FAST SOLUTION OF THE INVERSE ELECTROMAGNETIC CASTING PROBLEM

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Abstract. In this paper we deal with an inverse electromagnetic casting problem, which consists in designing a set of inductors in such a way that a liquid metal achieves a given shape. The inductors are assumed to be made of single solid-core wires with a negligible cross-section area. The inverse problem is rewritten in the form of an optimization problem. In particular, a Kohn-Vogelius based functional is minimized with respect to a set of admissible inductors, leading to a non-iterative second order optimization algorithm. Finally, several numerical examples are presented showing that the proposed approach is effective to design suitable inductors.

1. Introduction

This work deals with an inverse problem concerning the electromagnetic shaping technique used in molten metals casting processes. The advantage of the electromagnetic shaping is that it makes use of electromagnetic fields for contact-less heating, shaping and control of solidification of hot metals. This technique is appropriate for preparing very pure specimens in metallurgical experiments, as even small traces of impurities can affect the physical properties of the sample. One of the most interesting applications is the electromagnetic casting of AI alloys, see for example the references Evans (1995); Besson et al. (1991).

We consider a two-dimensional model corresponding to an infinitely long falling down molten metal column shaped by an electromagnetic field externally created by a set of vertical inductors. We assume that the frequency of the current is so high that we can neglect the penetration of the electromagnetic field into the molten metal (skin effect). The inverse shaping problem considered in this work consists of finding an appropriate set of vertical electrical currents such that the induced electromagnetic field makes a given mass of molten metal acquire a shape with a prescribed cross section.

The direct electromagnetic casting problem has already been investigated from the points of view of the modeling, the application and the numerical simulation. The two-dimensional model considered here has been studied in many papers, for instance (Shercliff, 1981; Sneyd and Moffatt, 1982; Moffatt, 1985; Brancher and Séro-Guillaume, 1985; Gagnoud et al., 1986; Henrot et al., 1989; Sero-Guillaume et al., 1992; Zouaoui et al., 1990; Pierre and Roche, 1993; Coulaud, 1998) and (Hayouni and Novruzi, 2002) where a sufficient condition for the existence of solutions is provided. Numerical analysis of the three-dimensional case can be found in Novruzi and Roche (2000).

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The inverse problem in the two-dimensional case was analyzed in (Henrot and Pierre, 1989; Felici, 1992; Shin et al., 2012), where necessary conditions of existence are presented and the non-uniqueness of the solution is proved. In the three-dimensional case we know only two papers, see (Felici, 1992; Pierre and Rouy, 1996). From the numerical point of view, the two-dimensional inverse problem has been studied by using shape and topological optimization methods. In (Canelas et al., 2009a,b; Roche et al., 2012) the authors optimized the position and shape of a fixed number of inductors. Topological optimization and level set methods used in (Canelas et al., 2011) and (Shin et al., 2012) allow the authors to determine the position and shape, as well the number of inductors. In (Canelas et al., 2014) the authors introduced a second order topological sensibility analysis to determine the configuration of inductors by minimizing a Kohn-Vogelius based shape functional.

In the present work we consider a new approach for the inverse electromagnetic shaping problem: given certain set of positions, the algorithm proposed looks for the best values of the electric current intensities. The main advantage of the new approach is that it is not iterative, the solution is found by solving a single linear system of equations, and therefore the solution method is considerably faster than all previous methods that find the solution by solving a nonlinear mathematical programming formulation, such as in (Canelas et al., 2009a,b; Roche et al., 2012), or by using heuristic iterative algorithms such as in (Canelas et al., 2011) and (Canelas et al., 2014). The algorithm presented in (Shin et al., 2012) is not iterative also, but the new proposed approach is more versatile, since it allows the user to predefine a feasible set of inductor positions, which could be important for obtaining a practical solution.

The paper is organized as follows. In Section 2 the problem we are dealing with is introduced. Section 2.1 presents the model problem which is given by a boundary value problem whose source-term is the electrical current density. The inverse problem is introduced in Section 2.2, where there is also a brief discussion concerning well-posedness of the associated direct problem. In Section 2.3 the inverse problem is rewritten in the form of an optimization problem, where a Kohn-Vogelius based functional is minimized with respect to a set of admissible inductors. The sensitivity analysis of the cost functional is derived in Section 3. The resulting optimization algorithm is presented in Section 4, where a regularization procedure is introduced. Some numerical examples are presented in Section 5 showing that the proposed approach is effective to design suitable inductors. Finally, the paper ends with some concluding remarks presented in Section 6.

