ROBUST COMPLIANCE TOPOLOGY OPTIMIZATION BASED ON THE TOPOLOGICAL DERIVATIVE CONCEPT

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Abstract. In this paper we present an approach for robust compliance topology optimization under volume constraint. The compliance is evaluated considering a point-wise worst case scenario. Analogously to Sequential Optimization and Reliability Assessment, the resulting robust optimization problem can be decoupled into a deterministic topology optimization step and a reliability analysis step. This procedure allows us to use topology optimization algorithms already developed with only small modifications. Here, the deterministic topology optimization problem is addressed with an efficient algorithm based on the topological derivative concept and a level-set domain representation method. The reliability analysis step is handled as in the Performance Measure Approach. Several numerical examples are presented showing the effectiveness of the proposed approach.

1. Introduction

Compliance topology optimization of continuum structures has been subject of intense research over the last decades. See for instance the book [9]. In this case, one seeks the optimum distribution of material inside a given domain that leads to a structure with minimal compliance for a given amount of material. Results obtained by topology optimization have a valuable field of application in the design of structures and mechanical components. To deal with this problems, several approaches for topology optimization have been proposed. Among these approaches, we highlight methods based on the concept of topological derivatives. This concept allows the development of efficient and robust topology optimization algorithms. See for instance the book [30].

On the other hand, in most cases of practical interest, the parameters of the optimization problem are not deterministic variables. Applied forces intensities, for example, may not be completely known or may present stochastic variations. Optimization considering uncertainties has been extensively studied in the last decades and several strategies to tackle the problem have been proposed, see for example the reviews presented in [10, 26, 31, 35] and references therein for details.

In particular, most works on the subject can be grouped into three main classes: Reliability Based Design Optimization, Robust Optimization and Risk Based Optimization. In the case of Reliability Based Design Optimization the constraints of the optimization problem are stated as to impose a maximum failure probability of the system [23, 27, 28]. In the case of Robust Optimization, the goal is to obtain an optimum design that is least sensitive to variations and uncertainties of the variables [6, 14, 33, 38]. In the case of Risk Based Optimization, the failure probability of the system is used to evaluate the total cost of failures, that together with other costs compose the objective function of the problem [8].

Topology optimization of continuum structures considering uncertainties has already been addressed in the literature. In the context of Reliability Based Design Optimization,

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volume minimization with stress constraints has been studied in [15, 27, 32], while design of Microelectromechanical systems has been presented in [24, 28]. Regarding the Robust Optimization, compliance based topology design under uncertain loads has been studied in [4, 14, 19, 39], synthesis of compliant mechanism together with design of structures with minimum compliance have been tackled in [12, 25], whereas boundary uncertainties have been investigated in [18]. Some applications of Risk Based Optimization to problems concerning structural optimization have been presented in [7, 8], while a general review of select applications have been presented in [16].

It has been observed that uncertainties consideration may lead to solutions conceptually different from deterministic optimization. This fact supports the application of optimization under uncertainties in several cases of practical interest. However, in general, uncertainty based optimization requires much more computation effort than its deterministic counterpart. This fact limits the range of practical applications of uncertainty based optimization.

In this context, the goal of this paper is to present an approach for robust compliance topology optimization that is both general and computationally efficient. The compliance is evaluated considering a point-wise worst case scenario, found within an event set of possible outcomes of the random parameters. Analogously to Sequential Optimization and Reliability Assessment [13], the resulting robust optimization problem can be decoupled into a deterministic topology optimization step with modified parameters and a reliability analysis step. Since the proposed approach decouples the nested robust optimization problem, it is computationally efficient and easy to implement provided that it requires only existing algorithms. In the reliability analysis step the point-wise worst case scenarios are found as in the Performance Measure Approach. The topology optimization algorithm proposed in [3] is used to solve the associated deterministic problem, which is based on the topological derivative concept and a level-set domain representation method. From the mathematical point of view, the topological derivative concept has been proved to be robust with respect to uncertainties on the data [22]. In this paper therefore, the theory developed in [22] is also confirmed from the numerical point of view. In addition, the resulting decoupled topology optimization problem with modified parameters can also be treated by using the topological derivative concept [30]. Therefore, the main contribution of our work consists in presenting a simple and general formulation for the robust topology optimization problem, by combining well established methods together with existing theoretical results. The proposed alternating algorithm is very efficient and of simple computational implementation. In particular, the numerical examples presented at the end of this paper address the case of topology optimization with load intensities as random variables. The general statement of the problem is also presented and can be extended to other random variables as well, which however may require some additional computational effort.

