting expression is trivially minimized with respect to the parameters under consideration which leads to a non-iterative second order reconstruction algorithm. As a result, the reconstruction process becomes very robust with respect to noisy data and independent of any initial guess. Finally, some numerical experiments are presented to show the effectiveness of the proposed reconstruction algorithm.

1. Introduction

In this paper, we deal with an inverse potential problem in $\mathbb{R}^2$ whose corresponding forward problem is governed by a modified Helmholtz equation. The inverse problem under consideration is about the reconstruction of a set of anomalies embedded into a geometrical domain with the help of partial measurements of the associated potential. This problem is motivated by applications in aerospace industry, geophysics and medical science where scientists try to detect anomalies embedded in a medium by measurements obtained from incident acoustic, elastic and electromagnetic waves [16]. In addition, another motivation for the current investigation comes from the open problem mentioned in [23, pp. 126, Problem 4.2], which has applications in semiconductor theory. According to Kovtunenko and Kunisch [30], from the mathematical point of view, object identification is an inverse problem which belongs to the field of shape and topology optimization, system identification, and parameter estimation. A number of researchers have been studied this kind of inverse problem governed by different partial differential equations. See, for instance, the works related to the Laplace [3, 11, 15, 22], Poisson [9, 10, 17, 24, 31], Schrödinger [5], Helmholtz [12, 13, 18, 29, 30] as well as modified Helmholtz [26] equations.

Iterative or non-iterative approaches can be used for solving inverse problems consisting in the reconstruction of an unknown number of geometric objects from given measurements. Among a variety of methods, we want to draw the attention of the readers on

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Key words and phrases. Potential inverse problem, modified Helmholtz equation, topological derivatives, topology optimization, non-iterative reconstruction method.

the level-set method and the methods based on asymptotic expansions. The level-set method can be seen as a first-order iterative approach where a geometric test object is reconstructed iteratively following a steepest descent direction of an objective function [8, 17, 24, 25]. With regard to the methods based on asymptotic analysis, we focus on those devised from the concept of topological derivatives [38]. According to [36], the use of the first-order topological derivative also leads to first-order iterative methods but in contrast to the level-set methods, they are free of initial guess. The simultaneous use of the first and second-order topological derivatives allows to devise a class of non-iterative methods whose solutions are also independent on the initial guess [7, 19]. It motivates us to solve the problem, we are dealing with, using higher-order topological derivatives.

In particular, let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain with smooth boundary $\partial\Omega$. We consider a subset Ω_o of Ω where measurements of a scalar field of interest are taken. As illustrated in Figure 1(a), there may be an unknown number (denoted by $N^* \in \mathbb{Z}^+$) of isolated anomalies ω_i^* within the domain Ω, i.e., there is a set $\omega^* = \cup_{i=1}^{N^*} \omega_i^*$, with open connected components ω_i^* which satisfy $\overline{\omega_i^*} \cap \overline{\omega_j^*} = \emptyset$ for $i \neq j$ and $\overline{\omega_i^*} \cap \partial\Omega = \emptyset$, $\overline{\omega_i^*} \cap \Omega_o = \emptyset$ for each $i, j \in \{1, \cdots, N^*\}$.

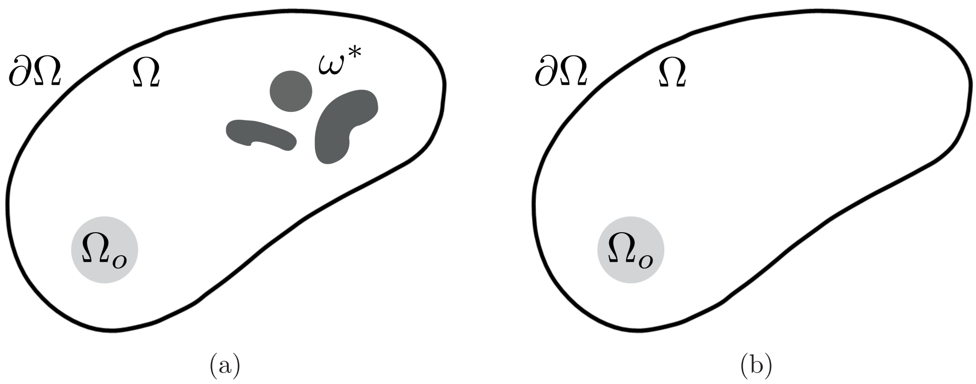


FIGURE 1. (a) Domain Ω with a set of anomalies ω^* and (b) Domain Ω without anomalies.

We consider the domain Ω as a bounded region representing a medium which contains a different substance within a subdomain ω^*. In this set up, the inverse problem consists in finding k_{ω^*} such that the potential z satisfies the following boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta z + k_{\omega^*} z & = & 0 & \text{in } \Omega, \\ z & = & g & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where the given Dirichlet data g is smooth and the parameter k_{ω^*} is defined as

$$k_{\omega^*} = \left\{ \begin{array}{ll} 0 & \text{in } \Omega \setminus \omega^*, \\ k & \text{in } \omega^*, \end{array} \right. \tag{1.2}$$

with $k \in \mathbb{R}^+$.

Now, for an initial guess k_ω of k_{ω^*}, we consider the potential u to be the solution to the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u + k_\omega u & = & 0 & \text{in } \Omega, \\ u & = & g & \text{on } \partial\Omega, \end{array} \right. \tag{1.3}$$

where

$$k_\omega = \begin{cases} 0 & \text{in } \Omega \setminus \omega, \\ k & \text{in } \omega. \end{cases} \tag{1.4}$$

The quantity k_{ω^*} is unknown and hence z but we assume that z can be measured in Ω_o. We would like to find k_{ω^*} with the help of measurements of z taken in Ω_o. If we want to look for an appropriate k_{ω^*}, we wish u to agree with z in Ω_o i.e. we want $u = z_{|\Omega_o}$.

The inverse problem (1.1) does not have a unique solution when we want to determine both, the topology of ω^* and the value k. Let us observe this phenomenon through a simple example. We consider a circular anomaly of radius $\rho < 1$ and material property k centred into a unit disk in $\mathbb{R}^2$. For simplicity, we take $g = 1$ in the problem (1.3). By introducing the polar coordinate system (r, θ), one can observe that the solution does not depend on θ. In fact, its explicit formula in terms of modified Bessel functions of the first kind I_0 and I_1 is given by

$$u(r) = \begin{cases} \lambda_1 I_0(\sqrt{k} r) & \text{if } 0 \le r \le \rho, \\ \lambda_2 \ln r + 1 & \text{if } \rho \le r \le 1. \end{cases} \tag{1.5}$$

The constants λ_1 and λ_2 are such that

$$\lambda_1 = \frac{1}{I_0(\sqrt{k}\rho) - I_1(\sqrt{k}\rho)\sqrt{k}\rho \ln \rho} \quad \text{and} \quad \lambda_2 = \frac{I_1(\sqrt{k}\rho)\sqrt{k}\rho}{I_0(\sqrt{k}\rho) - I_1(\sqrt{k}\rho)\sqrt{k}\rho \ln \rho}. \tag{1.6}$$

Notice that, for two different pairs (ρ_1, k_1) and (ρ_2, k_2), one can produce a unique λ_2 using the second formula in (1.6). For example, if we take $(\rho_1, k_1) = (0.5, 1.0)$ and $(\rho_2, k_2) = (0.25, 4.38)$, we get the unique $\lambda_2 = 0.112$ for both of the cases. In Figure 2, we plot the profiles of both solutions $u_1(r)$ and $u_2(r)$ from (1.5) corresponding to the pairs (ρ_1, k_1) and (ρ_2, k_2), respectively.

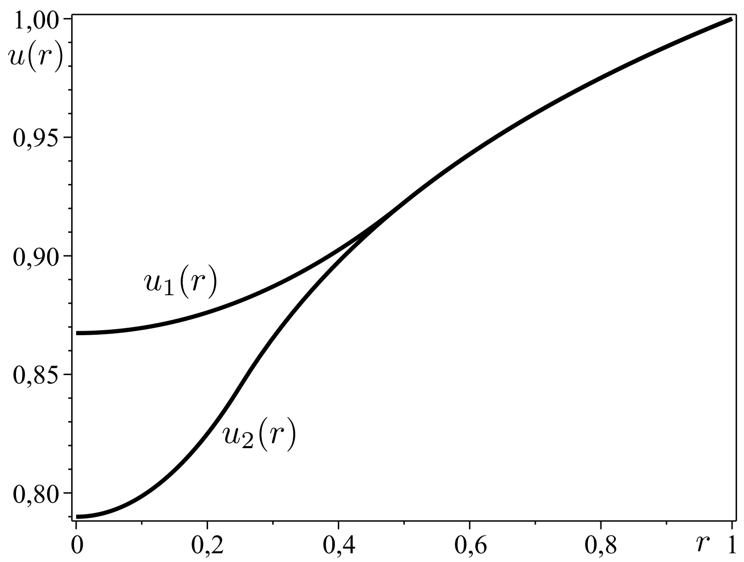


FIGURE 2. Counter-example of lack of uniqueness when both, the topology of ω^* and the material property k, are simultaneously unknown.

From the above example, it is clear that the inverse problem (1.1) cannot be solved uniquely when both, the topology of ω^* and the material property k, are unknown simultaneously and the measurements are taken in Ω_o, away from the hidden anomalies. Hence, in this article, we assume that the material property of the medium k is known and we reconstruct the support of the anomalies ω^* with the help of the measurements of z taken in Ω_o. It is also well known that the inverse problem of finding ω^* in (1.1) for a given k still leads to an ill-posed boundary value problem [23]. Therefore, the idea is to rewrite it as a topology optimization problem. For this purpose, we consider a weaker formulation of the inverse problem (1.1) which consists in solving the topology optimization problem

$$\underset{\omega \subset \Omega}{\text{Minimize}} \;\; \mathcal{J}_\omega\left(u^1, \cdots, u^M\right) = \sum_{m=1}^{M} \int_{\Omega_o} \left(u^m - z^m\right)^2, \tag{1.7}$$

where $M \in \mathbb{Z}^+$ is the number of observations, z^m denotes the measurement of the potential in Ω_o and u^m denotes the solution of the boundary value problem (1.3) corresponding to the Dirichlet data g^m for $m = 1, \cdots, M$. Notice that, the minimizer of the topology optimization problem (1.7) produces the best approximation to ω^*, solution of the inverse problem (1.1), in an appropriate sense. Since we are interested in approximating ω^* by a set of ball-shaped anomalies, the unknown parameters involved in the minimization problem (1.7) are given by the number of anomalies and their corresponding centers and radii.