2. The electromagnetic casting problem

The two-dimensional model considers the cross-section of a vertical column of molten metal, which is shaped by the electromagnetic field created by vertical inductors. We assume that a stationary horizontal cross-section is reached, so that the two-dimensional model is valid. We also assume that the frequency of the imposed current is very high, and hence the electromagnetic field does not penetrate into the molten metal. An equilibrium shape is then characterized by the static balance on the surface of the molten metal between the surface tension and the electromagnetic forces produced by pointwise inductors. The associated inverse problem consists in the design of the inductors so that a given shape for the liquid metal is achieved. The inverse problem is rewritten in the form of an optimization problem. Since we are dealing with pointwise inductors, a Kohn-Vogelius based functional is minimized with respect to their locations and current intensities, leading to a non-iterative second order optimization algorithm.

2.1. **The model problem.** Let the cross-section of the molten metal column be represented by the closed and simply connected domain $\omega \in \mathbb{R}^2$ with boundary Γ. Let $\Omega = \mathbb{R}^2 \setminus \omega$ be the open exterior domain. Let j_0 be the mean square value of the vertical component of the electrical current density vector. We assume that j_0 has a compact support in Ω and satisfies

$$\int_{\Omega} j_0 \, dx = 0 \,. \tag{1}$$

In that case the equilibrium domains are characterized by the following Bernoulli-type boundary equilibrium equation:

$$\frac{1}{2\mu_0} \left|\partial_\nu \varphi\right|^2 + \sigma \mathcal{C} = P_0 \quad \text{on } \Gamma \,, \tag{2}$$

with $\partial_\nu \varphi := \nabla \varphi \cdot \nu$, where ν is the normal vector to the boundary Γ pointing to the molten metal. In addition, $\mu_0 > 0$ is the vacuum permeability, $\sigma > 0$ is the surface tension of the molten metal, $\mathcal{C}$ is the curvature of Γ seen from the metal, P_0 is the constant that represents the difference between the interior and exterior pressures, and φ is the magnetic flux function which satisfies:

$$\left\{ \begin{array}{rll} -\Delta \varphi & = \mu_0 j_0 & \text{in } \Omega \,, \\ \varphi & = 0 & \text{on } \Gamma \,, \\ \varphi(x) & = O(1) & \text{as } \|x\| \to \infty \,. \end{array} \right. \tag{3}$$

The direct problem consists of looking for a domain $\omega \subset \mathbb{R}^2$ such that φ is a solution of (3) and satisfies (2) for certain real constant P_0.

We are interested in the particular case when the inductors are made of single solid-core wires with a negligible area of cross-section. In this case the function j_0 can be modeled as the sum of several Dirac masses, i.e. j_0 belongs to the following set of admissible currents

$$C_\delta(\Omega) = \left\{ j : \Omega \to \mathbb{R} \mid j(x) = \sum_{i=1}^{n} \alpha_i \delta_i(x), \text{ with } \sum_{i=1}^{n} \alpha_i = 0 \text{ and } n \in \mathbb{N} \right\}, \tag{4}$$

with $\alpha_i \in \mathbb{R}$ used to denote the inductors intensities and $\delta_i(x) := \delta(x - x_i)$, where all inductor locations $x_i \in \Omega$. In other words, we have a finite number of wires that do not touch the boundary Γ.

2.2. **The inverse problem.** Let the target shape $\omega \subset \mathbb{R}^2$ be a closed and simple connected domain with boundary Γ of class C^2. The purpose of the inverse problem is to find an electric current density j_0 and a real constant P_0 such that the solution φ of (3) satisfies also the equilibrium equation (2). The inverse problem is generally ill-posed, since small variations of the boundary Γ of ω cause dramatic variations of the inverse problem solution (Felici and

Brancher, 1991). In addition, the solution may not exist, or may be non-unique (Henrot and Pierre, 1989; Shin et al., 2012).

The authors of reference (Henrot and Pierre, 1989) provided the main known result about existence of solutions: given $P_0 \geq \sigma \max_\Gamma \mathcal{C}$, the inverse problem has a solution j_0 if and only if (i) Γ is an analytic curve and (ii) if

$$P_0 = \sigma \max_\Gamma \mathcal{C}\,, \tag{5}$$

then the maximum of the curvature $\mathcal{C}$ must be achieved in an even number of points (the points are counted with their multiplicity). In addition, the existence of a solution j_0 satisfying (1) requires the satisfaction of (5).

Hence, given a smooth target shape ω, if we compute P_0 by (5) and define the positive function $\overline{p} = \sqrt{2\mu_0(P_0 - \sigma\mathcal{C})}$, then the equilibrium equation (2) can be rewritten in the following way:

$$\partial_\nu \varphi = \varkappa \overline{p} \ \text{ on } \Gamma\,, \tag{6}$$

where $\varkappa = \pm 1$, with the sign changes of $\varkappa$ located where the curvature $\mathcal{C}$ reaches the maximum value. Note that $\varkappa \overline{p}$ is of class C^0 on Γ.