This paper is organized as follows. The robust design optimization problem is stated in Section 2. In particular, we describe how the Sequential Optimization and Reliability Assessment concept can be used to decouple the nested robust optimization problem into a deterministic optimization step and a reliability analysis step. The resulting deterministic topology optimization problem is presented in Section 3, which is solved using the topological derivative concept together with a level-set domain representation method. Some numerical examples are presented in Section 4, showing the efficiency and simplicity of the proposed approach as well as the importance of considering uncertainties in the topology optimization problem. Finally, some concluding remarks are presented in Section 5.

2. Proposed Approach

Let us introduce a hold-all structural domain $\mathcal{D} \subset \mathbb{R}^2$ and a subdomain $\Omega \subset \mathcal{D}$. We consider the minimization of the structural compliance under volume constraint, which can be written as:

$$\text{Find}\, \Omega^d \subset \mathcal{D}\,, \text{such that:}$$
$$\begin{cases} \underset{\Omega \subset \mathcal{D}}{\textbf{Minimize}} & \mathcal{K}_\Omega \;=\; \displaystyle\int_{\mathcal{D}} \sigma_\Omega \cdot \varepsilon_\Omega\,, \\ \textbf{Subject to} & |\Omega| \;\leq\; M\,, \end{cases} \tag{2.1}$$

where $|\Omega|$ is the Lebesgue measure of Ω (i.e. volume of the structure) and M is a prescribed amount of material. The quantity $\sigma_\Omega \cdot \varepsilon_\Omega$ is twice the strain energy density, with σ_Ω and ε_Ω used to denote the stress and the strain tensors, respectively, obtained from the solution of a linear elasticity system in Ω.

When some parameters of the problem (e.g. load intensities) are random variables, the compliance becomes a random variable itself. In this case the compliance is not known in the deterministic sense and the deterministic problem from Eq. (2.1) can result in inefficient designs in practice. In order to take into account such uncertainties it is necessary to apply some Robust Optimization strategy. An interesting approach is to minimize the expected value or the standard deviation of the compliance, for example [14]. However, we point out that the resulting problem is not a Reliability Based Design Optimization problem in the strict sense since there are no failure probabilities involved, and thus we classify it as a Robust Optimization problem.

Since uncertainties based optimization problems can easily become intractable from the computational effort point of view, it is essential to approach the problem in a way that allow efficient computational solution. A common approach is to transform the uncertainties based problem into an equivalent deterministic problem. In this way, once we find the equivalent deterministic problem, efficient existing algorithms can be applied.

According to [6, 17], Robust Optimization methods can be broadly classified into those based on probabilistic modeling and those based on non-probabilistic modeling. In probabilistic modeling we consider that the probability distribution functions (PDF) of some random variables are known and then take into account failure probabilities of the constraints, mean value of the objective function or variance of the objective function, for example. On the other hand, non-probabilistic approaches often assume that some parameters are unknown but belong to a bounded set, but no other statistical information is used.

A common approach used in the case of non-probabilistic modeling is known as Worst Case Design Optimization [17]. In this approach, the deterministic optimization problem is solved considering a Worst Case Scenario among the possible outcomes of the unknown parameters. The worst Case Scenario [21] is given by the combination of parameters that lead to the worst performance of the system being optimized. One important aspect that requires special attention in the context of Worst Case Design Optimization is related to the properties of the problem that gives the Worst Case Scenario. When the original problem is convex, the Worst Case Scenario problem frequently becomes non-convex. This issue is thoroughly discussed in [6, 17].

Another important aspect of Worst Case Design Optimization is how the set of possible outcomes is defined. Taking the set too large may result in too conservative designs. Taking the set too small, on the other hand, can lead to non robust solutions. In this work we follow the Worst Case Design Optimization approach, since it leads to a computationally efficient decoupling of the Robust Optimization problem. However, we use statistical information of the unknown parameters in order to build the bounded set of possible outcomes.

2.1. **The bounded set of possible outcomes.** Here we assume that the unknown parameters are represented by a random vector $\mathbf{x}$ with known statistical information. In order to write the bounded set of possible outcomes in a concise manner we assume that the random vector $\mathbf{x}$ is composed by independent Gaussian components x_i. Then we can obtain the normalized random vector $\mathbf{u}$ by using the transformation $T : \mathbf{x} \to \mathbf{u}$ given by

$$\mathrm{u}_i = \frac{\mathrm{x}_i - \mu_i}{s_i}, \tag{2.2}$$

where u_i are independent Normal random variables with mean equal to zero and unitary standard deviation and μ_i and s_i are the mean and the standard deviation of the random variable x_i.