In particular, problem (1.7) is minimized with respect to a set of ball-shaped anomalies by using the concept of topological derivatives. It means that the shape functional $\mathcal{J}_\omega\left(u^1, \cdots, u^M\right)$ is expanded asymptotically and then truncated up to the desired order term. The resulting expression is trivially minimized with respect to the parameters under consideration which leads to a non-iterative second order reconstruction algorithm. As a result, the reconstruction process becomes very robust with respect to noisy data and independent of any initial guess.

The paper is organized as follows. In Section 2, the mathematical formulation of the inverse potential problem is described as a topological optimization problem. In Section 3, some notations and auxiliary problems are introduced. The topological asymptotic expansion of the shape functional is presented in Section 4, which is the main result of this article. The *a priori* estimates of the remainders, obtained in Section 4, are presented in Section 5. The novel non-iterative reconstruction algorithm is devised in Section 6 and some numerical experiments showing the effectiveness of the proposed reconstruction algorithm are presented.

2. Topology Optimization Setting

The inverse problem (1.1) has been written in the form of a topology optimization problem (1.7). It is well known that a quite general approach for dealing with such class of problems is based on the concept of topological derivative, which consists in expanding the shape functional $\mathcal{J}_\omega\left(u^1, \cdots, u^M\right)$ with respect to the parameters depend upon a set of small inclusions. Since the topological derivative does not depend on the initial guess of the unknown topology ω^*, we start with the unperturbed domain by setting $\omega = \emptyset$, see Figure 1(b). More precisely, we consider

$$\mathcal{J}_0\left(u_0^1,\cdots,u_0^M\right)=\sum_{m=1}^{M}\int_{\Omega_o}\left(u_0^m-z^m\right)^2, \tag{2.1}$$

where u_0^m be the solution of the unperturbed boundary value problem

$$\begin{cases} -\Delta u_0^m &= 0 \quad \text{in } \Omega,\\ u_0^m &= g^m \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

In this article, we are considering the topology optimization problem (1.7) for the ball-shaped anomalies and hence we define the topologically perturbed counter-part of (2.2) by introducing $N\in\mathbb{Z}^+$ number of small circular inclusions $B_{\varepsilon_i}(x_i)$ with center at $x_i\in\Omega$ and radius ε_i for $i=1,\cdots,N$. The set of inclusions can be denoted as

$$B_\varepsilon(\xi)=\bigcup_{i=1}^{N}B_{\varepsilon_i}(x_i), \tag{2.3}$$

where $\xi=(x_1,\ldots,x_N)$ and $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_N)$. Moreover, we assume that $\overline{B_\varepsilon}\cap\partial\Omega=\emptyset$, $\overline{B_\varepsilon}\cap\Omega_o=\emptyset$ and $\overline{B_{\varepsilon_i}(x_i)}\cap\overline{B_{\varepsilon_j}(x_j)}=\emptyset$ for each $i\neq j$ and $i,j\in\{1,\cdots,N\}$. The shape functional associated with the topologically perturbed domain is written as

$$\mathcal{J}_\varepsilon\left(u_\varepsilon^1,\cdots,u_\varepsilon^M\right)=\sum_{m=1}^{M}\int_{\Omega_o}\left(u_\varepsilon^m-z^m\right)^2 \tag{2.4}$$

with u_ε^m be the solution of the perturbed boundary value problem

$$\begin{cases} -\Delta u_\varepsilon^m+k_\varepsilon u_\varepsilon^m &= 0 \quad \text{in } \Omega,\\ u_\varepsilon^m &= g^m \quad \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where the parameter k_ε is defined as

$$k_\varepsilon=\begin{cases} 0 & \text{in } \Omega\setminus B_\varepsilon(\xi),\\ k & \text{in } B_\varepsilon(\xi). \end{cases} \tag{2.6}$$

As mentioned earlier, the topological derivatives measure the sensitivity of the shape functional with respect to the parameters (ε,ξ) depending upon a set of small inclusions $B_\varepsilon(\xi)$. Therefore, in this article, our idea is to obtain the number, radius and location of the inclusions that produce the best approximation to the anomaly ω^* by using the concept of topological derivatives.

3. Notations and Auxiliary Problems

The first term of the asymptotic expansion of a given shape functional with respect to the small parameter which measures the size of singular domain perturbations, such as holes, inclusions, source-terms, cracks, etc., represents the first-order topological derivative. This concept, introduced by Sokołowski & Żochowski [38] and further developed by many authors [14, 20, 34, 37], can be seen as a particular case of the broader class of asymptotic methods fully developed in the books by Ammari et al. [1] and Ammari & Kang [4], for instance. See also related works [27, 32, 33]. For an account on the topological derivative concept the reader may refer to the book by Novotny & Sokołowski [35]. In the context of Helmholtz equation, the stability and resolution analysis for a imaging functional based on the first-order topological derivative has been presented in [2]. The case of high-order topological expansions for Helmholtz problems into two spatial dimensions has been studied in [28]. However, such an analysis is missing for the potential inverse problem we are dealing with. On the other hand, one can define nth order topological derivative. In fact, the second-order topological derivative concept started to play an important role in the resolution of a class of inverse problems. In particular, it has been successfully applied for solving the EIT [7, 19, 22] and gravimetry [9, 10] problems.

In general, an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is perturbed by introducing nonsmooth features confined in a small region $\omega_\varepsilon(\xi)$ of size $\varepsilon > 0$ centred at $\xi \in \Omega$ such that $\overline{\omega_\varepsilon(\xi)} \subset \Omega$. We define a characteristic function having support in the unperturbed domain Ω of the form $\chi = \mathbb{1}_\Omega$. Similarly, we introduce a characteristic function $\chi_\varepsilon(\xi)$ associated to the topologically perturbed domain. For example, in the case of holes as the perturbation $\omega_\varepsilon(\xi)$, we can write $\chi_\varepsilon(\xi) = \mathbb{1}_\Omega - \mathbb{1}_{\overline{\omega_\varepsilon(\xi)}}$ and the singularly perturbed domain can be represented by $\Omega_\varepsilon(\xi) = \Omega \backslash \overline{\omega_\varepsilon(\xi)}$. Further, one assumes that a given shape functional $\psi(\chi_\varepsilon(\xi))$ associated to the topologically perturbed domain $\Omega_\varepsilon(\xi)$ admits the following topological asymptotic expansion

$$\psi(\chi_\varepsilon(\xi)) = \psi(\chi) + f(\varepsilon) D_T\psi(\xi) + o(f(\varepsilon)), \tag{3.1}$$

where $\psi(\chi)$ is the shape functional associated to the reference (unperturbed) domain Ω and $f(\varepsilon)$ is a positive function depending upon the size ε of the topological perturbation such that $f(\varepsilon) \to 0$ when $\varepsilon \downarrow 0$. The function $\xi \mapsto D_T\psi(\xi)$ is called the first order topological derivative of the shape functional ψ at ξ. Mathematically, we can express it as

$$D_T\psi(\xi) := \lim_{\varepsilon \to 0} \frac{\psi(\chi_\varepsilon(\xi)) - \psi(\chi)}{f(\varepsilon)}. \tag{3.2}$$

Similarly, the second order topological derivative of the shape functional ψ at ξ can be obtained by expanding the remainder term $o(f(\varepsilon))$ in (3.1). More precisely, we will get the topological asymptotic expansion

$$\psi(\chi_\varepsilon(\xi)) = \psi(\chi) + f(\varepsilon) D_T\psi(\xi) + f_2(\varepsilon) D_T^2\psi(\xi) + o(f_2(\varepsilon)), \tag{3.3}$$

where $f_2(\varepsilon)$ is such that

$$\lim_{\varepsilon \to 0} \frac{f_2(\varepsilon)}{f(\varepsilon)} = 0. \tag{3.4}$$

Thus, the second order topological derivative can be defined as

$$D_T^2\psi(\xi) := \lim_{\varepsilon \to 0} \frac{\psi(\chi_\varepsilon(\xi)) - \psi(\chi) - f(\varepsilon) D_T\psi(\xi)}{f_2(\varepsilon)}. \tag{3.5}$$

Furthermore, one can define higher order topological derivatives by arguing analogically.

In this article, we are interested in expanding the shape functional $\mathcal{J}_\varepsilon\left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right)$ defined in (2.4) similar to (3.3). Therefore, we start by simplifying the difference between the perturbed shape functional $\mathcal{J}_\varepsilon\left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right)$ and its unperturbed counter-part $\mathcal{J}_0\left(u_0^1, \cdots, u_0^M\right)$ defined in (2.4) and (2.1), respectively, as follows

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = \sum_{m=1}^{M} \int_{\Omega_o} \left[2\left(u_\varepsilon^m - u_0^m\right)\left(u_0^m - z^m\right) + \left(u_\varepsilon^m - u_0^m\right)^2\right], \tag{3.6}$$

where $u_\varepsilon = \left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right)$ and $u_0 = \left(u_0^1, \cdots, u_0^M\right)$.

For $m = 1, \cdots, M$, let us consider the following *ansätz* to justify the asymptotic expansion of u_ε^m with respect to the parameters corresponding to the small circular inclusions as described in Section 2

$$u_\varepsilon^m(x) = u_0^m(x) + k \sum_{i=1}^{N} |B_{\varepsilon_i}(x_i)| h_i^{\varepsilon,m}(x) + k^2 \sum_{i=1}^{N} \sum_{j=1}^{N} |B_{\varepsilon_i}(x_i)| |B_{\varepsilon_j}(x_j)| h_{ij}^{\varepsilon,m}(x) + \tilde{u}_\varepsilon^m(x), \quad (3.7)$$

where $|B_{\varepsilon_i}(x_i)|$ is the Lebesgue measure (volume) of the two-dimensional ball $B_{\varepsilon_i}(x_i)$, i.e., $|B_{\varepsilon_i}(x_i)| = \pi \varepsilon_i^2$. Furthermore, for each $i = 1, \cdots, N$ and $m = 1, \cdots, M$, $h_i^{\varepsilon,m}$ is the solution of

$$\begin{cases} \Delta h_i^{\varepsilon,m} = \dfrac{u_0^m}{|B_{\varepsilon_i}(x_i)|} \chi_{B_{\varepsilon_i}(x_i)} & \text{in } \Omega, \\ h_i^{\varepsilon,m} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.8)$$

We write $h_i^{\varepsilon,m}$ as a sum of three functions p_i^ε, q_i and $\tilde{h}_i^{\varepsilon,m}$. In other words,

$$h_i^{\varepsilon,m} = u_0^m(x_i)(p_i^\varepsilon + q_i) + \tilde{h}_i^{\varepsilon,m}, \quad (3.9)$$

where p_i^ε is a particular solution obtained by the convolution of $|B_{\varepsilon_i}(x_i)|^{-1} \chi_{B_{\varepsilon_i}(x_i)}$ with the kernel of the Laplacian. More precisely,

$$p_i^\varepsilon(x) = \frac{1}{|B_{\varepsilon_i}(x_i)|} \int_{B_{\varepsilon_i}(x_i)} \frac{1}{2\pi} \log \|y - x\| dy. \quad (3.10)$$