Consequently, the inverse problem can be formulated in the following way: given a target shape ω, find an electric current density j_0 satisfying (1) such that there exists φ, solution to the following over-determined boundary value problem:

$$\begin{cases} -\Delta\varphi &= \mu_0 j_0 & \text{in } \Omega\,, \\ \varphi &= 0 & \text{on } \Gamma\,, \\ \partial_\nu\varphi &= \varkappa\overline{p} & \text{on } \Gamma\,, \\ \varphi(x) &= O(1) & \text{as } \|x\| \to \infty\,. \end{cases} \tag{7}$$

To find a solution j_0 the following approach is proposed. First, we remove the Dirichlet condition $\varphi = 0$ from (7) with the purpose to avoid the over-determination. Then we minimize the functional $\frac{1}{2}\int_\Gamma \varphi^2\, ds$ with respect to j_0 trying to enforce the Dirichlet condition that was left aside. In the general case the optimal value of the functional will not be zero, hence the optimal solution j_0 will be only an approximation to the inverse problem.

Consider (7) without the Dirichlet condition $\varphi = 0$ on Γ and with j_0 satisfying (1). Then, this problem has a solution if and only if the following compatibility condition is satisfied (Hsiao and Wendland, 2008):

$$\int_\Gamma \varkappa\overline{p}\, ds = 0\,. \tag{8}$$

It turns out that (8) is a necessary condition for the existence of solutions to the inverse problem. In the following we will assume that the target shape ω is such that (8) holds. Note that the Neumann problem is still not well-posed, since $u = \varphi + c$ is a solution for each real constant c provided that φ is a solution. However, there is only one solution that minimizes the functional $\frac{1}{2}\int_\Gamma u^2\, ds$ with respect to c. Note that this functional is well defined. In fact, since Γ is at least of class C^2 and $\varkappa\overline{p}$ is of class C^0 on Γ, hence any solution u of the Neumann problem is of class C^1 in a neighborhood of Γ provided that the inductors do not touch the boundary Γ (Atkinson, 1997). To find the optimal c, we compute

$$\frac{1}{2}\int_\Gamma u^2\,ds = \frac{1}{2}\int_\Gamma \varphi^2\,ds + c\int_\Gamma \varphi\,ds + \frac{1}{2}c^2|\Gamma|\,. \tag{9}$$

By differentiating the above expression with respect to c we obtain the optimal value $c^* = -|\Gamma|^{-1}\int_\Gamma \varphi\,ds$. By considering this result, we have that the optimal function u is the solution of the exterior Neumann problem that also satisfies the following condition:

$$\int_\Gamma u\,ds = \int_\Gamma \varphi + c^*\,ds = 0\,. \tag{10}$$

Remark 1. *The resulting Neumann problem obtained after removing the Dirichlet condition $\varphi = 0$ on Γ, and including the condition $\int_\Gamma \varphi\,ds = 0$, is a well-posed problem provided that the boundary data satisfies* (8), *see the book by Hsiao and Wendland (2008). The details are given in the next section.*

2.3. **The optimization problem.** In this paper we study the particular case in which j_0 is a finite linear combination of Dirac masses, namely $j_0 \in C_\delta(\Omega)$, which is defined through (4). Moreover, in this work we consider a set of points x_i, located in positions previously established in Ω, and look for suitable current intensities α_i, such that the resulting electric current density j_0 is a good approximation of a solution of the inverse problem. Consequently, this approach reduces the original problem to a finite dimensional one. The optimization formulation of the inverse problem consists of looking for a minimum $j_0 \in C_\delta(\Omega)$ of the functional

$$\mathcal{J}(u) = \frac{1}{2}\int_\Gamma u^2\,ds\,, \tag{11}$$

where $u : \Omega \to \mathbb{R}$ is the unique solution of the following auxiliary problem:

$$\left\{\begin{array}{rcll} -\Delta u &=& \mu_0 j_0 & \text{in} \quad \Omega, \\ \partial_\nu u &=& \varkappa\overline{p} - \mu_0\gamma(j_0) & \text{on} \quad \Gamma, \\ u(x) &=& O(1) & \text{as} \quad \|x\| \to \infty, \\ \displaystyle\int_\Gamma u\,ds &=& 0. & \end{array}\right. \tag{12}$$

