ASYMPTOTIC ANALYSIS OF VARIATIONAL INEQUALITIES WITH APPLICATIONS TO OPTIMUM DESIGN IN ELASTICITY

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Abstract. Contact problems with given friction are considered for plane elasticity in the framework of shape-topological optimization. The asymptotic analysis of the second kind variational inequalities in plane elasticity is performed for the purposes of shape-topological optimization. To this end, the saddle point formulation for the associated Lagrangian is introduced for the variational inequality. The non-smooth term in the energy functional is replaced by pointwise constraints for the multipliers. The one term expansion of the strain energy with respect to the small parameter which governs the size of the singular perturbation of geometrical domain is obtained. The topological derivatives of energy functional are derived in closed form adapted to the numerical methods of shape-topological optimization. In general, the topological derivative (TD) of the elastic energy is defined through a limit passage when the small parameter governing the size of the topological perturbation goes to zero. TD can be used as a steepest-descent direction in an optimization process like in any method based on the gradient of the cost functional. In this paper, we deal with the topological asymptotic analysis in the context of contact problems with given friction. Since the problem is nonlinear, the domain decomposition technique combined with the Steklov-Poincaré pseudodifferential boundary operator is used for asymptotic analysis purposes with respect to the small parameter associated with the size of the topological perturbation. As a fundamental result, the expansion of the strain energy coincides with the expansion of the Steklov-Poincaré operator on the boundary of the truncated domain, leading to the expression for TD. Finally, the obtained TD is applied in the context of topology optimization of mechanical structures under contact condition with given friction.

1. Introduction

The optimum design in structural mechanics for problems governed by variational inequalities is considered in the literature using the energy functionals. Such variational problems are non-smooth, therefore one cannot expect the existence of classical shape gradients for general shape functionals depending on solutions of variational inequalities. We refer to the monograph [1] for the so-called conical shape derivatives of solutions to variational inequalities of the second kind. To this end, the solutions are given by the saddle-points of Lagrangian. The multipliers associated with the nondifferentiable terms of the elastic energy functional are subject to pointwise inequality constraints. The results obtained in [1] on shape sensitivity analysis are extended to the framework of topological sensitivity analysis. Actually, the topological derivatives of the energy functionals for contact problems with given friction with respect to the singular domain perturbations by creation of holes or inclusions are obtained. In this way, the asymptotic analysis is applied to numerical solution of optimum design for contact problems.

The complete theory of topological derivatives for linear elasticity in three spatial dimensions from the point of view of asymptotic analysis is given in [2], see also [3, 4, 5] for further developments on polarization tensors in elasticity or piezoelasticity. The results obtained for linear elasticity cannot be directly extended to variational inequalities. The

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difficulty, is non-smooth nature of variational inequalities. We refer to [6] for a result obtained in the case of the Signorini problem by using the classical approach of compound asymptotic expansions under the hypothesis of strict complementarity for the unknown solution of variational inequality.

In order to circumvent this difficulty, the new method of asymptotic analysis for variational inequalities based on the domain decomposition technique combined with the compound asymptotic expansions is proposed in [7] and it is presented with all details in monograph [8]. In this way, the topological derivatives of the non-smooth energy functional can be obtained. We show also that the theoretical results on asymptotic analysis are useful for numerical solution of an important optimum design problem.

Located in Florianópolis-Brazil, the Hercílio Luz bridge shown in Fig. 1 is a rare and significant bridge on many different aspects. It is recognized as the longest suspension bridge in Brazil. It was also the longest spanning eyebars suspension bridge in the world when built, between 1922 and 1926. One of the most noteworthy features of Hercílio Luz bridge is that the main cables are formed by eyebars chain rather than wire cables [9].

The Hercílio Luz bridge links Florianópolis island to the continent. Because of the very aggressive environment over the ocean, it has started to suffer from a high corrosion process. In particular, some of the eyebars in the chain have collapsed according to the red line shown in Fig. 1. Based on safety concerns, the Hercílio Luz bridge was closed for the first time in 1982, and reopened again in 1988. After a technical report analyzing the feasibility of keeping the traffic over the bridge, presented 1990, it was completely closed in 1991. Nowadays there is an effort on the rehabilitation of the bridge. Another famous bridge of similar design, the Silver Bridge over the Ohio River in the U.S.A., collapsed in 1967 due to a failure of a single eyebar in the suspension chain.

In this paper, we are interested in the redesign of an eyebar belonging to the eyebars chain of the Hercílio Luz cable bridge. The eyebars are linked through pin-joints, which are under contact condition with friction. There is a vast literature dealing with contact problems in elasticity. For the mathematical and numerical analysis of variational inequalities, see for instance the following monographs [10, 11, 12].

In order to deal with the design problem, the topology optimization of elastic structures under contact condition with given friction (stick-sleep condition) is considered. From mathematical point of view the model considered takes the form of a variational inequality of the second kind. The convenience for topological sensitivity analysis is an equivalent variational formulation as a saddle point of the Lagrangian. Such a formulation is already analyzed in [1] for the purposes of the shape sensitivity analysis. In this article new results on the existence of topological derivatives for the energy functional are derived. What is also important, the results obtained by the asymptotic analysis are used for the numerical solution of the shape-topological optimization problem. The paper is written in such a way that it is also accessible to the engineering community. We combine the asymptotic analysis in singularly perturbed geometrical domains which belongs to pure mathematics, with the numerical methods of shape-topological optimization which belongs to applied mathematics.