Outside the ball $B_{\varepsilon_i}(x_i)$, we can simplify (3.10) to obtain

$$p_i(x) := p_i^\varepsilon(x) = \frac{1}{2\pi} \log \|x_i - x\| \quad \forall x \in \Omega \setminus \overline{B_{\varepsilon_i}(x_i)}. \quad (3.11)$$

Observe that $p_i(x)$ does not depend on ε_i. Additionally, q_i is the solution to the homogeneous boundary value problem

$$\begin{cases} \Delta q_i = 0 & \text{in } \Omega, \\ q_i = -\dfrac{1}{2\pi} \log \|x_i - x\| & \text{on } \partial\Omega \end{cases} \quad (3.12)$$

and $\tilde{h}_i^{\varepsilon,m}$ solves the boundary value problem

$$\begin{cases} \Delta \tilde{h}_i^{\varepsilon,m} = \dfrac{u_0^m - u_0^m(x_i)}{|B_{\varepsilon_i}(x_i)|} \chi_{B_{\varepsilon_i}(x_i)} & \text{in } \Omega, \\ \tilde{h}_i^{\varepsilon,m} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.13)$$

Taking into account the decomposition (3.9), we can introduce the notations

$$h_i := p_i + q_i \quad \text{and} \quad h_i^\varepsilon := p_i^\varepsilon + q_i \quad (3.14)$$

with p_i^ε, p_i as given in (3.10), (3.11), respectively. In (3.7), $h_{ij}^{\varepsilon,m}$ and $\tilde{u}_\varepsilon^m$ are the solutions of the following boundary value problems

$$\begin{cases} \Delta h_{ij}^{\varepsilon,m} = \dfrac{h_j^{\varepsilon,m}}{|B_{\varepsilon_i}(x_i)|} \chi_{B_{\varepsilon_i}(x_i)} & \text{in } \Omega, \\ h_{ij}^{\varepsilon,m} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.15)$$

and

$$\begin{cases} -\Delta \tilde{u}_\varepsilon^m + k_\varepsilon \tilde{u}_\varepsilon^m &= -\Phi_\varepsilon^m \quad \text{in } \Omega, \\ \tilde{u}_\varepsilon^m &= 0 \qquad\ \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

respectively, for each $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$. In problem (3.16), we have

$$\Phi_\varepsilon^m = k^3 \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N |B_{\varepsilon_j}(x_j)| |B_{\varepsilon_l}(x_l)| h_{jl}^{\varepsilon,m} \chi_{B_{\varepsilon_i}(x_i)}. \tag{3.17}$$

To compact the notation, let us denote

$$\alpha = (\alpha_1, \cdots, \alpha_N) \quad \text{with} \quad \alpha_i = |B_{\varepsilon_i}(x_i)|, \tag{3.18}$$

for $i = 1, \cdots, N$. Using (3.18), the expansion (3.7) will have the form

$$u_\varepsilon^m(x) = u_0^m(x) + k \sum_{i=1}^N \alpha_i h_i^{\varepsilon,m}(x) + k^2 \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j h_{ij}^{\varepsilon,m}(x) + \tilde{u}_\varepsilon^m(x). \tag{3.19}$$

In order to simplify further analysis, let us introduce an adjoint state v^m as the solution of the following auxiliary boundary value problem

$$\begin{cases} -\Delta v^m &= (u_0^m - z^m) \chi_{\Omega_o} \quad \text{in } \Omega, \\ v^m &= 0 \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

4. Main Theorem

In this section, we state our main result which describes the topological asymptotic expansion of the perturbed shape functional in terms of the parameters related to N number of ball-shaped inclusions as explained in Section 2.

Theorem 1. *Let q_i, h_i for $i = 1, \cdots, N$ and u_0^m, v^m for $m = 1, \cdots, M$ be the functions defined in* (3.12), (3.14) *and* (2.2), (3.20), *respectively. Then, for the vector α introduced in* (3.18), *we have the following asymptotic expansion for the topologically perturbed shape functional $\psi(\chi_\varepsilon(\xi)) = \mathcal{J}_\varepsilon(u_\varepsilon)$ defined in* (2.4)*:*

$$\psi(\chi_\varepsilon(\xi)) = \psi(\chi) - \alpha \cdot d(\xi) + G(\xi)\alpha \cdot \mathrm{diag}(\alpha \otimes \log \alpha) + \frac{1}{2} H(\xi)\alpha \cdot \alpha + o(|\alpha|^2), \tag{4.1}$$

where $\psi(\chi) = \mathcal{J}_0(u_0)$ is the topologically unperturbed shape functional from (2.1). *Moreover, the vector $d \in \mathbb{R}^N$, the matrix $G \in \mathbb{R}^N \times \mathbb{R}^N$ and the Hessian matrix $H \in \mathbb{R}^N \times \mathbb{R}^N$ in the above expansion are defined as*

$$d_i := 2k \sum_{m=1}^M u_0^m(x_i) v^m(x_i), \tag{4.2}$$

$$G_{ii} := -\frac{k^2}{2\pi} \sum_{m=1}^M u_0^m(x_i) v^m(x_i), \quad G_{ij} = 0, \quad \textit{if} \quad i \neq j \tag{4.3}$$

and

$$H_{ii} := \frac{1+\log\pi^2}{2\pi} k^2 \sum_{m=1}^{M} u_0^m(x_i) v^m(x_i) - 4k^2 \sum_{m=1}^{M} u_0^m(x_i) v^m(x_i) q_i(x_i) \\ - \frac{k}{\pi} \sum_{m=1}^{M} \nabla u_0^m(x_i) \cdot \nabla v^m(x_i) + 2k^2 \sum_{m=1}^{M} (u_0^m(x_i))^2 \int_{\Omega_o} h_i^2, \quad (4.4)$$

$$H_{ij} := -4k^2 \sum_{m=1}^{M} u_0^m(x_j) v^m(x_i) h_j(x_i) + 2k^2 \sum_{m=1}^{M} u_0^m(x_i) u_0^m(x_j) \int_{\Omega_o} h_i h_j, \quad \textit{if} \quad i \neq j, \quad (4.5)$$

respectively, for $i, j = 1, \cdots, N$.

5. Proof of the Main Result

The proof of Theorem 1 is demonstrated in three steps. Firstly, we develop the asymptotic expansion of the topologically perturbed shape functional. Next, we prove *a priori* estimates related to the auxiliary states $\tilde{h}_i^{\varepsilon,m}$, $h_i^{\varepsilon,m}$, $h_{ij}^{\varepsilon,m}$ and $\tilde{u}_\varepsilon^m$ for $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$. Finally, in the last part of this section, the previously obtained results are used to estimate the remainders appeared in the first step. These estimates justify our topological asymptotic expansion (4.1).

5.1. **Asymptotic development of the shape functional.** Let us use (3.19) in (3.6), to obtain

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = 2k \sum_{m=1}^{M} \sum_{i=1}^{N} \alpha_i \int_{\Omega_o} h_i^{\varepsilon,m} (u_0^m - z^m) \\ + 2k^2 \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \int_{\Omega_o} h_{ij}^{\varepsilon,m} (u_0^m - z^m) \\ + k^2 \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \int_{\Omega_o} h_i^{\varepsilon,m} h_j^{\varepsilon,m} + \sum_{m=1}^{M} \sum_{\ell=1}^{6} \mathcal{E}_\ell^m(\varepsilon), \quad (5.1)$$

where

$$\mathcal{E}_1^m(\varepsilon) = 2 \int_{\Omega_o} \tilde{u}_\varepsilon^m (u_0^m - z^m), \quad (5.2)$$

$$\mathcal{E}_2^m(\varepsilon) = 2k^3 \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} \alpha_i \alpha_j \alpha_l \int_{\Omega_o} h_{ij}^{\varepsilon,m} h_l^{\varepsilon,m}, \quad (5.3)$$

$$\mathcal{E}_3^m(\varepsilon) = 2k \sum_{i=1}^{N} \alpha_i \int_{\Omega_o} \tilde{u}_\varepsilon^m h_i^{\varepsilon,m}, \quad (5.4)$$

$$\mathcal{E}_4^m(\varepsilon) = k^4 \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} \sum_{p=1}^{N} \alpha_i \alpha_j \alpha_l \alpha_p \int_{\Omega_o} h_{ij}^{\varepsilon,m} h_{lp}^{\varepsilon,m}, \quad (5.5)$$

$$\mathcal{E}_5^m(\varepsilon) = 2k^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \int_{\Omega_o} \tilde{u}_\varepsilon^m h_{ij}^{\varepsilon,m} \quad (5.6)$$

and

$$\mathcal{E}_6^m(\varepsilon) = \int_{\Omega_o} (\tilde{u}_\varepsilon^m)^2 . \tag{5.7}$$

Now, let us introduce the weak formulation of the adjoint problem (3.20) to find $v^m \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla v^m \cdot \nabla \eta = \int_{\Omega_o} (u_0^m - z^m)\, \eta, \quad \forall \eta \in H_0^1(\Omega) . \tag{5.8}$$

The weak formulations of (3.8) and (3.15) are to find $h_i^{\varepsilon,m} \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla h_i^{\varepsilon,m} \cdot \nabla \eta = -\frac{1}{|B_{\varepsilon_i}(x_i)|} \int_{B_{\varepsilon_i}(x_i)} u_0^m \eta, \quad \forall \eta \in H_0^1(\Omega) \tag{5.9}$$

and $h_{ij}^{\varepsilon,m} \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla h_{ij}^{\varepsilon,m} \cdot \nabla \eta = -\frac{1}{|B_{\varepsilon_i}(x_i)|} \int_{B_{\varepsilon_i}(x_i)} h_j^{\varepsilon,m} \eta, \quad \forall \eta \in H_0^1(\Omega) , \tag{5.10}$$

respectively. By taking $\eta = h_i^{\varepsilon,m}$ in (5.8) and $\eta = v^m$ in (5.9) as test functions, we get

$$\int_{\Omega_o} h_i^{\varepsilon,m} (u_0^m - z^m) = -\frac{1}{|B_{\varepsilon_i}(x_i)|} \int_{B_{\varepsilon_i}(x_i)} u_0^m v^m . \tag{5.11}$$

Similarly, if we take $\eta = h_{ij}^{\varepsilon,m}$ in (5.8) and $\eta = v^m$ in (5.10) as test functions, it gives

$$\int_{\Omega_o} h_{ij}^{\varepsilon,m} (u_0^m - z^m) = -\frac{1}{|B_{\varepsilon_i}(x_i)|} \int_{B_{\varepsilon_i}(x_i)} h_j^{\varepsilon,m} v^m . \tag{5.12}$$