Using the normalized random vector $\mathbf{u} = T(\mathbf{x})$ we can define the event set as

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^m : \|T(\mathbf{x})\| \leq \beta_t\}, \tag{2.3}$$

where β_t is some prescribed parameter. Note that the event set is composed only by outcomes as distant as β_t from the mean value $T(\mathbf{x}) = \mathbf{0}$. In other words, only outcomes obtained within β_t standard deviations from the mean value are considered.

From the definition of the cumulative distribution function (CDF)

$$P(u \leq \beta_t) = \Phi(\beta_t) \tag{2.4}$$

where Φ is the cumulative distribution function (CDF) of a Normal random variable. Consequently, by imposing $\|T(\mathbf{x})\| \leq \beta_t$ we are actually imposing a target probability of occurrence

$$P_t = P(\|T(\mathbf{x})\| \leq \beta_t) = \Phi(\beta_t). \tag{2.5}$$

From Eq. (2.5) we note that by prescribing a given value β_t we are actually prescribing that the set $\mathcal{E}$ is comprised only by outcomes which CDF correspond to a target probability P_t. Consequently, we can obtain the value of β_t for a given P_t from

$$\beta_t = \Phi^{-1}(P_t). \tag{2.6}$$

consider a larger or smaller portion of the possible outcomes as needed. Finally, we note that β_t is related to the Hasofer-Lind reliability index [20, 29], in the sense that it defines a distance from the expected value in the normalized space.

2.2. **The robust optimization approach .** In order to build a Robust Optimization approach, we search for the Worst Case Scenario point-wisely inside the structural domain. At each point of the domain, the strain energy density Worst Case Scenario $\mathbf{x}^*$ for events in the set $\mathcal{E}$ can be found by solving the following problem:

$$\begin{aligned} &\text{Find } \mathbf{x}^* \in \mathbb{R}^m \text{, such that:} \\ &\begin{cases} \underset{\mathbf{x} \in \mathcal{E}}{\textbf{Maximize}} & \sigma_\Omega(\mathbf{x}) \cdot \varepsilon_\Omega(\mathbf{x}) \,, \\ \textbf{Subject to} & \|T(\mathbf{x})\| \leq \beta \,, \end{cases} \end{aligned} \tag{2.7}$$

where β is a given parameter representing the reliability index. The constraint $\|T(\mathbf{x})\| \leq \beta$ states that we are searching for a solution in the set $\mathcal{E}$ as defined in Eq. (2.3). Finally, the notation $\sigma_\Omega \cdot \varepsilon_\Omega = \sigma_\Omega(\mathbf{x}) \cdot \varepsilon_\Omega(\mathbf{x})$ represents the fact that the strain energy depends on unknown parameters $\mathbf{x}$.

Minimization of the strain energy density considering a point-wise Worst Case Scenario can be achieved by substitution of $\mathbf{x}^*$ into (2.1). The resulting optimization problem is:

$$\text{Find}\,\Omega^p \subset \mathcal{D}\,\text{, such that:}$$
$$\begin{cases} \underset{\Omega\subset\mathcal{D}}{\textbf{Minimize}} & \mathcal{K}_\Omega \;=\; \displaystyle\int_{\mathcal{D}} \sigma_\Omega(\mathbf{x}^*)\cdot\varepsilon_\Omega(\mathbf{x}^*)\;, \\ \textbf{Subject to} & |\Omega| \;\leq\; M\;, \end{cases} \tag{2.8}$$

where $\sigma_\Omega(\mathbf{x}^*)\cdot\varepsilon_\Omega(\mathbf{x}^*)$ indicates that the strain energy density is evaluated considering the point-wise Worst Case Scenario $\mathbf{x}^*$ as defined in (2.7).

In the context of this work it is important to note the difference between Ω^d, solution to (2.1), and Ω^p solution to (2.8). Note that Ω^p is the optimum solution of the Robust Optimization problem while Ω^d is the solution of the deterministic optimization problem.

2.3. **Sequential Optimization and Reliability Assessment .** Unfortunately, it is not efficient to address the problem from (2.8) directly, since $\mathbf{x}^*$ is defined implicitly by means of (2.7). This leads to a nested optimization problem, where it is necessary to solve the maximization problem (2.7) at each point of the domain before evaluating the objective function from (2.8).

However, it is possible to decouple these two optimization problems using concepts from Sequential Optimization and Reliability Assessment. We first define the solution of the problem from (2.8) as the pair $(\Omega^p, \mathbf{x}^p)$. We note that, $\mathbf{x}^*$ is defined as the Worst Case Scenario obtained for an arbitrary topology, while $\mathbf{x}^p$ is defined as the special Worst Case Scenario obtained with the optimum topology Ω^p.