Optimization of structures submitted to contact boundary conditions has received considerable attention in the last decades [14, 15, 16, 17, 18]. In particular, we are interested in the topological derivative concept [8], which is defined as the first term (correction) of the asymptotic expansion of a given shape functional with respect to a small parameter that measures the size of singular domain perturbations, such as holes, inclusions, defects, source-terms and cracks. The topological derivative can naturally be used as a steepest-descent direction in an optimization process like in any method based on the gradient of the cost functional. Therefore, this relatively new concept has applications in many different fields such as shape and topology optimization, inverse problems, imaging processing,

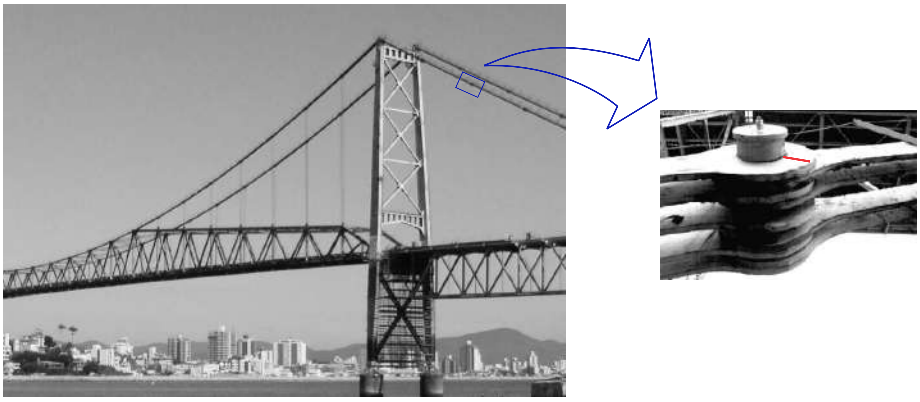

FIGURE 1. Hercílio Luz bridge in Florianópolis, Brazil. In highlight an eyebar chain (adapted from [13]).

multi-scale material design and mechanical modeling including damage, fracture evolution phenomena and control of crack propagation. See, for instance, [19, 20, 21, 22, 5, 23]. See also recent papers [24, 25, 26, 7] dealing with topological asymptotic analysis in the context of contact problems. An application of the same technique in the context of coupled electro-mechanical system can be found in [27]. These results are here extended to the case of given friction condition.

The asymptotic analysis of linear elasticity system in truncated domain has been performed in [28] by an application of the Green's function technique. In particular, the statement on a spherical hole can be found in Section 3.3, page 1766. In contrast to [28], the method developed in [7] has been designed for the purpose of asymptotic analysis in singularly perturbed domains for a class of nonlinear elasticity systems. It relies on the knowledge of the explicit solution of an auxiliary elasticity problem posed in a subdomain of simple geometry. In the ring in two spatial dimensions it is obtained by the complex Kolosov potentials [29]. By a result from the functional analysis on positive, self-adjoint operators, the expansion of the elastic energy in the ring gives rise to the expansion of the Steklov-Poincaré operator on the boundary of the topologically perturbed truncated domain, with the remainder uniformly bounded in the operator norm. In addition, the explicit solution allows us to replace the expression of the topological derivative unbounded in the energy norm by its equivalent form which is bounded in the energy norm. In this way, the truncated domain technique proposed in [28] was extended to the nonlinear contact problems in elasticity fully developed in [7]. We refer also [30] for the general case of elliptic systems and for the self-adjoint extensions of elliptic operators in punctured domains.

Therefore, following the original ideas presented by [7], in this paper the topological derivative is extended to the context of topology optimization of elastic structures under contact condition with given friction. Since the problem is nonlinear, the domain decomposition technique combined with the Steklov-Poincaré pseudo-differential boundary operator is used in the asymptotic analysis with respect to the small parameter associated with the size of the topological perturbation. As a fundamental result, the expansion of the strain energy coincides with the expansion of the Steklov-Poincaré operator on the boundary of the truncated domain, leading to the associated topological derivative. Finally, the obtained result is used in the redesign of the eyebar from Hercílio Luz cable bridge.

The paper is organized as follows. In Section 2 the mechanical problem we are dealing with is stated. The domain decomposition technique and the Steklov-Poincaré operator are presented in Section 3. The topological asymptotic analysis of the problem under

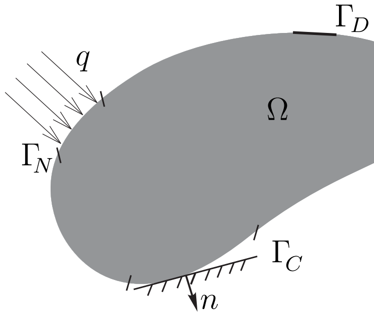

FIGURE 2. Domain representation.

consideration is developed in Section 4. The associated topological derivative, obtained in its closed form, is presented in Section 5, which represents the main theoretical result of the paper. In Section 6 the optimization problem we are dealing with is formulated and the redesign of the eyerbar under contact condition with given friction (stick-sleep condition) is presented. Finally, the paper ends with some concluding remarks in Section 7.

2. Problem formulation

Let us consider an open and bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary Γ, as shown in the sketch of Fig. 2. The boundary Γ consists of three mutually disjoint parts, namely, $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$. Displacements and boundary tractions are respectively prescribed on Γ_D and Γ_N, while on Γ_C there is a possible contact condition over a rigid foundation. We assume that the normal vector on Γ_C of both elastic and rigid surfaces are collinear, allowing to set just one normal vector n on the potential contact region Γ_C. Therefore, the mechanical problem consists in finding the minimizer $u \in \mathcal{K}$ of the following functional

$$\mathcal{J}(v) := \frac{1}{2}\int_\Omega \sigma(v)\cdot\nabla^s v - \int_{\Gamma_N} q\cdot v + \mu_a \int_{\Gamma_C} |v\cdot\tau|, \quad \forall v \in \mathcal{K}, \tag{2.1}$$

where $\sigma(v)$ is the Cauchy stress tensor, $q \in H^{1/2}(\Gamma_N;\mathbb{R}^2)$ is a given boundary traction and μ_a is a known friction coefficient. In addition, τ denotes the tangential vector on Γ and $\mathcal{K}$ is a convex and closed cone defined as

$$\mathcal{K} := \mathcal{K}(\Omega) = \{v \in H^1(\Omega;\mathbb{R}^2) : v = 0 \text{ on } \Gamma_D \text{ and } v\cdot n \leq 0 \text{ on } \Gamma_C\}. \tag{2.2}$$