By using (5.11) and (5.12) in (5.1), we get

$$\begin{aligned} \mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = &-2k \sum_{m=1}^M \sum_{i=1}^N \int_{B_{\varepsilon_i}(x_i)} u_0^m v^m - 2k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{j=1}^N \alpha_j \int_{B_{\varepsilon_i}(x_i)} h_j^{\varepsilon,m} v^m \\ &+ k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \int_{\Omega_o} h_i^{\varepsilon,m} h_j^{\varepsilon,m} + \sum_{m=1}^M \sum_{\ell=1}^6 \mathcal{E}_\ell^m(\varepsilon) . \end{aligned} \tag{5.13}$$

Taking into account the notations (3.14), we get

$$\begin{aligned} \mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = &-2k \sum_{m=1}^M \sum_{i=1}^N \int_{B_{\varepsilon_i}(x_i)} u_0^m v^m \\ &- 2k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{\substack{j=1 \\ j\neq i}}^N \alpha_j u_0^m(x_j) \int_{B_{\varepsilon_i}(x_i)} h_j v^m - 2k^2 \sum_{m=1}^M \sum_{i=1}^N \alpha_i u_0^m(x_i) \int_{B_{\varepsilon_i}(x_i)} h_i^\varepsilon v^m \\ &+ k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j u_0^m(x_i)\, u_0^m(x_j) \int_{\Omega_o} h_i h_j + \sum_{m=1}^M \sum_{\ell=1}^9 \mathcal{E}_\ell^m(\varepsilon) . \end{aligned} \tag{5.14}$$

Here, the three new remainders are defined as

$$\mathcal{E}_7^m(\varepsilon) = -2k^2 \sum_{i=1}^N \sum_{\substack{j=1 \\ j\neq i}}^N \alpha_j \int_{B_{\varepsilon_i}(x_i)} \tilde{h}_j^{\varepsilon,m} v^m , \tag{5.15}$$

$$\mathcal{E}_8^m(\varepsilon) = -2k^2 \sum_{i=1}^{N} \alpha_i \int_{B_{\varepsilon_i}(x_i)} \tilde{h}_i^{\varepsilon,m} v^m, \tag{5.16}$$

$$\mathcal{E}_9^m(\varepsilon) = k^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \int_{\Omega_o} \left(u_0^m(x_i) h_i \tilde{h}_j^{\varepsilon,m} + u_0^m(x_j) h_j \tilde{h}_i^{\varepsilon,m} + \tilde{h}_i^{\varepsilon,m} \tilde{h}_j^{\varepsilon,m} \right). \tag{5.17}$$

We can simplify (5.14) further by noting the following:

(i) In the first and the second terms of (5.14), we can consider the Taylor's expansions of the functions u_0^m, v^m and h_j around the point x_i, namely,

$$u_0^m(x) = u_0^m(x_i) + \nabla u_0^m(x_i) \cdot (x - x_i) + \frac{1}{2} \nabla^2 u_0^m(x_i)(x - x_i) \cdot (x - x_i) + D^3 u_0^m(\widehat{x})(x - x_i)^3, \tag{5.18}$$

$$v^m(x) = v^m(x_i) + \nabla v^m(x_i) \cdot (x - x_i) + \frac{1}{2} \nabla^2 v^m(x_i)(x - x_i) \cdot (x - x_i) + D^3 v^m(\widehat{x})(x - x_i)^3 \tag{5.19}$$

and

$$h_j(x) = h_j(x_i) + \nabla h_j(x_i) \cdot (x - x_i) + D^2 h_j(\widehat{x})(x - x_i)^2, \tag{5.20}$$

where $\widehat{x}$ is an intermediate point between x and x_i. Moreover, $D^n f(\widehat{x})(x - x_i)^n$, $n \geq 1$, $n \in \mathbb{N}$, denotes the last nth term of the Taylor's expansion of a function $f(x)$ around x_i.

(ii) In the third term of (5.14), we can use the explicit form of the analytical part p_i^ε of h_i^ε inside the ball $B_{\varepsilon_i}(x_i)$.

Finally, after taking into account the above mentioned observations along with the decomposition (3.9) with the fact that u_0^m and v^m are harmonic outside Ω_o, (5.14) takes the form

$$\begin{aligned}
\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = {} & -2k \sum_{m=1}^{M} \sum_{i=1}^{N} \alpha_i u_0^m(x_i) v^m(x_i) - \frac{k^2}{2\pi} \sum_{m=1}^{M} \sum_{i=1}^{N} \alpha_i^2 \log \alpha_i \, u_0^m(x_i) v^m(x_i) \\
& + \frac{1 + \log \pi^2}{4\pi} k^2 \sum_{m=1}^{M} \sum_{i=1}^{N} \alpha_i^2 u_0^m(x_i) v^m(x_i) - 2k^2 \sum_{m=1}^{M} \sum_{i=1}^{N} \alpha_i^2 u_0^m(x_i) v^m(x_i) q_i(x_i) \\
& - \frac{k}{2\pi} \sum_{m=1}^{M} \sum_{i=1}^{N} \alpha_i^2 \nabla u_0^m(x_i) \cdot \nabla v^m(x_i) - 2k^2 \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \alpha_i \alpha_j u_0^m(x_j) v^m(x_i) h_j(x_i) \\
& + k^2 \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j u_0^m(x_i) u_0^m(x_j) \int_{\Omega_o} h_i(x) h_j(x) + \sum_{m=1}^{M} \sum_{\ell=1}^{19} \mathcal{E}_\ell^m(\varepsilon).
\end{aligned} \tag{5.21}$$

Now we have new remainders, namely,

$$\mathcal{E}_{10}^m(\varepsilon) = -2k \sum_{i=1}^{N} \int_{B_{\varepsilon_i}(x_i)} \left[\nabla u_0^m(x_i) \cdot (x - x_i) \right] \left[D^3 v^m(\widehat{x})(x - x_i)^3 \right], \tag{5.22}$$

$$\mathcal{E}_{11}^{m}(\varepsilon)=-2k\sum_{i=1}^{N}\int_{B_{\varepsilon_i}(x_i)}\left[D^2u_0^m(x_i)(x-x_i)^2\right]\left[D^2v^m(x_i)(x-x_i)^2\right], \tag{5.23}$$

$$\mathcal{E}_{12}^{m}(\varepsilon)=-2k\sum_{i=1}^{N}\int_{B_{\varepsilon_i}(x_i)}\left[\nabla v^m(x_i)\cdot(x-x_i)\right]\left[D^3u_0^m(\widehat{x})(x-x_i)^3\right], \tag{5.24}$$

$$\mathcal{E}_{13}^{m}(\varepsilon)=-2k\sum_{i=1}^{N}\int_{B_{\varepsilon_i}(x_i)}\left[D^3u_0^m(\widehat{x})(x-x_i)^3\right]\left[D^3v^m(\widehat{x})(x-x_i)^3\right], \tag{5.25}$$

$$\mathcal{E}_{14}^{m}(\varepsilon)=-2k^2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\alpha_j u_0^m(x_j)\,v^m(x_i)\int_{B_{\varepsilon_i}(x_i)}D^2h_j(\widehat{x})(x-x_i)^2, \tag{5.26}$$

$$\mathcal{E}_{15}^{m}(\varepsilon)=-2k^2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\alpha_j u_0^m(x_j)\,h_j(x_i)\int_{B_{\varepsilon_i}(x_i)}D^2v^m(\widehat{x})(x-x_i)^2, \tag{5.27}$$

$$\mathcal{E}_{16}^{m}(\varepsilon)=-2k^2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\alpha_j u_0^m(x_j)\int_{B_{\varepsilon_i}(x_i)}\left[\nabla v^m(x_i)\cdot(x-x_i)\right]\left[\nabla h_j(x_i)\cdot(x-x_i)\right], \tag{5.28}$$

$$\mathcal{E}_{17}^{m}(\varepsilon)=-2k^2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\alpha_j u_0^m(x_j)\int_{B_{\varepsilon_i}(x_i)}\left[D^2v^m(\widehat{x})(x-x_i)^2\right]\left[D^2h_j(\widehat{x})(x-x_i)^2\right], \tag{5.29}$$

$$\mathcal{E}_{18}^{m}(\varepsilon)=-2k^2\sum_{m=1}^{M}\sum_{i=1}^{N}\alpha_i u_0^m(x_i)\int_{B_{\varepsilon_i}(x_i)}(q_i v^m-q_i(x_i)\,v^m(x_i)), \tag{5.30}$$

$$\mathcal{E}_{19}^{m}(\varepsilon)=-2k^2\sum_{m=1}^{M}\sum_{i=1}^{N}\alpha_i u_0^m(x_i)\int_{B_{\varepsilon_i}(x_i)}p_i^{\varepsilon}(v^m-v^m(x_i)). \tag{5.31}$$

5.2. **Preliminary lemmas.** In order to simplify the presentation, we denote all the constants independent of ε, i and m as C for $i=1,\cdots,N$ and $m=1,\cdots,M$, whose value changes according to the place it is used.

Lemma 2. *For $i=1,\cdots,N$ and $m=1,\cdots,M$, let $\tilde{h}_i^{\varepsilon,m}$ be the weak solution of the variational problem to find $\tilde{h}_i^{\varepsilon,m}\in H_0^1(\Omega)$ such that*

$$\int_{\Omega}\nabla\tilde{h}_i^{\varepsilon,m}\cdot\nabla\eta=-\frac{1}{|B_{\varepsilon_i}(x_i)|}\int_{B_{\varepsilon_i}(x_i)}(u_0^m-u_0^m(x_i))\eta,\quad\forall\eta\in H_0^1(\Omega). \tag{5.32}$$

Then, there exists a positive constant C independent of ε such that

$$\|\tilde{h}_i^{\varepsilon,m}\|_{H^1(\Omega)}\leq C\varepsilon_i^{\delta_i},\quad\forall i=1,\cdots,N\quad\text{and}\quad m=1,\cdots,M, \tag{5.33}$$

for any $0<\delta_i<1$.