Note that we have imposed (10) as an additional condition in the exterior Neumann problem in order to have an unique solution for any given j_0. The function $\gamma(j_0)$ is a compatibility constant defined by:

$$\gamma(j_0) = \frac{1}{|\Gamma|}\int_\Omega j_0\,dx\,. \tag{13}$$

The term $\gamma(j_0)$ added in the Neumann boundary conditions guarantees the compatibility condition required for the existence of solutions of (12) when (1) is not satisfied. This procedure will allow us to enlarge the domain of definition of the objective function and treat (1) as a constraint of the optimization problem. If a solution satisfying (1) is found, e.g. $j_0 \in C_\delta(\Omega)$, then $\gamma(j_0) = 0$, which means that all conditions in (7) will be satisfied with the exception of the Dirichlet condition, which will be satisfied approximately with an accuracy depending on the final value of the objective function.

Remark 2. *Note that we are not dealing with the Kohn-Vogelius criterion in its strict sense. Actually, our approach is just based (inspired) on it. As mentioned above, the cost functional* (11) *is well defined provided that the inductors do not touch the boundary* Γ*.*

3. Sensitivity analysis

The next step consists in measuring the sensitivity of the functional (11) with respect to the insertion of perturbations in the electric current j_0 of the auxiliary problem (12). The idea is to obtain an explicit form for the sensitivities with respect to the control parameters α_i, which shall be crucial to devise a simple and fast optimization algorithm to be presented in the next section. For this purpose, we add to the source j_0 a number m of Dirac masses with arbitrary locations and intensities, i.e.

$$j_\delta(x) = j_0(x) + \sum_{i=1}^{m} \alpha_i \delta_i(x), \tag{14}$$

Note that if $j_0 \in C_\delta(\Omega)$, then $j_\delta \in C_\delta(\Omega)$ only if:

$$\sum_{i=1}^{m} \alpha_i = 0. \tag{15}$$

In order to evaluate the effect that the perturbation (14) causes on the functional (11), it is necessary to make the sensitivity analysis of the functional with respect to the perturbation parameters, namely: m, α_i and x_i. From (14) we get the following exterior boundary value problem:

$$\left\{ \begin{array}{rcll} -\Delta u_\delta &=& \mu_0 j_\delta & \text{in} \quad \Omega, \\ \partial_\nu u_\delta &=& \varkappa \overline{p} - \mu_0 \gamma(j_\delta) & \text{on} \quad \Gamma, \\ u_\delta(x) &=& O(1) & \text{as} \quad \|x\| \to \infty, \\ \displaystyle\int_\Gamma u_\delta \, ds &=& 0. & \end{array} \right. \tag{16}$$

Then, the perturbed shape functional is given by:

$$\mathcal{J}(u_\delta) = \frac{1}{2} \int_\Gamma u_\delta^2 \, ds \, . \tag{17}$$

The next step is to calculate the difference between the perturbed and unperturbed shape functional. Moreover, it is required that the resulting expression be written in terms of the perturbation parameters. So we propose a relation between the solutions of (16) and (12). This relation is given by:

$$u_\delta = u + \sum_{i=1}^{m} \alpha_i h_i, \tag{18}$$

where each function h_i is solution to the following exterior boundary value problem:

$$\begin{cases} -\Delta h_i &= \mu_0 \delta_i \quad \text{in} \quad \Omega, \\ \partial_\nu h_i &= -\dfrac{\mu_0}{|\Gamma|} \quad \text{on} \quad \Gamma, \\ h_i(x) &= O(1) \quad \text{as} \quad \|x\| \to \infty, \\ \displaystyle\int_\Gamma h_i \, ds &= 0. \end{cases} \tag{19}$$

It should be noted that there is a direct relationship between the functions h_i and the perturbation points x_i, since $\delta_i(x) = \delta(x - x_i)$. However, the functions h_i are independent of the intensity values α_i.

Proposition 3. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)^\top \in \mathbb{R}^m$ *such that* $\sum_{i=1}^m \alpha_i = 0$ *and let* $v = \sum_{i=1}^m \alpha_i h_i$ *where the functions* h_i *are the solutions of* (19) *for a set of distinct points* x_i*. Then the trace of* v *on* Γ *is zero only if* $\boldsymbol{\alpha} = 0$*.*

Proof. Take $\boldsymbol{\alpha}$ such that $\sum_{i=1}^m \alpha_i = 0$, and assume that the trace of $v = \sum_{i=1}^m \alpha_i h_i$ on Γ is zero. Then v is a solution to:

$$\begin{cases} -\Delta v &= \sum_{i=1}^m \mu_0 \alpha_i \delta_i \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \Gamma, \\ \partial_\nu v &= 0 \quad \text{on} \quad \Gamma, \\ v(x) &= O(1) \quad \text{as} \quad \|x\| \to \infty, \end{cases} \tag{20}$$