In particular, the unique minimizer $u \in \mathcal{K}$ of (2.1) is solution of the following variational inequality

$$\int_\Omega \sigma(u)\cdot\nabla^s(v-u) - \int_{\Gamma_N} q\cdot(v-u) + \mu_a\int_{\Gamma_C}(|v\cdot\tau| - |u\cdot\tau|) \geq 0, \quad \forall v \in \mathcal{K}. \tag{2.3}$$

From the inequality (2.3) it follows that the strong form of the elasticity equilibrium problem under contact and stick-sleep conditions is stated as [31]: Find $u : \Omega \to \mathbb{R}^2$ such

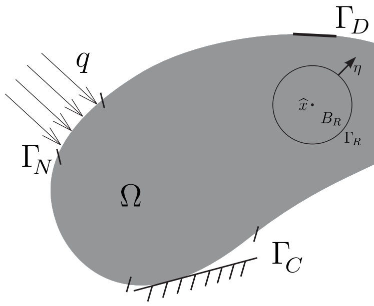

FIGURE 3. Domain decomposition representation.

that,

$$
\left\{
\begin{array}{rcll}
-\mathrm{div}(\sigma(u)) & = & 0 & \text{in } \Omega, \\
\sigma(u) & = & \mathbb{C}\nabla^s u & \text{in } \Omega, \\
u & = & 0 & \text{on } \Gamma_D, \\
\sigma(u)n & = & q & \text{on } \Gamma_N, \\
\left.
\begin{array}{rcl}
u \cdot n & \leq & 0 \\
\sigma^{nn}(u) & \leq & 0 \\
\sigma^{nn}(u)(u\cdot n) & = & 0 \\
\sigma^{n\tau}(u)(u\cdot\tau) + \mu_a|u\cdot\tau| & = & 0 \\
-\mu_a \leq \quad \sigma^{n\tau}(u) & \leq & \mu_a
\end{array}
\right\} & & & \text{on } \Gamma_C.
\end{array}
\right. \tag{2.4}
$$

Some terms in the above expressions still require explanation. The strain tensor $\nabla^s u$ is given by

$$
\nabla^s u := (\nabla u)^s = \frac{1}{2}(\nabla u + (\nabla u)^\top). \tag{2.5}
$$

The fourth order elastic tensor $\mathbb{C} = \mathbb{C}^\top$ is written as

$$
\mathbb{C} = 2\mu\mathbb{I} + \lambda(\mathrm{I}\otimes\mathrm{I}), \tag{2.6}
$$

with $\mathbb{I}$ and I representing the fourth and second order identity tensors, respectively, while $\mu,\ \lambda$ are used to denote the Lamé's coefficients. In addition, the normal component $\sigma^{nn}(u)$ of the stress tensor is defined as

$$
\sigma^{nn}(u) := \sigma(u)n\cdot n, \tag{2.7}
$$

while the shear component on the tangential plane $\sigma^{n\tau}(u)$ is given by

$$
\sigma^{n\tau}(u) := \sigma(u)n\cdot\tau. \tag{2.8}
$$

Note that the contact problem we are dealing with is a simplified model where the friction coefficient is assumed to be known. The reader may refer to the papers [32, 24, 33, 16] and the book [34] for an account on similar as well as more sophisticated models.

3. Domain Decomposition Technique

We start by decomposing Ω into two parts, namely $\Omega = \Omega_R \cup \overline{B_R}$, where $\Omega_R := \Omega\setminus\overline{B_R}$ and B_R, with boundary Γ_R, is used to denote a ball of radius $R > 0$ centered at an arbitrary point $\widehat{x}\in\Omega$. See sketch in Fig. 3. Then, we consider the following linear elasticity system in B_R: Given $\psi\in H^{1/2}(\Gamma_R;\mathbb{R}^2)$, find the displacement $w : B_R \mapsto \mathbb{R}^2$, such that

$$
\left\{
\begin{array}{rcll}
-\mathrm{div}(\sigma(w)) & = & 0 & \text{in } B_R, \\
\sigma(w) & = & \mathbb{C}\nabla^s w & \text{in } B_R, \\
w & = & \psi & \text{on } \Gamma_R.
\end{array}
\right. \tag{3.1}
$$

Using (3.1), we can define the Steklov-Poincaré pseudo-differential boundary operator:

$$\mathcal{A} : \psi \in H^{1/2}(\Gamma_R; \mathbb{R}^2) \mapsto \sigma(w)\eta \in H^{-1/2}(\Gamma_R; \mathbb{R}^2) \tag{3.2}$$

where η is the outward normal vector to the boundary Γ_R. Observe that by the definition of the operator $\mathcal{A}$, the solution w of (3.1) satisfies:

$$\int_{B_R} \sigma(w) \cdot \nabla^s w = \int_{\Gamma_R} \mathcal{A}(\psi) \cdot \psi. \tag{3.3}$$