Proof. By taking $\eta = \tilde{h}_i^{\varepsilon,m}$ as a test function in (5.32), we have

$$\int_\Omega |\nabla \tilde{h}_i^{\varepsilon,m}|^2 = -\frac{1}{|B_{\varepsilon_i}(x_i)|}\int_{B_{\varepsilon_i}(x_i)} (u_0^m - u_0^m(x_i))\tilde{h}_i^{\varepsilon,m}. \tag{5.34}$$

From the Cauchy-Schwarz inequality and the interior elliptic regularity of the function u_0^m, there exists a positive constant C independent of ε, i and m such that

$$\begin{aligned}\int_\Omega |\nabla \tilde{h}_i^{\varepsilon,m}|^2 &\leq C\varepsilon_i^{-2}\|u_0^m - u_0^m(x_i)\|_{L^2(B_{\varepsilon_i})}\|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} \\ &\leq C\varepsilon_i^{-2}\|x - x_i\|_{L^2(B_{\varepsilon_i})}\|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} \leq C\|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})}.\end{aligned} \tag{5.35}$$

Notice that, Hölder inequality and the Sobolev embedding theorem can be used to derive

$$\|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} \leq C\varepsilon_i^{1/q}\|\tilde{h}_i^{\varepsilon,m}\|_{L^{2p}(B_{\varepsilon_i})} \leq C\varepsilon_i^{\delta_i}\|\tilde{h}_i^{\varepsilon,m}\|_{H^1(\Omega)}, \tag{5.36}$$

for any $1 < q < \infty$ with $1/p + 1/q = 1$. Let us denote $\delta_i = 1/q$ which implies $0 < \delta_i < 1$. Using (5.36) in (5.35), we get

$$\int_\Omega |\nabla \tilde{h}_i^{\varepsilon,m}|^2 \leq C\varepsilon_i^{\delta_i}\|\tilde{h}_i^{\varepsilon,m}\|_{H^1(\Omega)}. \tag{5.37}$$

From Poincaré inequality, we have

$$C\|\tilde{h}_i^{\varepsilon,m}\|_{H^1(\Omega)}^2 \leq \int_\Omega |\nabla \tilde{h}_i^{\varepsilon,m}|^2. \tag{5.38}$$

Combining (5.34)-(5.38), we get the required estimate (5.33). □

Corollary 3. *For $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$, let $\tilde{h}_i^{\varepsilon,m}$ be the weak solution of* (5.32)*. Then, there exists a positive constant C independent of ε such that*

$$\|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_j})} \leq C\varepsilon_i^{\delta_i}\varepsilon_j^{\delta_j}, \quad \forall i, j = 1, \cdots, N \quad \text{and} \quad m = 1, \cdots, M, \tag{5.39}$$

for any $0 < \delta_i, \delta_j < 1$.

Proof. Similar to (5.36), from Hölder inequality and the Sobolev embedding theorem, we get

$$\|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_j})} \leq C\varepsilon_j^{\delta_j}\|\tilde{h}_i^{\varepsilon,m}\|_{H^1(\Omega)} \leq C\varepsilon_j^{\delta_j}\varepsilon_i^{\delta_i}, \tag{5.40}$$

for any $0 < \delta_i, \delta_j < 1$. We obtain the last inequality by using Lemma 2. Hence the fact. □

Lemma 4. *For $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$, let $h_i^{\varepsilon,m}$ be written as* (3.9)*. Then, there exists a positive constant C independent of ε such that*

$$\|h_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} \leq C(\varepsilon_i|\log\varepsilon_i| + \varepsilon_i^{2\delta_i}), \tag{5.41}$$

$$\|h_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_j})} \leq C(\varepsilon_j + \varepsilon_i^{\delta_i}\varepsilon_j^{\delta_j}), \quad \text{if} \quad i \neq j, \tag{5.42}$$

for any $0 < \delta_i, \delta_j < 1$ with $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$.

Proof. From the decomposition (3.9) and the triangular inequality, we have

$$\|h_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_j})} \leq C\left(\|p_i^\varepsilon\|_{L^2(B_{\varepsilon_j})} + \|q_i\|_{L^2(B_{\varepsilon_j})} + \|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_j})}\right). \tag{5.43}$$

The explicit form of the analytical part p_i^ε in (3.10) and (3.11) together with the interior elliptic regularity of function q_i and Corollary 3 allow us to derive

$$\begin{aligned} \|h_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} &\leq C(\varepsilon_i|\log\varepsilon_i| + \varepsilon_i + \varepsilon_i^{2\delta_i}), & (5.44)\\ \|h_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_j})} &\leq C(\varepsilon_j + \varepsilon_i^{\delta_i}\varepsilon_j^{\delta_j}), \quad \text{if} \quad i \neq j, & (5.45)\end{aligned}$$

which leads to the required estimates (5.41) and (5.42), respectively, for any $0 < \delta_i, \delta_j < 1$ with $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$. □

Lemma 5. *For $i = 1, \cdots, N$ and $m = 1, \cdots, M$, let $h_i^{\varepsilon,m}$ be the weak solution of the variational problem* (5.9)*. Then, there exists a positive constant C independent of ε such that*

$$\|h_i^{\varepsilon,m}\|_{H^1(\Omega)} \leq C(\sqrt{|\log\varepsilon_i|} + \varepsilon_i^{\delta_i - 1/2}), \tag{5.46}$$

for any $0 < \delta_i < 1$ with $i = 1, \cdots, N$ and $m = 1, \cdots, M$.

Proof. By taking $\eta = h_i^{\varepsilon,m}$ as a test function in (5.9), we have

$$\int_\Omega |\nabla h_i^{\varepsilon,m}|^2 = -\frac{1}{|B_{\varepsilon_i}(x_i)|}\int_{B_{\varepsilon_i}(x_i)} u_0^m h_i^{\varepsilon,m}. \tag{5.47}$$

From the Cauchy-Schwarz inequality together with the interior elliptic regularity of the function u_0^m and Lemma 4, we obtain

$$\begin{aligned} \int_\Omega |\nabla h_i^{\varepsilon,m}|^2 &\leq C\varepsilon_i^{-2}\|u_0^m\|_{L^2(B_{\varepsilon_i})}\|h_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} \\ &\leq C\varepsilon_i^{-1}\|h_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} \leq C(|\log\varepsilon_i| + \varepsilon_i^{2\delta_i-1}). & (5.48)\end{aligned}$$

From the Poincaré inequality, we have

$$C\|h_i^{\varepsilon,m}\|_{H^1(\Omega)}^2 \leq \int_\Omega |\nabla h_i^{\varepsilon,m}|^2. \tag{5.49}$$

Combining (5.47)-(5.49), we get the required estimate (5.46). □

Lemma 6. *For $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$, let $h_{ij}^{\varepsilon,m}$ be the weak solution of the variational problem* (5.10)*. Then, there exists a positive constant C independent of ε such that*

$$\begin{aligned} \|h_{ii}^{\varepsilon,m}\|_{H^1(\Omega)} &\leq C\varepsilon_i^{\delta_i-1}(|\log\varepsilon_i| + \varepsilon_i^{2\delta_i-1}), & (5.50)\\ \|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)} &\leq C\varepsilon_i^{\delta_i-1}(1 + \varepsilon_i^{\delta_i-1}\varepsilon_j^{\delta_j}), \quad \textit{if} \quad i \neq j, & (5.51)\end{aligned}$$

for any $0 < \delta_i, \delta_j < 1$ with $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$.

Proof. By taking $\eta = h_{ij}^{\varepsilon,m}$ as test a function in (5.10), we have

$$\int_\Omega |\nabla h_{ij}^{\varepsilon,m}|^2 = -\frac{1}{|B_{\varepsilon_i}(x_i)|}\int_{B_{\varepsilon_i}(x_i)} h_j^{\varepsilon,m} h_{ij}^{\varepsilon,m}. \tag{5.52}$$

From the Cauchy-Schwarz inequality, we obtain

$$\int_\Omega |\nabla h_{ij}^{\varepsilon,m}|^2 \leq C\varepsilon_i^{-2}\|h_j^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})}\|h_{ij}^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})}. \tag{5.53}$$

By using Lemma 4, we can write

$$\int_\Omega |\nabla h_{ii}^{\varepsilon,m}|^2 \leq C\varepsilon_i^{-1}(|\log \varepsilon_i| + \varepsilon_i^{2\delta_i-1})\, \|h_{ii}^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})}, \tag{5.54}$$

$$\int_\Omega |\nabla h_{ij}^{\varepsilon,m}|^2 \leq C\varepsilon_i^{-1}(1 + \varepsilon_i^{\delta_i-1}\varepsilon_j^{\delta_j})\, \|h_{ij}^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})}, \quad \text{if} \quad i \neq j. \tag{5.55}$$

Notice that, Hölder inequality and the Sobolev embedding theorem can be used to derive

$$\|h_{ij}^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} \leq C\varepsilon_i^{1/q}\|h_{ij}^{\varepsilon,m}\|_{L^{2p}(B_{\varepsilon_i})} \leq C\varepsilon_i^{\delta_i}\|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)}, \tag{5.56}$$

for any $1 < q < \infty$ with $1/p + 1/q = 1$. Like earlier, let us denote $\delta_i = 1/q$ which implies $0 < \delta_i < 1$. Using (5.56) into (5.54) and (5.55), we get

$$\int_\Omega |\nabla h_{ii}^{\varepsilon,m}|^2 \leq C\varepsilon_i^{\delta_i-1}(|\log \varepsilon_i| + \varepsilon_i^{2\delta_i-1})\, \|h_{ii}^{\varepsilon,m}\|_{H^1(\Omega)}, \tag{5.57}$$

$$\int_\Omega |\nabla h_{ij}^{\varepsilon,m}|^2 \leq C\varepsilon_i^{\delta_i-1}(1 + \varepsilon_i^{\delta_i-1}\varepsilon_j^{\delta_j})\, \|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)}, \quad \text{if} \quad i \neq j. \tag{5.58}$$

From Poincaré inequality, we have

$$C\|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)}^2 \leq \int_\Omega |\nabla h_{ij}^{\varepsilon,m}|^2. \tag{5.59}$$

Combining (5.57)-(5.59), we get the required estimates (5.50) and (5.51). □

Lemma 7. *For $m = 1, \cdots, M$, let $\tilde{u}_\varepsilon^m$ be the weak solution of the variational problem to find $\tilde{u}_\varepsilon^m \in H_0^1(\Omega)$ such that*

$$\int_\Omega \nabla \tilde{u}_\varepsilon^m \cdot \nabla \eta + \int_\Omega k_\varepsilon \tilde{u}_\varepsilon^m \eta = -\int_\Omega \Phi_\varepsilon^m \eta, \quad \forall \eta \in H_0^1(\Omega), \tag{5.60}$$

where Φ_ε^m is given by (3.17)*. Then, there exists a positive constant C independent of ε such that*

$$\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)} \leq C \sum_{i,j,l=1}^{N} \varepsilon_l^{2\delta_l}\varepsilon_i^{\delta_i+1}(\varepsilon_i^2|\log \varepsilon_i| + \varepsilon_i^{2\delta_i+1} + \varepsilon_j^2 + \varepsilon_i^{\delta_i-1}\varepsilon_j^{\delta_j+2}), \tag{5.61}$$

for any $0 < \delta_i, \delta_j, \delta_l < 1$ with $i, j, l = 1, \cdots, N$ and $m = 1, \cdots, M$.