In addition, there exist funtions $\phi_1, \ldots, \phi_m$, satisfying (see the demonstration of Theorem 2 in (Shin et al., 2012)):

$$\begin{cases} -\Delta \phi_j &= 0 \quad \text{in} \quad \Omega, \\ \phi_j(x_i) &= \delta_{ij}, \\ |\partial_r \phi_j| &= O(r^{-2}) \quad \text{as} \quad r \to \infty, \end{cases} \tag{21}$$

where r is the coordinate of the polar system, $r = \|x\|$, and δ_{ij} is the Kronecker delta. Then, the second Green identity applied to v and ϕ_j provides:

$$\sum_{i=1}^m \mu_0 \alpha_i \phi_j(x_i) = \mu_0 \alpha_j = 0 \,, \tag{22}$$

which proves the proposition. □

Now we can calculate the sensitivity of the shape functional. Subtracting Eqs. (17) and (11) we get the following:

$$\begin{aligned} \mathcal{J}(u_\delta) - \mathcal{J}(u) &= \frac{1}{2} \int_\Gamma (u_\delta^2 - u^2) \, ds \\ &= \frac{1}{2} \int_\Gamma (u_\delta - u)(u_\delta + u) \, ds \\ &= \frac{1}{2} \int_\Gamma \left(\sum_{i=1}^m \alpha_i h_i \right) \left(2u + \sum_{i=1}^m \alpha_i h_i \right) ds \\ &= \int_\Gamma u \left(\sum_{i=1}^m \alpha_i h_i \right) ds + \frac{1}{2} \int_\Gamma \left(\sum_{i=1}^m \alpha_i h_i \right)^2 ds. \end{aligned} \tag{23}$$

Note that the expression on the RHS of (23) depends directly on the parameters m, α_i and x_i (the latter through the functions h_i). It is also observed that this expression is a quadratic form with respect to the intensities α_i.

Let $\boldsymbol{d} = (d_1, d_2, \dots, d_m)^\top \in \mathbb{R}^m$ be the vector with the following entries:

$$d_i := \int_\Gamma u h_i \, ds \tag{24}$$

and $\mathbf{H}$ the symmetric matrix with entries:

$$H_{ij} := \int_\Gamma h_i h_j \, ds. \tag{25}$$

Then we obtain:

$$\mathcal{J}(u_\delta) = \mathcal{J}(u) + \boldsymbol{\alpha}^\top \boldsymbol{d} + \frac{1}{2}\boldsymbol{\alpha}^\top \mathbf{H}\boldsymbol{\alpha}\,, \tag{26}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)^\top \in \mathbb{R}^m$.

Proposition 4. *The matrix* $\mathbf{H}$ *is symmetric positive semidefinite, and it is positive definite on the linear subspace* $\{\boldsymbol{\alpha} \in \mathbb{R}^m : \sum_{i=1}^m \alpha_i = 0\}$.

Proof. The symmetry of $\mathbf{H}$ results directly from its definition (25). Furthermore, for any $\boldsymbol{\alpha} \in \mathbb{R}^m$ we have

$$\boldsymbol{\alpha}^\top \mathbf{H}\boldsymbol{\alpha} = \int_\Gamma \left(\sum_{i=1}^m \alpha_i h_i \right)^2 ds\,, \tag{27}$$

and the thesis of the proposition follows from Proposition 3. □

4. Procedure of solution

Since (26) is a quadratic form in $\boldsymbol{\alpha}$, the next step consists in minimizing the variation of the functional with respect to this variable. In other words, we want to solve the following minimization problem:

$$\left\{ \begin{array}{ll} \underset{\boldsymbol{\alpha} \in \mathbb{R}^m}{\text{Minimize}} & J(\boldsymbol{\alpha}) := \boldsymbol{\alpha}^\top \boldsymbol{d} + \frac{1}{2}\boldsymbol{\alpha}^\top (\mathbf{H} + \lambda \mathbf{I})\,\boldsymbol{\alpha} \\ \text{Subject to} & \sum_{i=1}^m \alpha_i = \boldsymbol{e}^\top \boldsymbol{\alpha} = 0 \end{array} \right. \tag{28}$$

where $\lambda > 0$ is a regularizing parameter and $\boldsymbol{e} = (1, 1, \dots, 1)^\top \in \mathbb{R}^m$. The regularization term is necessary to reduce the condition number of the problem. Note that the inverse problem in the continuum setting is ill-posed, so that any attempt to approximate a continuum solution by considering a large number of points covering a neighborhood of the liquid metal will result in a bad conditioned problem. A bad location of the points can also affect dramatically the condition number, e.g. if two points are too close to each other or all the points are too far from the liquid metal, then we have observed that $\mathbf{H}$ presents a high condition number. In fact, the matrix $\mathbf{H}$ becomes singular if two points coincide. The regularizing term, that can be written as $(\lambda/2)\|\boldsymbol{\alpha}\|^2$, has also the beneficial effect of penalizing the high electric current intensities, which leads to a reduction of the total electric energy consumption. Then, the regularizing term is convenient also from the economical point of view. Connections with the Tikhonov regularization and the Levemberg-Marquardt algorithm can also be recognized.