That is, the energy inside B_R is equal to the energy associated with the Steklov-Poincaré operator on the boundary Γ_R. In addition, by setting $\psi = u_{|\Gamma_R}$, we have $w = u_{|B_R}$ and $u^R = u_{|\Omega_R}$. Thus,

$$\mathcal{J}(u) = \mathcal{J}^R(u^R), \tag{3.4}$$

where the functional $\mathcal{J}^R(u^R)$ is defined as

$$\mathcal{J}^R(u^R) := \frac{1}{2}\int_{\Omega_R} \sigma(u^R) \cdot \nabla^s u^R - \int_{\Gamma_N} q \cdot u^R + \mu_a \int_{\Gamma_C} |u^R \cdot \tau| + \frac{1}{2}\int_{\Gamma_R} \mathcal{A}(u^R) \cdot u^R, \tag{3.5}$$

with the minimizer $u^R \in \mathcal{K}^R := \mathcal{K}(\Omega_R)$ solution to the following variational inequality

$$\int_{\Omega_R} \sigma(u^R) \cdot \nabla^s(v - u^R) - \int_{\Gamma_N} q \cdot (v - u^R) + \mu_a \int_{\Gamma_C} (|v \cdot \tau| - |u^R \cdot \tau|) \\ + \int_{\Gamma_R} \mathcal{A}(u^R) \cdot (v - u^R) \geq 0, \quad \forall v \in \mathcal{K}^R. \tag{3.6}$$

4. Topological asymptotic analysis

We are interested on the topological derivative of the energy shape functional (2.1) with respect to the nucleation of a small inclusion. Therefore, let us consider that the domain Ω is subjected to a topological perturbation confined in a small arbitrary shaped set $\omega_\varepsilon(\widehat{x})$ of size ε and center at an arbitrary point $\widehat{x}$ of Ω, such that $\overline{\omega_\varepsilon(\widehat{x})} \subset \Omega$. In the case of a perforation, for example, the topologically perturbed domain is obtained as follows $\Omega_\varepsilon(\widehat{x}) = \Omega \setminus \overline{\omega_\varepsilon(\widehat{x})}$. Then, we assume that a given shape functional $J(\Omega_\varepsilon(\widehat{x}))$, associated to the topologically perturbed domain, admits the following topological asymptotic expansion [8]

$$J(\Omega_\varepsilon(\widehat{x})) = J(\Omega) + f(\varepsilon) D_T J(\widehat{x}) + o(f(\varepsilon)), \tag{4.1}$$

where $D_T J(\widehat{x})$ is the topological derivative of the shape functional J at $\widehat{x}$, $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0$ and ε is the parameter governing the size of the perturbation. According to the classical definition of the topological derivative [35], from (4.1) we have

$$D_T J(\widehat{x}) := \lim_{\varepsilon \to 0} \frac{J(\Omega_\varepsilon(\widehat{x})) - J(\Omega)}{f(\varepsilon)}. \tag{4.2}$$

In this work, the topological perturbation is characterized by the nucleation of a small circular inclusion B_ε of radius $0 < \varepsilon < R$ and center at $\widehat{x} \in \Omega$, which is assumed to be far enough from the potential contact region Γ_C. This inclusion is filled with different material property from the background represented by a piecewise constant function γ_ε defined as

$$\gamma_\varepsilon = \gamma_\varepsilon(x) := \begin{cases} 1, & \text{if } x \in \Omega \setminus \overline{B_\varepsilon}, \\ \gamma, & \text{if } x \in B_\varepsilon, \end{cases} \tag{4.3}$$

with $\gamma \in \mathbb{R}^+$ used to represent the contrast on the material properties. Then, the corresponding perturbed problem consists in finding the minimizer $u_\varepsilon \in \mathcal{K}$ of the functional

$$\mathcal{J}_\varepsilon(v) := \frac{1}{2}\int_\Omega \gamma_\varepsilon \sigma(v) \cdot \nabla^s v - \int_{\Gamma_N} q \cdot v + \mu_a \int_{\Gamma_C} |v \cdot \tau|, \quad \forall v \in \mathcal{K}. \tag{4.4}$$

The element $u_\varepsilon \in \mathcal{K}$ is solution of the following perturbed variational inequality

$$\int_\Omega \gamma_\varepsilon \sigma(u_\varepsilon) \cdot \nabla^s(v - u_\varepsilon) - \int_{\Gamma_N} q \cdot (v - u_\varepsilon) + \mu_a \int_{\Gamma_C} (|v \cdot \tau| - |u_\varepsilon \cdot \tau|) \geq 0, \quad \forall v \in \mathcal{K}. \tag{4.5}$$

The associated strong system reads: Find $u_\varepsilon : \Omega \to \mathbb{R}^2$ such that,

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma_\varepsilon \sigma(u_\varepsilon)) & = & 0 & \text{in } \Omega, \\ \sigma(u_\varepsilon) & = & \mathbb{C}\nabla^s u_\varepsilon & \text{in } \Omega, \\ u_\varepsilon & = & 0 & \text{on } \Gamma_D, \\ \sigma(u_\varepsilon)n & = & q & \text{on } \Gamma_N, \\ [\![u_\varepsilon]\!] & = & 0 & \text{on } \partial B_\varepsilon, \\ [\![\gamma_\varepsilon \sigma(u_\varepsilon)]\!]n & = & 0 & \text{on } \partial B_\varepsilon, \\ u_\varepsilon \cdot n & \leq & 0 & \\ \sigma^{nn}(u_\varepsilon) & \leq & 0 & \\ \sigma^{nn}(u_\varepsilon)(u_\varepsilon \cdot n) & = & 0 & \text{on } \Gamma_C. \\ \sigma^{n\tau}(u_\varepsilon)(u_\varepsilon \cdot \tau) + \mu_a|u_\varepsilon \cdot \tau| & = & 0 & \\ -\mu_a \leq \quad \sigma^{n\tau}(u_\varepsilon) & \leq & \mu_a & \end{array} \right. \tag{4.6}$$