Proof. By taking $\eta = \tilde{u}_\varepsilon^m$ as a test function in (5.60), we have

$$\int_\Omega |\nabla \tilde{u}_\varepsilon^m|^2 + \int_\Omega k_\varepsilon |\tilde{u}_\varepsilon^m|^2 = -\int_\Omega \Phi_\varepsilon^m \tilde{u}_\varepsilon^m. \tag{5.62}$$

From the Cauchy-Schwarz inequality, we obtain

$$\int_\Omega |\nabla \tilde{u}_\varepsilon^m|^2 + \int_\Omega k_\varepsilon |\tilde{u}_\varepsilon^m|^2 \leq C \sum_{l=1}^{N} \|\tilde{u}_\varepsilon^m\|_{L^2(B_{\varepsilon_l})} \sum_{i=1}^{N}\sum_{j=1}^{N} \varepsilon_i^2\varepsilon_j^2 \|h_{ij}^{\varepsilon,m}\|_{L^2(B_{\varepsilon_l})}. \tag{5.63}$$

From Poincaré inequality, we have

$$C\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}^2 \leq \int_\Omega |\nabla \tilde{u}_\varepsilon^m|^2 + \int_\Omega k_\varepsilon |\tilde{u}_\varepsilon^m|^2. \tag{5.64}$$

Using this fact with the Hölder inequality and the Sobolev embedding theorem in (5.63), we get

$$\begin{aligned}\|\tilde{u}_\varepsilon^m\|^2_{H^1(\Omega)} &\leq C\sum_{l=1}^{N}\varepsilon_l^{2\delta_l}\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_i^2\varepsilon_j^2\|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)} \\ &\leq C\sum_{i,j,l=1}^{N}\varepsilon_l^{2\delta_l}\varepsilon_i^{\delta_i+1}(\varepsilon_i^2|\log\varepsilon_i|+\varepsilon_i^{2\delta_i+1}+\varepsilon_j^2+\varepsilon_i^{\delta_i-1}\varepsilon_j^{\delta_j+2})\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}. \quad (5.65)\end{aligned}$$

We use Lemma 6 to derive the second inequality of (5.65), which holds true for any $0<\delta_i,\delta_j,\delta_l<1$ with $i,j,l=1,\cdots,N$ and $m=1,\cdots,M$. Hence the fact. ☐

5.3. **A priori estimates of the remainders.** We shall prove that $\mathcal{E}_\ell^m(\varepsilon)=o\left(|\varepsilon|^4\right)$ for $\ell=1,\ldots,19$, where $|\varepsilon|:=\varepsilon_1+\cdots+\varepsilon_N$. For simplicity, we use the symbol C to denote any constant independent of ε. The estimate for the remainders are obtained in two steps. We start by using the Cauchy-Schwarz inequality, then

- for the remainders $\mathcal{E}_\ell^m(\varepsilon)$, $\ell=1,\ldots,9$, we use the appropriate lemmas of Section 5.2;
- for the remainders $\mathcal{E}_\ell^m(\varepsilon)$, $\ell=10,\ldots,18$, we use the fact $\|x-x_i\|^n_{L^2(B_{\varepsilon_i})}=O(|\varepsilon|^{n+1})$, where $n\in\mathbb{Z}^+$;
- for the remainder $\mathcal{E}_{19}^m(\varepsilon)$, we use the estimate of the explicit solution p_i^ε in the ball $B_{\varepsilon_i}(x_i)$ for $i=1,\cdots,N$.

Proceeding in this way, we obtain

$$|\mathcal{E}_1^m(\varepsilon)| \leq C\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega_o)}\|u_0^m-z^m\|_{H^1(\Omega_o)}\leq C\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}=o\left(|\varepsilon|^4\right), \quad (5.66)$$

for any $2/5<\delta<1$, where we have used Lemma 7;

$$|\mathcal{E}_2^m(\varepsilon)|\leq C|\varepsilon|^6\sum_{i=1}^{N}\sum_{j=1}^{N}\|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)}\sum_{l=1}^{N}\|h_l^{\varepsilon,m}\|_{H^1(\Omega)}=o\left(|\varepsilon|^4\right), \quad (5.67)$$

for any $1/8<\delta<1$, where we have used Lemmas 5 and 6;

$$|\mathcal{E}_3^m(\varepsilon)|\leq C|\varepsilon|^2\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}\sum_{i=1}^{N}\|h_i^{\varepsilon,m}\|_{H^1(\Omega)}=o\left(|\varepsilon|^4\right), \quad (5.68)$$

for any $1/12<\delta<1$, where we have used Lemmas 5 and 7;

$$|\mathcal{E}_4^m(\varepsilon)|\leq C|\varepsilon|^8\sum_{i=1}^{N}\sum_{j=1}^{N}\|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)}\sum_{l=1}^{N}\sum_{p=1}^{N}\|h_{lp}^{\varepsilon,m}\|_{H^1(\Omega)}=o\left(|\varepsilon|^4\right), \quad (5.69)$$

for any $0<\delta<1$, where we have used Lemma 6;

$$|\mathcal{E}_5^m(\varepsilon)|\leq C|\varepsilon|^4\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}\sum_{i=1}^{N}\sum_{j=1}^{N}\|h_{ij}^{\varepsilon,m}\|_{H^1(\Omega)}=o\left(|\varepsilon|^4\right), \quad (5.70)$$

for any $0<\delta<1$, where we have used Lemmas 6 and 7;

$$|\mathcal{E}_6^m(\varepsilon)|\leq C\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}\|\tilde{u}_\varepsilon^m\|_{H^1(\Omega)}=o\left(|\varepsilon|^4\right), \quad (5.71)$$

for any $0 < \delta < 1$, where we have used Lemma 7;

$$|\mathcal{E}_7^m(\varepsilon)| \leq C|\varepsilon|^3 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \|\tilde{h}_j^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} = o\left(|\varepsilon|^4\right), \tag{5.72}$$

for any $1/2 < \delta < 1$, where we have used Corollary 3 together with the interior elliptic regularity of functions v^m;

$$|\mathcal{E}_8^m(\varepsilon)| \leq C|\varepsilon|^3 \sum_{i=1}^N \|\tilde{h}_i^{\varepsilon,m}\|_{L^2(B_{\varepsilon_i})} = o\left(|\varepsilon|^4\right), \tag{5.73}$$

for any $1/2 < \delta < 1$, where we have use the same arguments as before;

$$|\mathcal{E}_9^m(\varepsilon)| \leq C|\varepsilon|^4 \sum_{i=1}^N \sum_{j=1}^N \left(\|\tilde{h}_j^{\varepsilon,m}\|_{H^1(\Omega)} + \|\tilde{h}_i^{\varepsilon,m}\|_{H^1(\Omega)} + \|\tilde{h}_i^{\varepsilon,m}\|_{H^1(\Omega)}\|\tilde{h}_j^{\varepsilon,m}\|_{H^1(\Omega)}\right) = o\left(|\varepsilon|^4\right), \tag{5.74}$$

for any $0 < \delta < 1$, where we have used Lemma 2;

$$|\mathcal{E}_{10}^m(\varepsilon)| \leq C \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})} \|x - x_i\|_{L^2(B_{\varepsilon_i})}^3 = O(|\varepsilon|^6); \tag{5.75}$$

$$|\mathcal{E}_{11}^m(\varepsilon)| \leq C \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})}^2 \|x - x_i\|_{L^2(B_{\varepsilon_i})}^2 = O(|\varepsilon|^6); \tag{5.76}$$

$$|\mathcal{E}_{12}^m(\varepsilon)| \leq C \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})} \|x - x_i\|_{L^2(B_{\varepsilon_i})}^3 = O(|\varepsilon|^6); \tag{5.77}$$

$$|\mathcal{E}_{13}^m(\varepsilon)| \leq C \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})}^3 \|x - x_i\|_{L^2(B_{\varepsilon_i})}^3 = O(|\varepsilon|^8); \tag{5.78}$$

$$|\mathcal{E}_{14}^m(\varepsilon)| \leq C|\varepsilon|^2 \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})}^2 \|1\|_{L^2(B_{\varepsilon_i})} = O(|\varepsilon|^6); \tag{5.79}$$

$$|\mathcal{E}_{15}^m(\varepsilon)| \leq C|\varepsilon|^2 \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})}^2 \|1\|_{L^2(B_{\varepsilon_i})} = O(|\varepsilon|^6); \tag{5.80}$$

$$|\mathcal{E}_{16}^m(\varepsilon)| \leq C|\varepsilon|^2 \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})} \|x - x_i\|_{L^2(B_{\varepsilon_i})} = O(|\varepsilon|^6); \tag{5.81}$$

$$|\mathcal{E}_{17}^m(\varepsilon)| \leq C|\varepsilon|^2 \sum_{i=1}^N \|x - x_i\|_{L^2(B_{\varepsilon_i})}^2 \|x - x_i\|_{L^2(B_{\varepsilon_i})}^2 = O(|\varepsilon|^8). \tag{5.82}$$

Finally, the last two remainders can be estimated as follows

$$
\begin{aligned}
|\mathcal{E}^m_{18}(\varepsilon)| &\leq C|\varepsilon|^2 \sum_{i=1}^{N} \|q_i v^m - q_i(x_i)v^m(x_i)\|_{L^2(B_{\varepsilon_i})} \|1\|_{L^2(B_{\varepsilon_i})} \\
&\leq C|\varepsilon|^3 \sum_{i=1}^{N} \|x - x_i\|_{L^2(B_{\varepsilon_i})} = O(|\varepsilon|^5); \qquad (5.83)
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{E}^m_{19}(\varepsilon)| &\leq C|\varepsilon|^2 \sum_{i=1}^{N} \|p_i^\varepsilon\|_{L^2(B_{\varepsilon_i})} \|v^m - v^m(x_i)\|_{L^2(B_{\varepsilon_i})} \\
&\leq C|\varepsilon|^2 \sum_{i=1}^{N} \varepsilon_i |\log \varepsilon_i| \, \|x - x_i\|_{L^2(B_{\varepsilon_i})} = o(|\varepsilon|^4). \qquad (5.84)
\end{aligned}
$$

6. Numerical Results

The expression on the right-hand side of (4.1) depends on the number of anomalies N, their sizes α and locations ξ. Thus, from (4.1), we can define

$$
\delta J(\alpha, \xi, N) := -\alpha \cdot d(\xi) + G(\xi)\alpha \cdot \operatorname{diag}(\alpha \otimes \log \alpha) + \frac{1}{2} H(\xi)\alpha \cdot \alpha \ . \qquad (6.1)
$$

The derivative of the function $\delta J(\alpha, \xi, N)$ with respect to the variable α yields the first order optimality condition

$$
\langle D_\alpha \delta J, \beta \rangle = [(H(\xi) + G(\xi))\,\alpha + 2G(\xi)\operatorname{diag}(\alpha \otimes \log \alpha) - d(\xi)] \cdot \beta = 0, \quad \forall \beta \ , \qquad (6.2)
$$

which leads to the non-linear system of the form

$$
(H(\xi) + G(\xi))\,\alpha + 2G(\xi)\operatorname{diag}(\alpha \otimes \log \alpha) = d(\xi) \qquad (6.3)
$$

with the entries of the vector $d \in \mathbb{R}^N$ and the matrices $G,\ H \in \mathbb{R}^N \times \mathbb{R}^N$ defined in (4.2), (4.3) and (4.4)-(4.5), respectively.