If, for some reason, $\mathbf{x}^p$ was known beforehand, it would not be necessary to solve the problem from (2.7) before solving the problem from (2.8). In this case, the optimum topology Ω^p can be found directly by taking the Worst Case Scenario $\mathbf{x}^p$ in the problem from (2.8). This puts in evidence that, once the Worst Case Scenario $\mathbf{x}^p$ is known, the problem (2.8) becomes a deterministic topology optimization problem. On the other hand, if the optimum topology Ω^p is known, the Worst Case Scenario $\mathbf{x}^p$ can be found directly by solving (2.7) point-wisely with the optimum topology Ω^p.

The problem from (2.7) is stated in the same form frequently encountered in Reliability Based Design Optimization using the Performance Measure Approach [26, 34, 37]. Consequently, it can be solved by efficient schemes already developed in the literature. Here this problem is solved using the Advanced Mean Value algorithm, originally proposed in [36] and described in details in [37]. A brief explanation of the algorithm is presented in Appendix A.

Obviously, neither Ω^p nor $\mathbf{x}^p$ are known beforehand, unless the problem is already solved. However, it is possible to start from initial approximations and iterate until accurate solutions are found. In fact, let us indicate some iteration number with (k), then an iterative algorithm for solving the Robust Optimization problem can be summarized as presented in the alternating Algorithm 1. The procedure is started with an arbitrary topology $\Omega^{(0)}$. With this topology we find the Worst Case Scenario $\mathbf{x}^{(0)}$ at each point of interest by solving (2.7) with $\Omega^{(0)}$. With the Worst Case Scenario $\mathbf{x}^{(0)}$ we find a new topology $\Omega^{(1)}$ and so on, until convergence is achieved.

Convergence of the solution can be checked on changes of Ω or $\mathbf{x}$ between two successive iterations. We claim however that there are no mathematical proofs showing that the proposed alternating algorithm converges. In any case, at least for the numerical experiments presented in Section 4, we did not observe any convergence or stability issues. Besides, the point-wise Worst Case Scenario $\mathbf{x}$ is evaluated only at interest points of the domain. In this work the Worst Case Scenario is evaluated at the integration points of the finite elements mesh.

It is also interesting to note that, conceptually, at each iteration of the alternating algorithm we require the solution of a deterministic topology optimization problem. However, in practice it seems to be more efficient to carry only a few iterations of the deterministic topology optimization algorithm before proceeding to the next step of the Robust Optimization algorithm. In this work we carry only a single step of the deterministic topology algorithm before updating the Worst Case Scenario point-wisely.

Algorithm 1: Alternating Robust Optimization Algorithm

input : Initial topology $\Omega^{(0)}$
output: $(\Omega^p,\, \mathbf{x}^p)$

Set $k \leftarrow 0$;
Compute $\mathbf{x}^{(0)}$ by solving (2.7) with $\Omega = \Omega^{(0)}$;
while *convergence is not achieved* **do**
    Make $k \leftarrow k + 1$;
    Compute $\Omega^{(k)}$ by solving (2.8) with $\mathbf{x}^* = \mathbf{x}^{(k-1)}$;
    Compute $\mathbf{x}^{(k)}$ by solving (2.7) with $\Omega = \Omega^{(k)}$;
end while
Return $(\Omega^p,\, \mathbf{x}^p)$ as solution.

3. Structural Topology Optimization Problem

The robust topology optimization we are dealing with consists in solving problem (2.8) evaluated at the worst case scenario, solution to the problem (2.7). Without loss of generality, we restrict ourselves to uncertainties on the intensity of the applied loads (note that the problem stated through (2.7) and (2.8) is general and can be applied when other parameters are random variables). In this case, we can use linear superposition of the effects in order to save computational effort. Let us consider that the applied load can be written as a linear combination of q_i independent loads, with $i = 1, ..., m$. Then, the stress σ_Ω and strain ε_Ω tensors at each point can be written as

$$\sigma_\Omega := \sum_{i=1}^{m} \mathrm{x}_i^* \sigma(u_i) \quad \text{and} \quad \varepsilon_\Omega := \sum_{i=1}^{m} \mathrm{x}_i^* \varepsilon(u_i) \;, \tag{3.1}$$

where, in this context, x_i^* can be interpreted as load scale factors. The canonical strain $\varepsilon(u_i)$ and stress $\sigma(u_i)$ tensors are obtained from each individual load q_i. For this purpose, we must to solve the following set of canonical variational problems related to the structural response when each load q_i is applied:

$$\begin{aligned} &\text{Find } u_i \in \mathcal{V}\text{, such that} \\ &\left\{ \begin{array}{l} \displaystyle\int_{\mathcal{D}} \sigma(u_i) \cdot \varepsilon(\eta) = \int_{\Gamma_N} q_i \cdot \eta \quad \forall \eta \in \mathcal{V}\;, \\ \qquad\qquad\qquad \text{with } \sigma(u_i) := \rho \mathbb{C} \varepsilon(u_i), \end{array} \right. \end{aligned} \tag{3.2}$$

where

$$\varepsilon(\varphi) = \frac{1}{2}(\nabla \varphi + (\nabla \varphi)^\top) \tag{3.3}$$

is the linearized Green tensor,

$$\mathbb{C} = \frac{E}{1-\nu^2}((1-\nu)\mathbb{I} + \nu \mathrm{I} \otimes \mathrm{I}) \tag{3.4}$$

is the elasticity tensor, I and $\mathbb{I}$ are the second and fourth identity tensors, respectively, E is the Young modulus and ν the Poisson ratio. The piecewise constant function ρ

$$\rho(x) = \begin{cases} 1, & \text{if} \quad x \in \Omega \, , \\ \rho_0, & \text{if} \quad x \in \mathcal{D} \setminus \overline{\Omega} \, , \end{cases} \tag{3.5}$$

with $0 < \rho_0 \ll 1$, is used to mimic voids. That is, the original structural problem, where the structure itself consists of the domain Ω of given elastic properties and the remaining part $\mathcal{D} \setminus \overline{\Omega}$ of the hold-all is empty (has no material), is approximated by means of the two-phase material distribution given by (3.5) over $\mathcal{D}$ where the empty region $\mathcal{D} \setminus \overline{\Omega}$ is occupied by a material (the soft phase) with Young's modulus, $\rho_0 E$, much lower than the given Young's modulus, E, of the structure material (the hard phase). The space $\mathcal{V}$ is defined as

$$\mathcal{V} := \{ \varphi \in H^1(\Omega; \mathbb{R}^2) : \varphi_{|\Gamma_D} = 0 \} \, . \tag{3.6}$$

Here, Γ_D and Γ_N are Dirichlet and Neumann boundaries, respectively, such that $\partial\mathcal{D} = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ with $\Gamma_D \cap \Gamma_N = \emptyset$, and q_i is the prescribed traction on Γ_N. See details in Fig. 1.

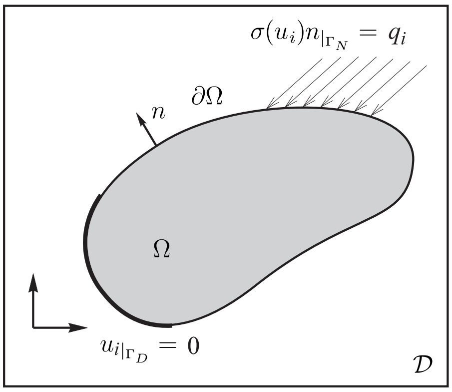


FIGURE 1. Sketch of the elasticity problem.

The volume constraint in problem (2.8) is trivially imposed via the Augmented Lagrangian Method [11]. In particular, it can be rewritten as an unconstrained optimization problem as follows:

$$\underset{\Omega \subset \mathcal{D}}{\textbf{Minimize}} \; \mathcal{J}_\Omega = \mathcal{K}_\Omega + \lambda_1 g_\Omega^+ + \frac{\lambda_2}{2} (g_\Omega^+)^2 \, , \tag{3.7}$$

where $g_\Omega^+ := \max\{g_\Omega, -\lambda_1/\lambda_2\}$, with $g_\Omega = (|\Omega| - M)/M$, $\lambda_2 > 0$ is a fixed multiplier and λ_1 is updated according to the following recursive formula:

$$\begin{aligned} \lambda_1^{(0)} &= 0 \\ \lambda_1^{(n+1)} &= \max\{0, \lambda_1^{(n)} + \lambda_2 g_\Omega\} \, . \end{aligned} \tag{3.8}$$

The deterministic structural compliance optimization problem (2.8) is solved by using the topological derivative concept, which has been shown to be robust with respect to uncertainties on the data [22]. The topological derivative measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation, such as the insertion of holes, inclusions or source-terms [30]. It is defined through a limit passage when the small parameter governing the size of the topological perturbation goes to zero. Therefore, the topological derivative can be used as a steepest-descent direction in an optimization process like in any method based on the gradient of the cost functional. In contrast to traditional topology optimization methods, the topological derivative formulation does not require a material model concept based on intermediary densities. Thus, interpolation schemes are unnecessary. In addition, it has the advantage of providing an analytical form for the topological sensitivity which allows to obtain the optimal design in few iterations. In fact, the resulting approach leads to a topology design algorithm of remarkably efficiency and of simple computational implementation, which does not require post-processing procedures of any kind and features only a minimal number of user-defined algorithmic parameters. In particular, the topological derivative of the shape functional $\mathcal{J}_\Omega$ with respect to the nucleation of a small circular inclusion with different material property from the background, represented by a contrast γ, is given by the sum