Now, we apply the domain decomposition technique to the above topologically perturbed problem, as shown in the sketch of Fig. 4. This decomposition allows us to proceed with the topological asymptotic analysis in a simple geometrical domain, which is separated from the analysis of the shape function (4.4) endowed with a nondifferentiable term. Therefore, in the ball B_R we consider the following linear elasticity system associated with the perturbed domain: Given $\psi \in H^{1/2}(\Gamma_R; \mathbb{R}^2)$, find the displacement $w_\varepsilon : B_R \mapsto \mathbb{R}^2$, such that

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma_\varepsilon \sigma(w_\varepsilon)) & = & 0 & \text{in } B_R, \\ \sigma(w_\varepsilon) & = & \mathbb{C}\nabla^s w_\varepsilon & \text{in } B_R, \\ w_\varepsilon & = & \psi & \text{on } \Gamma_R, \\ [\![w_\varepsilon]\!] & = & 0 & \text{on } \partial B_\varepsilon, \\ [\![\gamma_\varepsilon \sigma(w_\varepsilon)]\!]n & = & 0 & \text{on } \partial B_\varepsilon. \end{array} \right. \tag{4.7}$$

Using problem (4.7), we define the topologically perturbed counterpart of the Steklov-Poincaré boundary operator $\mathcal{A}_\varepsilon : H^{1/2}(\Gamma_R; \mathbb{R}^2) \mapsto H^{-1/2}(\Gamma_R; \mathbb{R}^2)$ as follows

$$\mathcal{A}_\varepsilon(\psi) = \sigma(w_\varepsilon)\eta \quad \text{on} \quad \Gamma_R. \tag{4.8}$$

By the definition of the operator $\mathcal{A}_\varepsilon$, the solution w_ε of (4.7) satisfies:

$$\int_{B_R} \gamma_\varepsilon \sigma(w_\varepsilon) \cdot \nabla^s w_\varepsilon = \int_{\Gamma_R} \mathcal{A}_\varepsilon(\psi) \cdot \psi. \tag{4.9}$$

That is, the energy inside the ball $B_R \supset B_\varepsilon$ is equal to the energy associated with the Steklov-Poincaré operator $\mathcal{A}_\varepsilon$ on Γ_R. We observe that by setting $\psi = u_{\varepsilon|_{\Gamma_R}}$ we have $w_\varepsilon = u_{\varepsilon|_{B_R}}$ and $u_\varepsilon^R = u_{\varepsilon|_{\Omega_R}}$, which implies the equality

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_\varepsilon^R(u_\varepsilon^R), \tag{4.10}$$

where the functional $\mathcal{J}_\varepsilon^R(u_\varepsilon^R)$ is written as

$$\mathcal{J}_\varepsilon^R(u_\varepsilon^R) := \frac{1}{2}\int_{\Omega_R} \sigma(u_\varepsilon^R) \cdot \nabla^s u_\varepsilon^R - \int_{\Gamma_N} q \cdot u_\varepsilon^R + \mu_a \int_{\Gamma_C} |u_\varepsilon^R \cdot \tau| + \frac{1}{2}\int_{\Gamma_R} \mathcal{A}_\varepsilon(u_\varepsilon^R) \cdot u_\varepsilon^R, \tag{4.11}$$

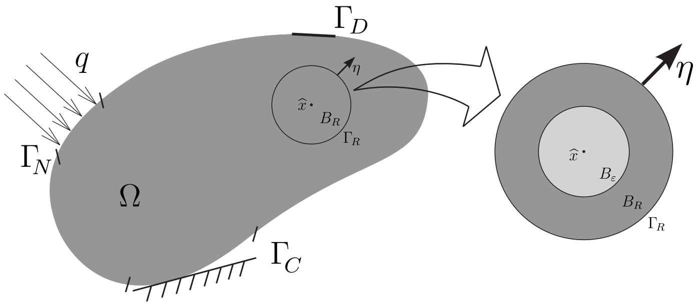

FIGURE 4. Perturbed domain decomposition.

with the minimizer $u_\varepsilon^R \in \mathcal{K}^R$ given by the unique solution of the variational inequality:

$$\int_{\Omega_R} \sigma(u_\varepsilon^R) \cdot \nabla^s(v - u_\varepsilon^R) - \int_{\Gamma_N} q \cdot (v - u_\varepsilon^R) + \mu_a \int_{\Gamma_C} (|v \cdot \tau| - |u_\varepsilon^R \cdot \tau|) + \int_{\Gamma_R} \mathcal{A}_\varepsilon(u_\varepsilon^R) \cdot (v - u_\varepsilon^R) \geq 0, \quad \forall v \in \mathcal{K}^R. \quad (4.12)$$

Before evaluate the topological derivative, we present two important results. The first one ensures the existence of the topological derivative associated with the problem under analysis. The second result proves the topological differentiability of the energy shape functional.