The quantity α, solution of (6.3), becomes a function of the locations ξ, namely $\alpha = \alpha(\xi)$, and its value is obtained by using the Newton's method. Let us now replace the solution of (6.3) into $\delta J(\alpha, \xi, N)$ defined by (6.1). Therefore, the optimal locations $\xi^\star$ can be trivially obtained from a combinatorial search over the domain Ω, solution to the following minimization problem

$$
\xi^\star = \underset{\xi \in X}{\operatorname{argmin}} \left\{ \delta J(\alpha(\xi), \xi, N) = -\frac{1}{2} \left(d(\xi) + G(\xi)\alpha(\xi)\right) \cdot \alpha(\xi) \right\} \ , \qquad (6.4)
$$

where X is the set of admissible anomalies locations, and the optimal sizes are given by $\alpha^\star = \alpha(\xi^\star)$. In summary, for a given N number of anomalies, our method is able to find their optimal sizes $\alpha^\star$ and locations $\xi^\star$ in a non-iterative scheme. When the number of anomalies N^* is unknown, we can start our algorithm based on an assumption that there exists $N > N^*$ and then we should find a $(N - N^*)$ number of trial balls with negligible sizes (see [19]). Since a combinatorial search over all the n-points of the set X has to be performed, then this exhaustive search becomes rapidly infeasible for $n \gg N$ as N increases [31]. In addition, the proposed method approximates the unknown set of hidden anomalies by several balls which can be seen as a limitation of our approach. However, it can be used to get a good initial guess for more sophisticated iterative approaches based on level-sets methods [6, 21, 24], for instance. In order to deal with a high number of anomalies we refer to [31] where a multi-grid strategy has been proposed. The above procedure written in pseudo-code format can be found in [31]. For further applications of this algorithm we refer to [9, 10, 19, 36], for instance.

Now, we present some numerical examples in order to demonstrate the effectiveness of the method proposed in the earlier sections of this paper. We consider the geometric domain $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$ which is discretized using three-node finite element scheme. The mesh is generated as a grid of 160×160 squares. Each square is divided into four triangles which leads to 102400 number of finite elements. The subdomain $\Omega_o \subset \Omega$ where the measurements of the potential are taken is defined differently for each example given below. Considering the mesh and the subdomain Ω_o, we define a uniform subgrid with a set of feasible nodes X within which a combinatorial search is performed in order to find the optimal size $\alpha^\star$ and the appropriate center $\xi^\star$ of the embedded anomalies. Moreover, we define three functions to be considered as Dirichlet data on the boundary $\partial\Omega$, namely, $g^1 = 1$, $g^2 = x$ and $g^3 = y$. In the Figures 4-9, we represent anomalies by black, the subdomain Ω_o by gray and the remaining domain $\Omega \setminus \Omega_o$ by white colors.

6.1. **Example 1: Sensitivity with respect to the material property.** In this example, we analyse the sensitivity of the reconstruction of the anomalies when different values of the parameter k are considered. Suppose that, there is a small anomaly ω^* located at $x^* = (0, 0)$, with radius $\varepsilon^* = 5 \times 10^{-3}$. The potential is measured in $\Omega_o = \Omega \setminus B_\rho(0, 0)$ with $B_\rho(0, 0) = \{x \in \mathbb{R}^2 \,:\, \|x\| < \rho\}$, where $\rho = 0.3$. In the current setting, we take only one observation produced by the Dirichlet data g^1. We reconstruct the anomaly ω^* by considering $k = 10^s$ with $s \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. The combinatorial search was conducted on the subgrid of 57 nodes within $\Omega \backslash \Omega_o$. We successfully find the exact location of the center x^* of the anomaly ω^* for all values of k. We plot the size of the anomaly $\varepsilon^\star$ on vertical axis against the value of k on horizontal axis in Figure 3. We observe that the exact radius ε^* of the anomaly was accurately predicted by $\varepsilon^\star$ with $k \in (0, 10]$, while for $k > 10$ the radius ε^* was underestimated. This phenomena occurs because the parameter k present in topological derivatives and the coefficient α are of similar order in equation (4.1). Hence, we take $k = 1$ for the forthcoming examples.

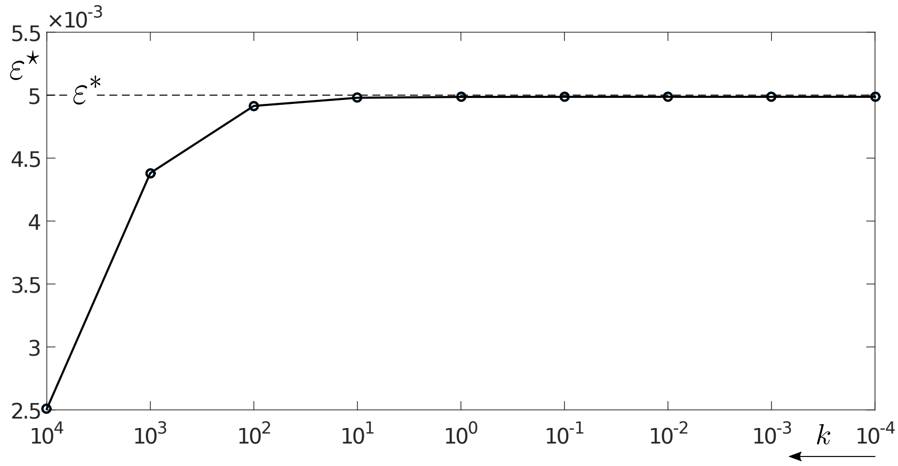


FIGURE 3. Example 1: The approximated solution $\varepsilon^\star$ for different values of k.

6.2. **Example 2: Sensitivity with respect to the set of admissible locations.** Here, the sensitivity of the reconstruction with respect to the set of admissible locations X is investigated. For this purpose, we consider the target anomaly as a circular region with radius $\varepsilon^* = 0.05$ and center located at $x^* = (-0.2125, 0.1625)$. The subdomain Ω_o is given by a closed region around the target anomaly. The domain Ω, subdomain Ω_o and anomaly ω^* are illustrated in Figure 4. Like in Example 1, we take only one observation produced by the Dirichlet data g^1. Three subgrids within $\Omega \setminus \Omega_o$ are considered as the set of admissible locations X. The first two with 113 and 481 points, respectively, such that $x^* \notin X$. The third subgrid has 1985 points, but with $x^* \in X$. The results associated with each subgrid are respectively shown in Figures 5(a)-5(c). From a qualitative comparison of the results presented in Figure 5, it can be seen that the more the subgrid is refined the better is the reconstruction. In particular, when $x^* \notin X$ (the center of the target does not coincide with any point of the subgrid), the algorithm returns a location $x^\star$ in X which is closest to x^*, as shown in Figures 5(a) and 5(b). Finally, when $x^* \in X$, the algorithm is able to return the exact location of the anomaly, as can be seen in Figure 5(c). In all cases, the resulting radius $\varepsilon^\star$ is very close to the true value ε^*. See the quantitative results in Table 1. From this example we confirm that our algorithm can be used to produce a good and reliable initial guess to other well-known reconstruction methods, as expected. In the forthcoming examples, we assume that the center of each anomaly to be reconstructed coincides with one point of the subgrid X.

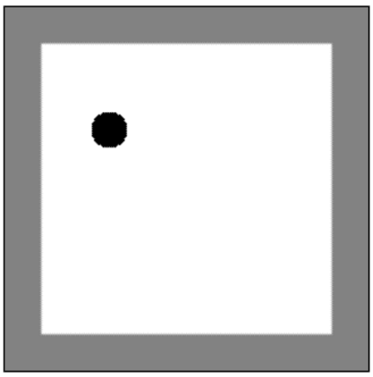
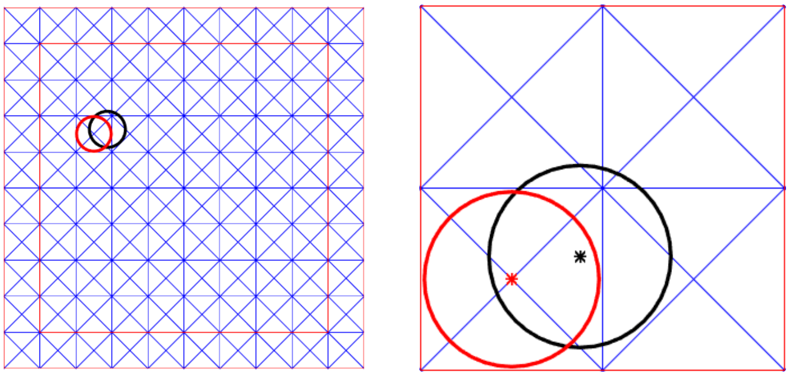
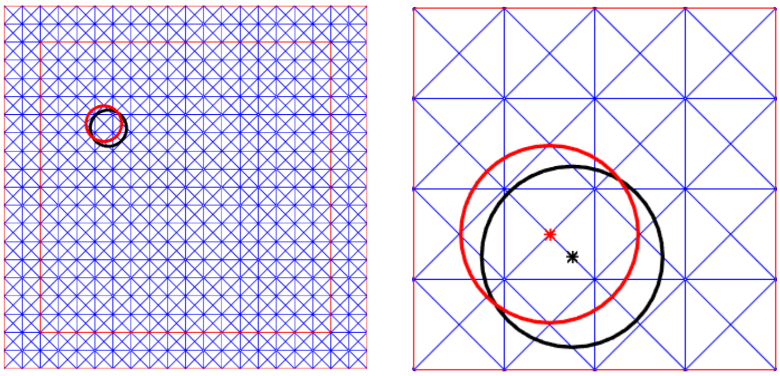


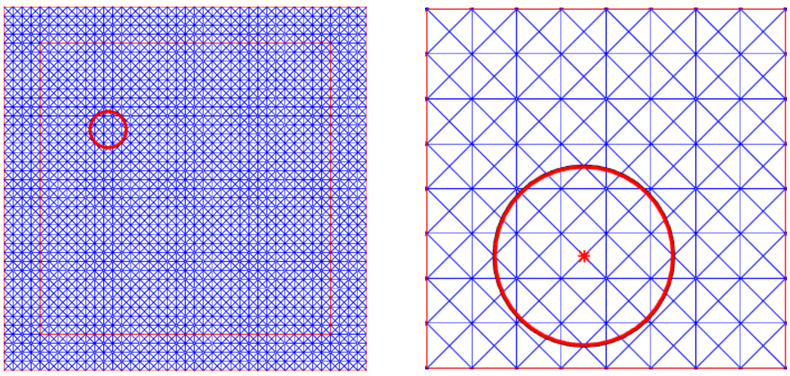
FIGURE 4. Example 2: Target.