$$D_T\mathcal{J}_\Omega = D_T\mathcal{K}_\Omega \mp \max\{0, \lambda_1 + \lambda_2 g_\Omega\} \ , \tag{3.9}$$

where $\mp$ means that if we remove material the volume becomes smaller, on the other hand, if we insert material the volume increases. The topological derivative for the compliance denoted by $D_T\mathcal{K}_\Omega$ is know and can be found in the book [30, ch.5 pp.158], for instance. It is given by:

$$D_T\mathcal{K}_\Omega = \mathbb{P}\sigma_\Omega \cdot \varepsilon_\Omega \ , \tag{3.10}$$

where the Pólya-Szegö polarization tensor $\mathbb{P}$ is

$$\mathbb{P} = \frac{1-\gamma}{1+\gamma a_2}\left((1+a_2)\mathbb{I} + \frac{1}{2}(a_1 - a_2)\frac{1-\gamma}{1+\gamma a_1}\mathrm{I}\otimes\mathrm{I}\right) \ , \tag{3.11}$$

with

$$a_1 = \frac{1+\nu}{1-\nu} \quad \text{and} \quad a_2 = \frac{3-\nu}{1+\nu} \ . \tag{3.12}$$

In addition, the contrast γ in the material property, is defined as follows

$$\gamma(x) = \begin{cases} \rho_0, & \text{if} \quad x \in \Omega \ , \\ \frac{1}{\rho_0}, & \text{if} \quad x \in \mathcal{D} \setminus \overline{\Omega} \ . \end{cases} \tag{3.13}$$

Remark 1. *The polarization tensor $\mathbb{P}$ is here given by an isotropic fourth order tensors because we are dealing with circular inclusions as topological perturbations. For arbitrary shaped inclusions the reader may refer to the book* [1]*, for instance. On the other hand, there are two main advantages in using circular inclusions in the context of topology optimization, which are:*

- *The associated topological derivative is given by a closed formula depending on the solution to the original unperturbed problem.*
- *There are optimality conditions rigorously derived in* [2]*, allowing for using the topological derivative together with a level-set domain representation method as a steepest-descent direction in a topology optimization algorithm* [3]*.*

In fact, the above result (3.10) *together with a level-set domain representation method proposed in* [3] *(see also* [5] *for more details) is used for solving the deterministic compliance topology optimization problem* (2.8) *with $\mathbf{x}^* = \mathbf{x}^{(k)}$ fixed, necessary in step (3) of the algorithm proposed in Section 2.3.*

4. Numerical Examples

In the numerical examples we assume that the structures are under a plane stress state. The Young's modulus and the Poisson ratio are respectively given by $E = 1.0$ and $\nu = 0.3$, while the contrast $\rho_0 = 10^{-4}$. The angle θ defined in [3], representing the optimality condition, has converged to a value smaller than 1^o in all cases. The mechanical problem is discretized into linear triangular finite elements and two steps of uniform mesh refinement were performed during the iterative process. Since only load intensities are taken as random variables, linear superposition of effects is used wherever it is possible. We point out that in order to compute the topological derivative associated with each iteration of the deterministic optimization algorithm [3], one Finite Element Analysis (FEA) is required per iteration. The robust algorithm, on the other hand, requires one FEA for each applied load per iteration, since linear superposition of the effects is used. Finally, the mean compliance and the compliance standard deviation of the structures are computed using Monte Carlo Simulation with 10^4 samples [20].

4.1. **Example 1.** Let us consider a square panel of size 1×1 clamped on the top and submitted to a pair of loads, as shown in Fig. 2. The loading consists of two forces $q_1 = (2.0, -1.0)$ and $q_2 = (-2.0, -1.0)$ applied on the middle of the bottom edge. The hold-all domain is discretized into a uniform mesh with 6400 elements and 3281 nodes. The required volume fraction is set as $M = 25\%$, while the parameter $\lambda_2 = 1.0$.

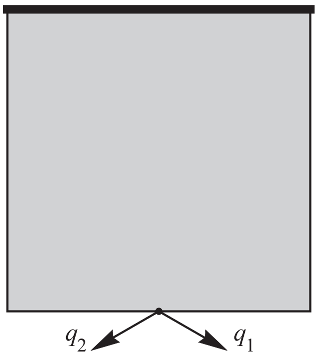


FIGURE 2. Example 1: initial guess and boundary conditions.