To solve (28) we consider the Lagrangian $L : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$:

$$L(\boldsymbol{\alpha}, \beta) = J(\boldsymbol{\alpha}) + \beta(\boldsymbol{e}^\top \boldsymbol{\alpha}). \tag{29}$$

Since J is a strictly convex quadratic function, and the constraint is linear, then there exists a unique Lagrangian multiplier β such that $(\boldsymbol{\alpha}, \beta)$ is a stationary point of the Lagrangian function. This point can be found by solving the following linear system:

$$\begin{pmatrix} \mathbf{H} + \lambda \mathbf{I} & \boldsymbol{e} \\ \boldsymbol{e}^\top & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \beta \end{pmatrix} = - \begin{pmatrix} \boldsymbol{d} \\ 0 \end{pmatrix}. \tag{30}$$

It is know that the linear system (30) has a unique solution for any $\lambda \geq 0$ since $\mathbf{H}$ has the properties indicated in Proposition 4. In addition, this linear system can be solved accurately if $\mathbf{H} + \lambda \mathbf{I}$ is well conditioned, see for instance (Luenberger, 2003). In addition, note that the residual of the solution of (30) in the unregularized system is:

$$R(\boldsymbol{\alpha}, \beta) = \mathbf{H}\boldsymbol{\alpha} + \beta \boldsymbol{e} + \boldsymbol{d}. \tag{31}$$

By assuming that the inductors locations are fixed, the resulting optimization algorithm becomes strikingly simple and very fast, which consists in solving the linear system (30) once. In contrast, if the locations also belong to the set of unknown design variables, then expression (26) would have to be minimized with respect to the intensities as well as to the locations of the inductors. This last step of the optimization problem leads to a combinatorial search in the set of admissible locations. These ideas have been fully developed in the paper by Machado et al. (2017) in the context of inverse gravimetry problem, where a discussion concerning the complexity of the resulting reconstruction algorithm can be found, together with alternatives to accelerate it.

5. Numerical examples

We show here some examples to illustrate the efficiency of the algorithm proposed. For details of the boundary element method used to solve the boundary value problems see (Canelas et al., 2011). In the examples we express all physical magnitudes in the same consistent system of units. In this system we set for all the examples $\sigma = 1.0 \times 10^{-4}$ and $\mu_0 = 1.0$. The regularizing parameter considered for the examples is $\lambda = 1.0 \times 10^{-8}$ unless otherwise indicated. In all examples the initial guess is set as $j_0 = 0$.

The first example considers the same target shape of Example 1 in (Canelas et al., 2014). This target shape is found as a solution of a direct free-surface problem for a given distribution of electric current. We first try to arrange the electric currents in a circle surrounding the target shape, which has radius $R = 1.5$, and after that in a fine regular grid of inductors. The solutions found when considering $m = 30$ possible inductors in the circle and $m = 1188$ in the regular grid are shown in Figure 1. Note that the arrange of 30 inductors is enough to find an equilibrium shape that almost exactly match the target shape. This configuration has also the advantage of requiring the solution of a smaller linear system which is not ill-conditioned even when considering