(a) subgrid of 113 points



(b) subgrid of 481 points



(c) subgrid of 1985 points

Figure 5. Example 2: Result obtained (left) and a zoom of the solution (right) for each set of admissible locations X. The red and black circles represent the solution and the target, respectively.

TABLE 1. Example 2: Results obtained for different set of admissible locations X

		Number of points of the subgrid		
		113	481	1985
$\omega^\star$	$x^\star$	$(-0.25, 0.15)$	$(-0.225, 0.175)$	$(-0.2125, 0.1625)$
	$\varepsilon^\star$	0.0480	0.0490	0.0496

6.3. **Example 3: Sensitivity with respect to the number of observations.** Reconstruction of two anomalies of different sizes is demonstrated in this example. Two circular regions ω_1^* and ω_2^* located at $x_1^* = (-0.1, 0.2)$ and $x_2^* = (-0.3, -0.3)$ with radii $\varepsilon_1^* = 0.02$ and $\varepsilon_2^* = 0.05$, respectively, are considered as the target anomalies. The subdomain Ω_o is given by a circular region within Ω. The reference domain Ω, subdomain Ω_o and target anomalies ω_1^* and ω_2^* are shown in Figure 6. A subgrid of 156 points is defined as the combinatorial search region inside the subdomain $\Omega \setminus \Omega_o$. We start by considering only one observation associated with the Dirichlet data g^2. After comparing Figures 6 and 7(a), we observe that the algorithm fails in reconstructing the target domain. This happens because of the lack of information. Therefore, we improve the number of measurements by considering all the Dirichlet data g^1, g^2 and g^3 simultaneously. In this case we obtain the centers $x_1^\star = (-0.1, 0.2)$ and $x_2^\star = (-0.3, -0.3)$ with the associated radii $\varepsilon_1^\star = 0.0195$ and $\varepsilon_2^\star = 0.0491$ of the target anomalies ω_1^* and ω_2^*, which are approximately equal to the true values ε_1^* and ε_2^*, respectively. We demonstrate the numerical result in the Figure 7(b). We conclude by noticing the need of more than one observation in the case of insufficient information. This motivates us to collect data through three boundary excitations g^1, g^2 and g^3 in the forthcoming example of reconstructing three anomalies simultaneously which is the case when we have comparatively less information.

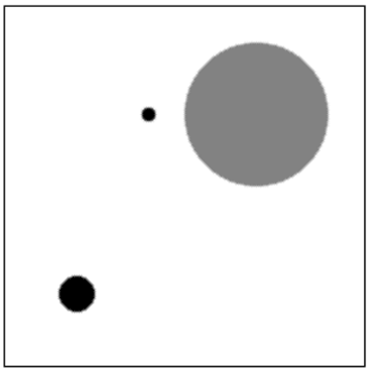
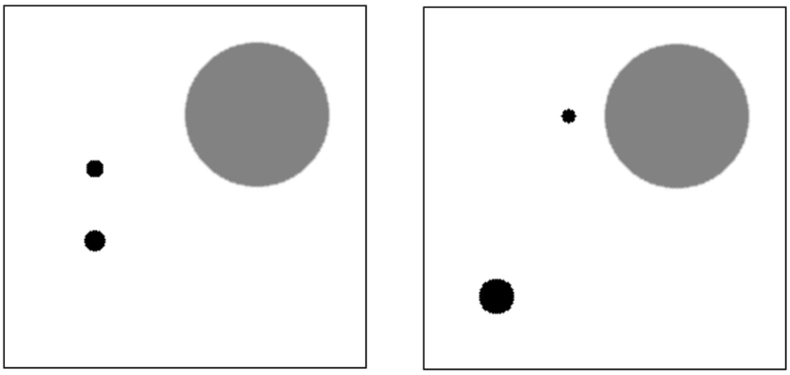


FIGURE 6. Example 3: Target.



(a) $M = 1$ (b) $M = 3$

FIGURE 7. Example 3: Results.

6.4. **Example 4: Sensitivity with respect to partial noisy data.** Reconstruction of three embedded anomalies is demonstrated in this example. Additionally, we verify the robustness of the method proposed in the earlier part of this paper with respect to the noisy data. Three circular regions with centers located at $x_1^* = (-0.2, 0.2)$, $x_2^* = (-0.1, -0.3)$, $x_3^* = (0.3, 0.1)$ and with radius $\varepsilon_1^* = \varepsilon_2^* = \varepsilon_3^* = 0.05$ are considered as the target anomalies. In the current setting, we take measurements in the subdomain Ω_o which is an union of four small regions in the neighborhood of the corners of the square domain Ω. The domain Ω, subdomain Ω_o and three anomalies ω_1^*, ω_2^* and ω_3^* are illustrated in Figure 8. Here, we consider, the subgrid for the combinatorial search inside the subdomain $\Omega \setminus \Omega_o$ which consists of 177 uniformly distributed nodes. In order to obtain the noisy synthetic data, the parameter k_{ω^*} is replaced by $k_{\omega^*}^{\mu}(x) = k_{\omega^*}(x) + \mu\tau(x)\|k_{\omega^*}(x)\|_{L^2(\Omega)}$, where $\tau(x)$ is a random variable taking values in $(0, 1)$ and μ corresponds to the noise level. The results obtained for different levels of noise are shown in Figure 9. It can be seen in Figure 9(a) that the anomalies are reconstructed accurately in the absence of noise. By comparing Figures 9(b)-9(c), we can observe that the reconstruction scheme proposed in this paper works efficiently up to 40% of noise in the parameter k_{ω^*}. For more noisy input, though we are not sure about the accuracy but the functionality of the proposed scheme is ensured which can be observed in the Figure 9(d).

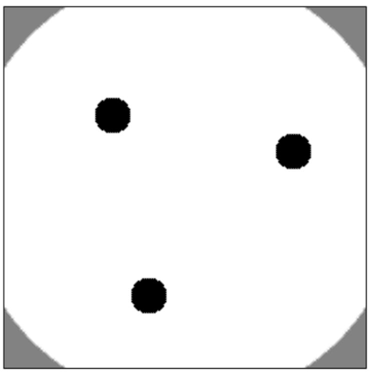
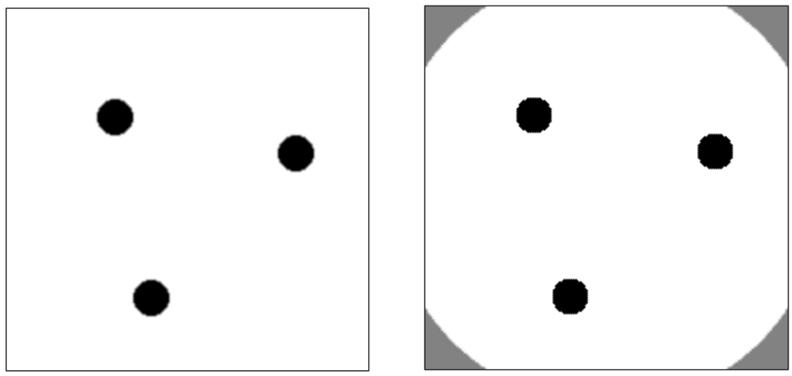
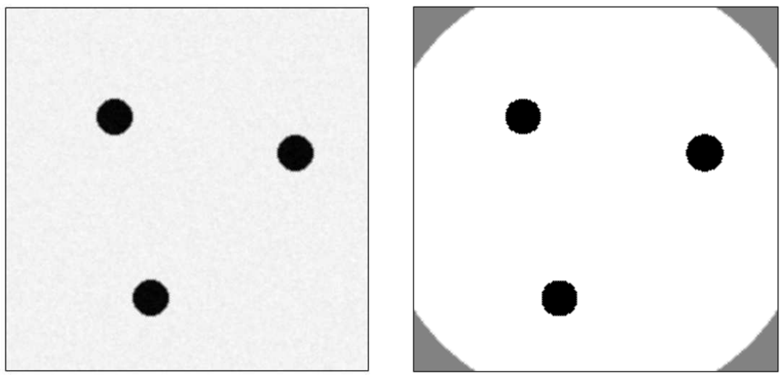


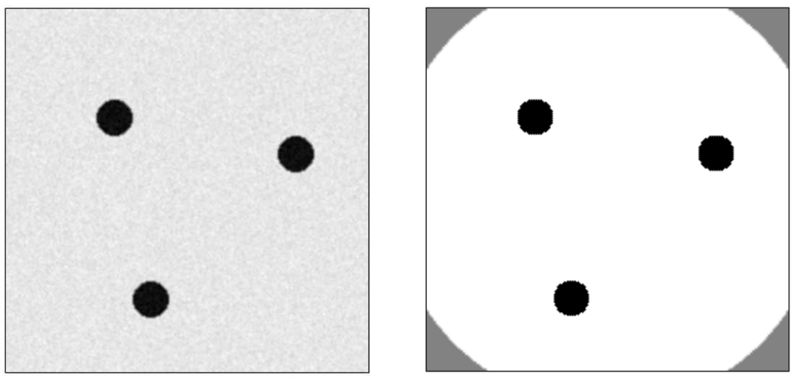
FIGURE 8. Example 4: Target.



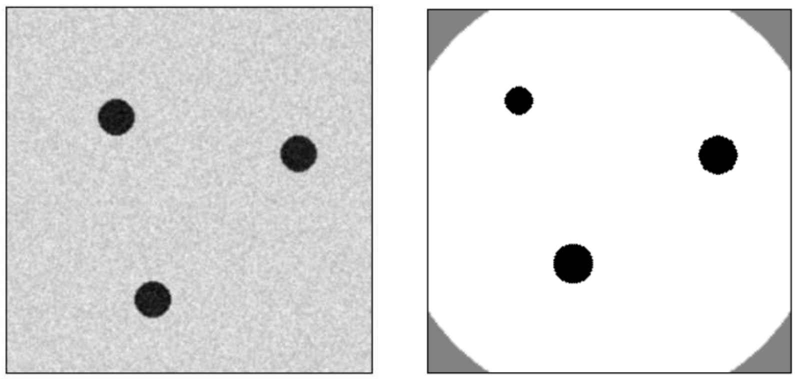
(a) $\mu = 0\%$



(b) $\mu = 20\%$



(c) $\mu = 40\%$



(d) $\mu = 80\%$

FIGURE 9. Example 4: Target (left) and the respective result (right) for different levels of noise.

Acknowledgments

This research was partly supported by CNPq (Brazilian Research Council), CAPES (Brazilian Higher Education Staff Training Agency) and FAPERJ (Research Foundation of the State of Rio de Janeiro). These supports are gratefully acknowledged. These research results have also received funding from the EU H2020 Programme and from MCTI/RNP-Brazil under the HPC4E Project, grant agreement n° 689772. The third author would like to thank the Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción (Chile) for their financial support through PROYECTOS VRID INICIACIÓN n° 216.013.0.41-1.0IN.

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