If topology optimization is made considering all parameters as deterministic, the optimal topology obtained at iteration 34 is that presented in Fig. 3, which is a benchmark solution to this problem.

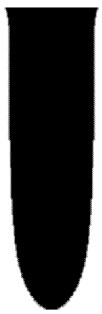


FIGURE 3. Example 1: result at iteration 34 for the deterministic case.

We now assume that the load scale factors are represented by Gaussian random variables. The event set is defined with $\beta = 2.0$ and the load scale factors have unitary mean, namely $\mu_1 = \mu_2 = 1.0$. In the first case, the standard deviations are equal and given by $s_1 = s_2 = 0.20$. The optimum topology obtained at iteration 48 is presented in Fig. 4. As expected, the optimum solution of the robust optimization problem leads to a V-bracket structure, able to handle the horizontal components of the loads that result for cases where the two forces do not have the same intensities.

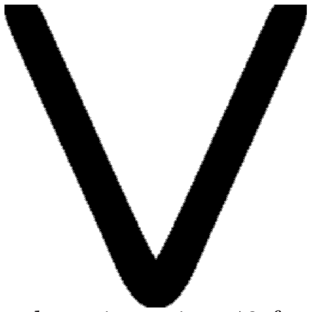
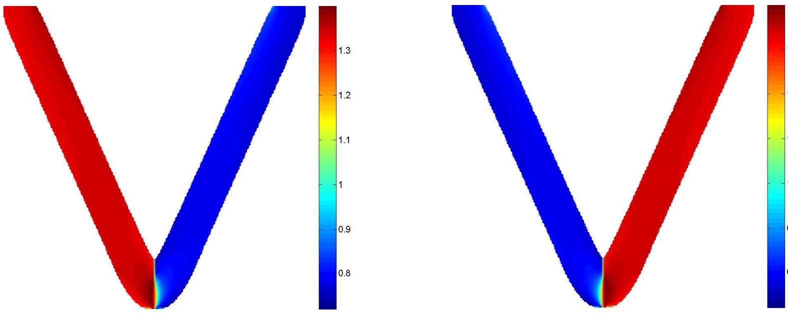


FIGURE 4. Example 1: result at iteration 48 for the robust case with $s_1 = s_2 = 0.20$.

The distributions of the load scale factors in the worst case scenario x_1 and x_2 are shown in Fig. 5.



(a) index x_1 (b) index x_2

FIGURE 5. Example 1: load scale factors x_1 and x_2 for the robust case with $s_1 = s_2 = 0.20$.

Finally, we repeat the same experiment by setting $s_1 = 0.20$ and $s_2 = 0.02$. As expected, in this case the final topology, obtained at iteration 47, is not symmetric as can be seen in Fig. 6.

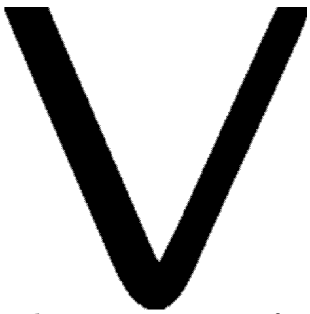
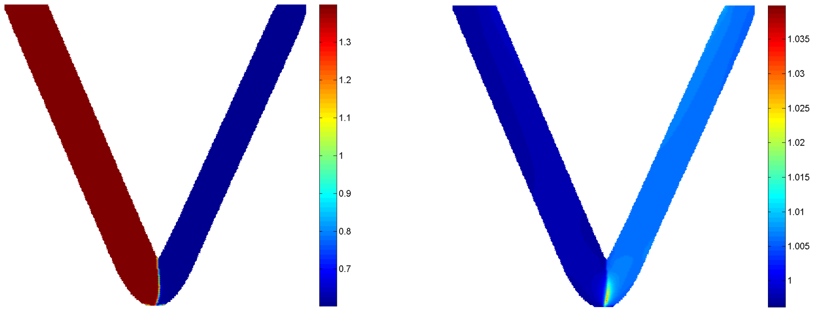


FIGURE 6. Example 1: result at iteration 47 for the robust case with $s_1 = 0.20$ and $s_2 = 0.02$.

The distributions of the load scale factors in the worst case scenario x_1 and x_2 are shown in Fig. 7.



(a) index x_1 (b) index x_2

FIGURE 7. Example 1: load scale factors x_1 and x_2 for the robust case with $s_1 = 0.20$ and $s_2 = 0.02$.