Proposition 1. *Let u^R and u_ε^R be solution to* (4.5) *and* (4.12)*, respectively, and assume that*

$$\mathcal{A}_\varepsilon = \mathcal{A} - \varepsilon^2 \mathcal{B} + \mathcal{R}_\varepsilon, \quad (4.13)$$

in the operator norm $\mathcal{L}(H^{1/2}(\Gamma_R;\mathbb{R}^2); H^{-1/2}(\Gamma_R;\mathbb{R}^2))$, with $\mathcal{B}$ used to denote a bounded linear operator and

$$\|\mathcal{R}_\varepsilon\|_{\mathcal{L}(H^{1/2}(\Gamma_R;\mathbb{R}^2);H^{-1/2}(\Gamma_R;\mathbb{R}^2))} = o(\varepsilon^2). \quad (4.14)$$

Then, the following estimate holds true:

$$\|u_\varepsilon^R - u^R\|_{H^1(\Omega^R;\mathbb{R}^2)} \leq C\varepsilon^2. \quad (4.15)$$

Proof. By taking $v = u_\varepsilon^R$ in (4.5) and $v = u^R$ in (4.12), we obtain:

$$\int_{\Omega_R} \sigma(u^R)\cdot\nabla^s(u_\varepsilon^R - u^R) - \int_{\Gamma_N} q\cdot(u_\varepsilon^R - u^R) + \mu_a \int_{\Gamma_C} (|u_\varepsilon^R\cdot\tau| - |u^R\cdot\tau|) + \int_{\Gamma_R} \mathcal{A}(u^R)\cdot(u_\varepsilon^R - u^R) \geq 0 \quad (4.16)$$

and

$$\int_{\Omega_R} \sigma(u_\varepsilon^R)\cdot\nabla^s(u^R - u_\varepsilon^R) - \int_{\Gamma_N} q\cdot(u^R - u_\varepsilon^R) + \mu_a \int_{\Gamma_C} (|u^R\cdot\tau| - |u_\varepsilon^R\cdot\tau|) + \int_{\Gamma_R} \mathcal{A}_\varepsilon(u_\varepsilon^R)\cdot(u^R - u_\varepsilon^R) \geq 0. \quad (4.17)$$

Then, after adding (4.16) and (4.17), we write

$$\int_{\Omega_R} \sigma(u_\varepsilon^R - u^R) \cdot \nabla^s(u_\varepsilon^R - u^R) + \int_{\Gamma_R} [\mathcal{A}_\varepsilon(u_\varepsilon^R) - \mathcal{A}(u^R)] \cdot (u_\varepsilon^R - u^R) \leq 0. \quad (4.18)$$

Now, assuming (4.13), we have that

$$\int_{\Omega_R} \sigma(u_\varepsilon^R - u^R)\cdot\nabla^s(u_\varepsilon^R - u^R) + \int_{\Gamma_R} \mathcal{A}(u_\varepsilon^R - u^R)\cdot(u_\varepsilon^R - u^R) \leq \varepsilon^2 \int_{\Gamma_R} \mathcal{B}(u_\varepsilon^R)\cdot(u_\varepsilon^R - u^R) + o(\varepsilon^2), \quad (4.19)$$

and by the coercivity of the bilinear form on the left-hand side of (4.19) it follows

$$C_0\|u_\varepsilon^R - u^R\|^2_{H^1(\Omega_R;\mathbb{R}^2)} \leq \varepsilon^2 \int_{\Gamma_R} \mathcal{B}(u_\varepsilon^R)\cdot(u_\varepsilon^R - u^R) + o(\varepsilon^2). \tag{4.20}$$

And also,

$$\begin{aligned}\|u_\varepsilon^R - u^R\|^2_{H^1(\Omega_R;\mathbb{R}^2)} &\leq C_1\varepsilon^2\|\mathcal{B}(u_\varepsilon^R)\|_{H^{-1/2}(\Gamma_R;\mathbb{R}^2)}\|u_\varepsilon^R - u^R\|_{H^{1/2}(\Gamma_R;\mathbb{R}^2)} \\ &\leq C_2\varepsilon^2\|u_\varepsilon^R - u^R\|_{H^1(\Omega_R;\mathbb{R}^2)}.\end{aligned} \tag{4.21}$$

Finally, we obtain (4.15) with $C = C_2/C_0$ independent of the small parameter ε. □

Lemma 2. *The perturbed energy shape functional $\mathcal{J}_\varepsilon^R(u_\varepsilon^R)$ in (4.11) is differentiable with respect to $\varepsilon \to 0$. In particular, it admits the asymptotic expansion*

$$\mathcal{J}_\varepsilon^R(u_\varepsilon^R) = \mathcal{J}^R(u^R) - \frac{\varepsilon^2}{2}\langle\mathcal{B}(u^R), u^R\rangle_{\Gamma_R} + o(\varepsilon^2), \tag{4.22}$$

where $\langle\phi,\varphi\rangle_{\Gamma_R}$ is used to denote the inner product on Γ_R, that is

$$\langle\phi,\varphi\rangle_{\Gamma_R} = \int_{\Gamma_R}\phi\cdot\varphi. \tag{4.23}$$

Proof. By taking into account that $u_\varepsilon^R \in \mathcal{K}^R$ is the minimizer of (4.11) and $u^R \in \mathcal{K}^R$ is the minimizer of (3.5), the following inequalities hold true

$$\mathcal{J}_\varepsilon^R(u_\varepsilon^R) - \mathcal{J}^R(u_\varepsilon^R) \leq \mathcal{J}_\varepsilon^R(u_\varepsilon^R) - \mathcal{J}^R(u^R) \leq \mathcal{J}_\varepsilon^R(u^R) - \mathcal{J}^R(u^R). \tag{4.24}$$

Using the definitions of $\mathcal{J}_\varepsilon^R$ and $\mathcal{J}^R$, considering the expansion (4.13) and after organizing all the terms, we have

$$\frac{\mathcal{J}_\varepsilon^R(u^R) - \mathcal{J}^R(u^R)}{\varepsilon^2} = -\frac{1}{2}\langle\mathcal{B}(u^R), u^R\rangle_{\Gamma_R} + \langle\mathcal{R}_\varepsilon(u^R), u^R\rangle_{\Gamma_R}. \tag{4.25}$$