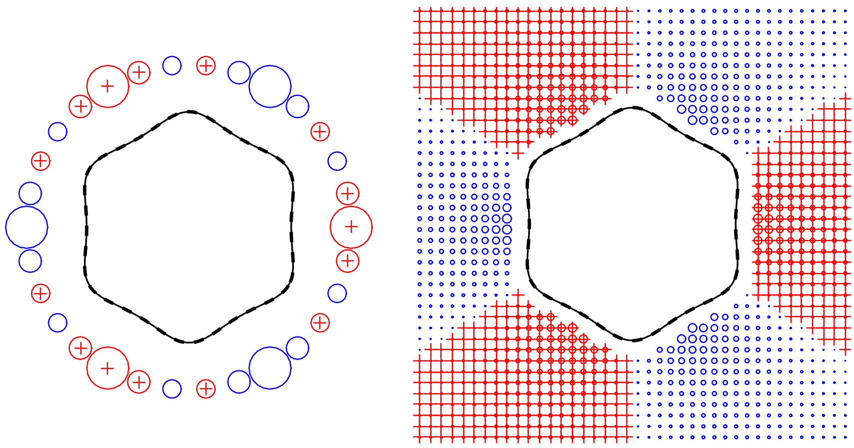

FIGURE 1. Example 1, solutions found when considering $m = 30$ electric currents arranged in a circle of radius $R = 1.5$ and $m = 1188$ inductors arranged in a square regular grid. Dashed line: target shape, thin solid line: equilibrium shape for the currents found. Currents are indicated by a circle with area proportional to the current intensity, the positive currents are indicated with a plus sign.

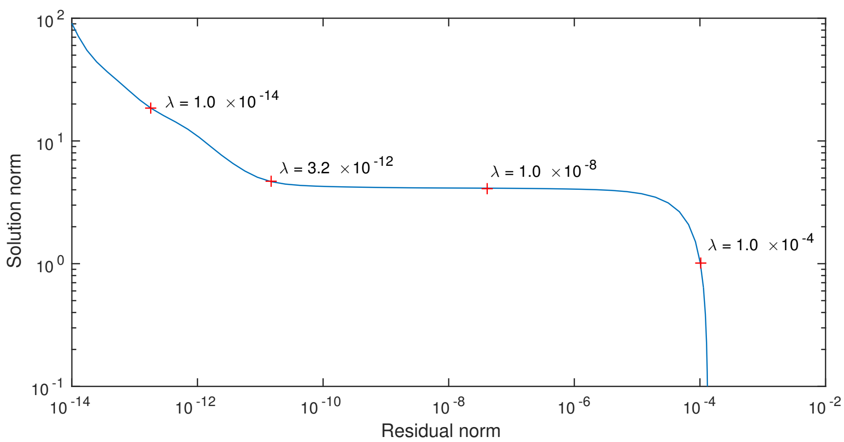

FIGURE 2. Example 1, L-curve obtained for the case of Figure 1-right. The solution norm $\|\boldsymbol{\alpha}\|$ is plotted against the residual norm $\|R(\boldsymbol{\alpha}, \beta)\|$. Four values of the parameter λ are indicated. The value $\lambda = 1.0 \times 10^{-14}$ provides a too high solution norm, the value $\lambda = 3.2 \times 10^{-12}$ would be chosen by the L-curve criterion, the value $\lambda = 1 \times 10^{-8}$ provides the still acceptable solution of Figure 1-right, and the value $\lambda = 1.0 \times 10^{-4}$ provides a too high residual norm.

$\lambda = 0$. The solution of 1188 inductors cannot be obtained without applying regularization. Figure 2 shows that when $\lambda < 3.2 \times 10^{-12}$ the solution implies high electric currents and is not acceptable. The L-curve criterion (Hansen and O'Leary, 1993) can then be used to choose an appropriate value for the regularization parameter.

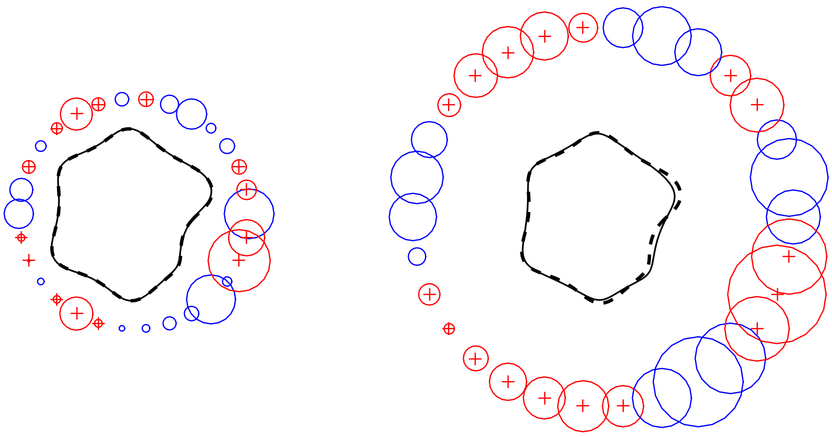

Figure 3. Example 2, solutions found when considering $m = 30$ electric currents arranged in circles of radius $R = 1.5$ and $R = 2.5$.

The second example considers the target shape of Example 2 in (Canelas et al., 2014). The concavity of target shape has the effect of increasing the intensities of the surrounding inductors. When experimenting with inductors located in a circle of larger radius the accuracy is deteriorated. If a radius $R = 2.5$ is considered then the concavity is lost as is shown in Figure 3.