Thus, it follows that

$$\lim_{\varepsilon\to 0}\left(\frac{\mathcal{J}\varepsilon^R(u^R) - \mathcal{J}^R(u^R)}{\varepsilon^2}\right) = -\frac{1}{2}\langle\mathcal{B}(u^R), u^R\rangle_{\Gamma_R}. \tag{4.26}$$

Similarly, we write

$$\frac{\mathcal{J}_\varepsilon^R(u_\varepsilon^R) - \mathcal{J}^R(u_\varepsilon^R)}{\varepsilon^2} = -\frac{1}{2}\langle\mathcal{B}(u_\varepsilon^R), u_\varepsilon^R\rangle_{\Gamma_R} + \langle\mathcal{R}_\varepsilon(u_\varepsilon^R), u_\varepsilon^R\rangle_{\Gamma_R}. \tag{4.27}$$

Now, taking into account the strong convergence of the minimizers in the energy space, Proposition 1, we obtain

$$\lim_{\varepsilon\to 0}\left(\frac{\mathcal{J}_\varepsilon^R(u_\varepsilon^R) - \mathcal{J}^R(u_\varepsilon^R)}{\varepsilon^2}\right) = -\frac{1}{2}\langle\mathcal{B}(u^R), u^R\rangle_{\Gamma_R}. \tag{4.28}$$

From the above limits and (4.24) we conclude that

$$\lim_{\varepsilon\to 0}\left(\frac{\mathcal{J}_\varepsilon^R(u_\varepsilon^R) - \mathcal{J}^R(u^R)}{\varepsilon^2}\right) = -\frac{1}{2}\langle\mathcal{B}(u^R), u^R\rangle_{\Gamma_R}. \tag{4.29}$$

Therefore, we can write (4.22). □

5. Topological Derivative Formula

In this section, the topological derivative associated with the problem under analysis is obtained in its closed form. Before proceeding, let us state the following important result, whose proof can be found in [8]:

Lemma 3. *The energy inside B_R admits the asymptotic expansion:*

$$\int_{B_R} \sigma_\varepsilon(w_\varepsilon) \cdot \nabla^s(w_\varepsilon) = \int_{B_R} \sigma(w) \cdot \nabla^s(w) - \varepsilon^2 \mathbb{P}_\gamma \sigma(w) \cdot \nabla^s w + o(\varepsilon^2), \tag{5.1}$$

where the polarization tensor $\mathbb{P}_\gamma$ is given by the following fourth order isotropic tensor

$$\mathbb{P}_\gamma = \frac{\pi(1-\gamma)}{1+\gamma a_2}\left((1+a_2)\mathbb{I} + \frac{1}{2}(a_1 - a_2)\frac{1-\gamma}{1+\gamma a_1}\mathrm{I}\otimes\mathrm{I}\right), \tag{5.2}$$

with $0 < \gamma < \infty$ and the parameters a_1, a_2 given by

$$a_1 = \frac{\lambda+\mu}{\mu} \quad \text{and} \quad a_2 = \frac{\lambda + 3\mu}{\lambda+\mu}. \tag{5.3}$$

Now, note that by Proposition 1 we write

$$\langle \mathcal{A}_\varepsilon(\phi), \varphi\rangle = \langle \mathcal{A}(\phi), \varphi\rangle - \varepsilon^2 \langle \mathcal{B}(\phi), \varphi\rangle + o(\varepsilon^2), \quad \forall \phi,\ \varphi. \tag{5.4}$$

Then, from Lemma 3 and expansion (5.4) we conclude that the expansion of the strain energy in B_R coincides with the expansion of the Steklov-Poincaré operator on the boundary Γ_R. Therefore, it follows that

$$\int_{\Gamma_R} \mathcal{B}(\phi) \cdot \psi = \mathbb{P}_\gamma \sigma(\phi) \cdot \nabla^s \varphi, \quad \forall \phi,\ \varphi. \tag{5.5}$$

Thus, by Lemma 2 together with equation (5.5) we have

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}(u) = -\frac{1}{2}\varepsilon^2 \mathbb{P}_\gamma \sigma(u) \cdot \nabla^s u + o(\varepsilon^2), \tag{5.6}$$

where we have also considered the equalities (3.4) and (4.10). Finally, by choosing $f(\varepsilon) = \varepsilon^2$, we have the main result of the paper, namely:

Theorem 4. *The topological derivative of the shape functional $\mathcal{J}(u)$, defined in* (2.1)*, is given by*

$$D_T \mathcal{J}(x) = -\frac{1}{2}\mathbb{P}_\gamma \sigma(u(x)) \cdot \nabla^s u(x), \quad \forall x \in \Omega. \tag{5.7}$$

6. Numerical application

In this section the obtained topological derivative (5.7) is applied in the context of topology optimization of structures under contact condition with given friction. The idea is to redesign an eyebar belonging to the eyebars chain of the Hercílio Luz cable bridge, as presented in Section 1.

Therefore, let us consider a hold-all domain $\mathcal{D} \subset \mathbb{R}^2$ such that $\Omega \subset \mathcal{D}$. The topology optimization problem we are dealing with consists in minimizing the total potential energy for a given amount of material, that is:

$$\begin{cases} \underset{\Omega\subset\mathcal{D}}{\text{Minimize}}\ \mathcal{J}(u) \\ \text{subject to} \quad |\Omega| \le M, \end{cases} \tag{6.1}$$

where $|\Omega|$ is the Lebesgue measure of Ω and M is the desired volume at the end of the optimization process. To deal with the volume constraint we use the Linear Penalization Method. Thus, problem (6.1) is rewritten as following:

$$\underset{\Omega\subset\mathcal{D}}{\text{Minimize}}\ \mathcal{F}_\Omega(u) := \mathcal{J}(u) + \beta|\Omega|, \tag{6.2}$$

where $\beta = \beta^\star/V_0$ is a penalization parameter with $\beta^\star > 0$, and V_0 denotes the initial volume of the structure. The optimization problem (6.2) is solved by using the topology optimization algorithm proposed in [36], which relies on the topological derivative concept and a level-set domain representation method. In particular, the topological derivative is used as a feasible descent direction to minimize the cost functional (6.2). The reader may refer to [27, 36, 37] for more details and applications of this algorithm.

The numerical implementation was performed by using FEniCS software [38]. Once we are interested in redesigning the eyebar shown in Fig. 1, we consider as original design the domain shown in Fig. 5(a). On the other hand, the initial guess is given by a rectangle of dimensions 400×1651 mm^2 with a semicircular hole of radius r_1, as shown in Fig. 5(b). See also [13]. We consider vertical symmetry conditions, represented by dashed lines. The potential contact region is given by the boundary of the semicircle of radius r_1. The structure is submitted to a distributed load q as shown in the sketch of Fig 5, whose resultant is denoted by $Q = 24000\,N$. In addition, the eyebar is made with steel, whose Young's modulus E, Poisson ratio ν, friction coefficient μ_a and stress limit of the material $\overline{\sigma}$ are given in Table 1, together with additional geometrical properties. Finally, we set the volume penalization parameter $\beta^\star = 4 \times 10^4$.

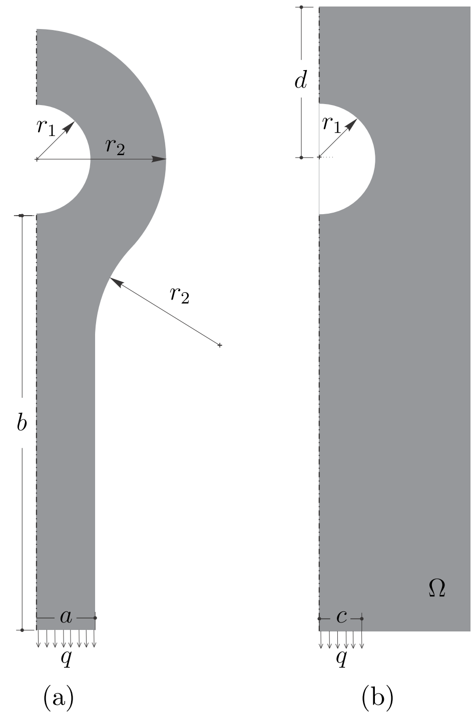

FIGURE 5. Original design (a) and initial guess (b).

TABLE 1. Material properties and geometrical parameters.

ν	μ_a	E	$\overline{\sigma}$	Q	a	b	c	d	r_1	r_2
0.3	0.2	210×10^3	440	24000	152	850	127	451	146	350
–	–	MPa	MPa	N	mm	mm	mm	mm	mm	mm

The obtained result is presented in Fig. 6. The normalized von Mises stresses distribution are shown in Fig. 7. Note that in both cases the maximal value is bounded by one. However, the original design has approximately 46.08% (orange area) while the obtained optimized eyebar has approximately 38.62% of volume fraction, which corresponds to more than 16% of volume reduction without violate the stress limit of the material.

FIGURE 6. Original design (a) and obtained result (b).

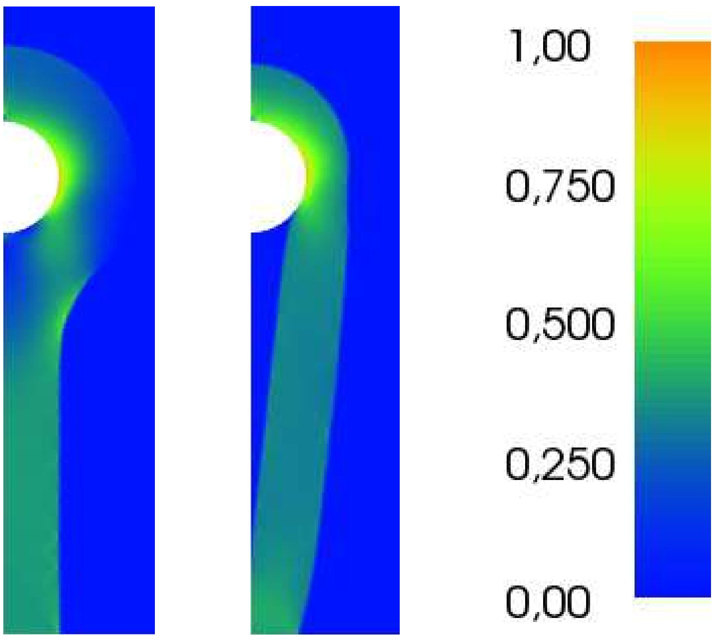

FIGURE 7. Stresses distributions: original design (left) and optimized result (right).

7. Conclusion

In this paper, the topological derivative concept has been applied in the context of contact problems in elasticity with given friction. Since the problem is nonlinear, the domain decomposition technique together with the Steklov-Poincaré pseudo-differential boundary operator were used in the topological asymptotic analysis of the energy shape functional with respect to the nucleation of a small circular inclusion. From such an analysis, the associated topological derivative has been derived in its closed form. The obtained result has been applied in a case study concerning the redesign of an eyebar belonging to the eyebars chain of the Hercílio Luz cable bridge. As a result, the obtained optimal design is much more efficient from the mechanical point of view in comparison with the original one, since its volume has been reduced about 16% while the maximal von Mises stress does not exceed the stress limit of the material.

The proposed method is general and it can be applied for numerical solution of shape-topology optimization of contact problems in three spatial dimensions. On the other hand, the topological sensitivity analysis of contact with the Coulomb friction is still an unsolved and difficult problem.

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