The next two examples consider target shapes that were not found by solving direct equilibrium problems. Example 3 correspond to an ellipse of principal diameters equal to 2.5 and 1.6. Example 4 considers the same target shape of Example 3 in Canelas et al. (2014). The solutions found when considered $m = 30$ inductors arranged in a circle of radius $R = 1.5$ for the ellipse and radius $R = 3.0$ for Example 4 are depicted in Figure 4. Note that the ellipse is accurately obtained, while the target shape of Example 4 cannot be shaped accurately by considering inductors arranged in a circle. Note also that in both examples the inductors located close to the target shape have less intensities that the inductors located far from it. This fact suggests that we could find more accurate and more economical solutions by locating the inductors close to the target shape.

We then solve Example 4 considering a regular grid of $m = 152$ possible inductors, and considering a subset of the inductors of the grid which contains only the closest $m = 48$ inductors to the target shape. The results obtained are shown in Figure 5. It can be appreciated that the solutions found are in both cases very accurate and preferable over the one depicted in Figure 4. Note that the intensities in some of the remaining inductors have changed in value and also in the sign. Finally, note that the solution with only $m = 48$ inductors has the advantage of requiring the solution of a smaller linear system of equations.

6. Conclusions

A new method for determining the optimal locations and intensities of point-wise inductors in a inverse electromagnetic casting problem has been proposed in this paper. The main idea consists in minimizing a Kohn-Vogelius based

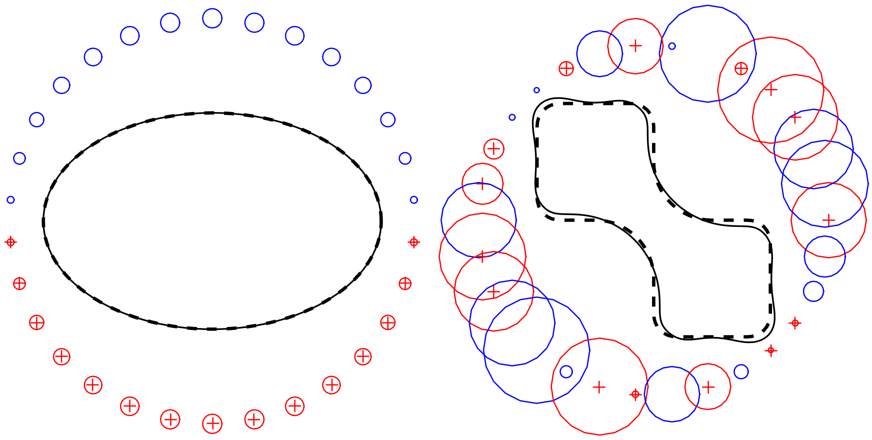

FIGURE 4. Examples 3 and 4, solutions found when considering $m = 30$ electric currents arranged in circles of radius $R = 1.5$ and $R = 3.0$, respectively.

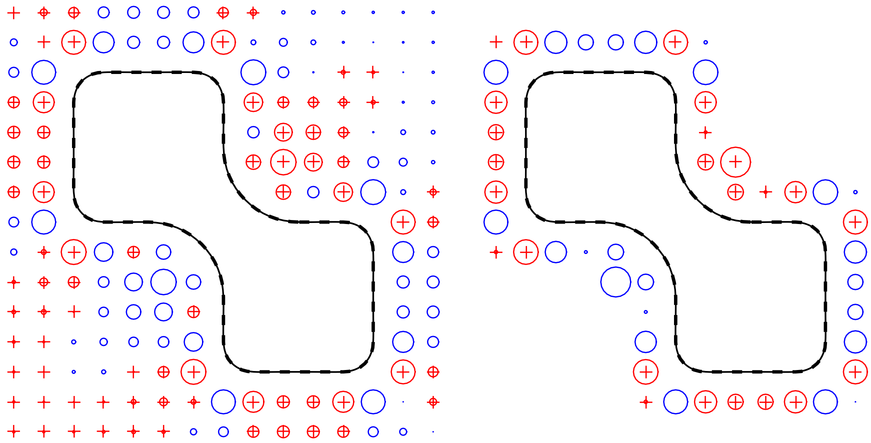

FIGURE 5. Examples, solutions found when considering $m = 152$ and $m = 42$ electric currents.

functional with respect to a set of admissible inductors, leading to a very simple and efficient (fast) non-iterative second-order algorithm. Since the associated Hessian matrix becomes ill-conditioned, a regularizing term has been introduced. As an adjacent result, it penalizes the higher electric current intensities, which leads to a reduction of the total electric energy consumption. Finally, some numerical examples have been resented, showing the efficiency of the proposed method to design suitable inductors.

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