4.2. **Example 2.** Now, let us consider the design of a tower clamped on the bottom and submitted to a pair of loads, as shown in Fig. 8. The hold-all domain is given by a T-bracket structure, whose lengths of the horizontal and vertical branches are respectively 0.4 and 0.6 measured along their center lines and both have identical width of 0.2. It is discretized into a uniform mesh with 25600 elements and 13001 nodes. The loading consists of a pair of forces $q_1 = (0.0, -1.0)$ and $q_2 = (0.0, -1.0)$ applied on the two opposite bottom corners of the horizontal branch. The required volume fraction is set as $M = 40\%$, while the parameter $\lambda_2 = 2.0$. The load scale factors have unitary mean $\mu_1 = \mu_2 = 1.0$ and identical standard deviations given by $s_1 = s_2 = 0.20$.

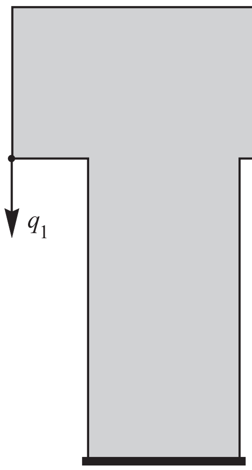


FIGURE 8. Example 2: initial guess and boundary conditions.

The optimal topology considering all parameters as deterministic obtained at iteration 36 is presented in Fig. 9. The mean compliance and the compliance standard deviation of this structure were found to be 89.26 and 40.32, respectively. This results in a coefficient of variation of 0.45.

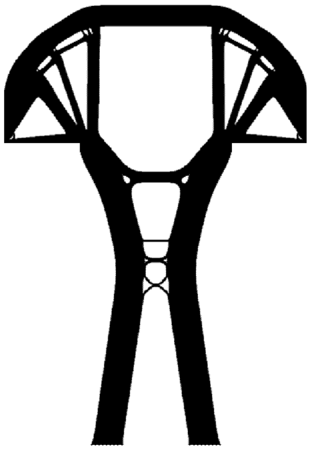


FIGURE 9. Example 2: result at iteration 36 for the deterministic case.

Now we consider the robust case, with the load scale factors represented again by Gaussian random variables. The event set is defined with $\beta = 2.0$. The optimum topology obtained at iteration 45 is presented in Fig. 10. The mean compliance and the compliance standard deviation of this structure are given respectively by 71.34 and 19.74, which results in a coefficient of variation of 0.28. This result indicates that the proposed robust approach is able to obtain much better solutions than its deterministic counterpart.

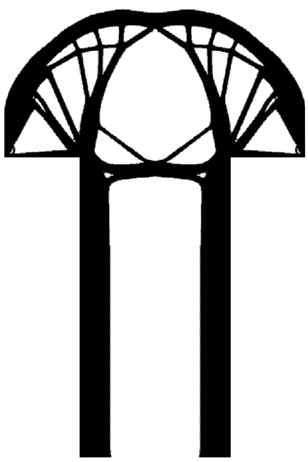
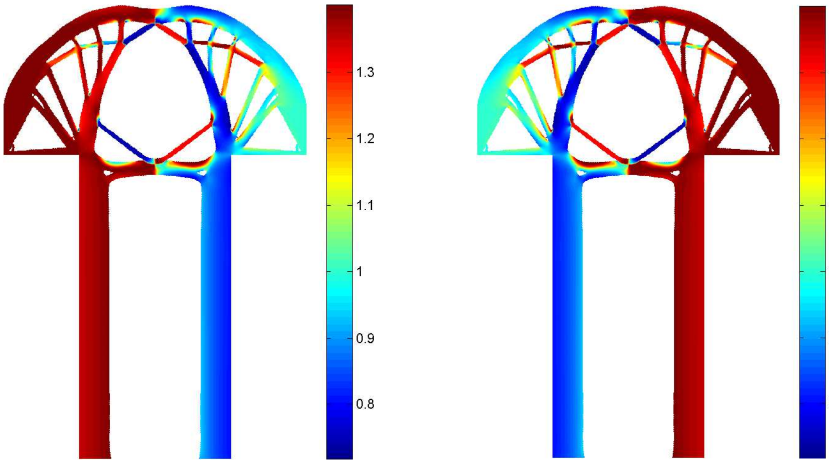


FIGURE 10. Example 2: result at iteration 45 for the robust case.

The distributions of the load scale factors in the worst case scenario x_1 and x_2 are shown in Fig. 11. It is interesting to note that the point-wise worst case scenarios present significant variation over the structural domain.



(a) index x_1 (b) index x_2

FIGURE 11. Example 2: load scale factors x_1 and x_2 for the robust case.

4.3. **Example 3.** The third example considers the design of a simply supported beam submitted to the loads shown in Fig. 12. The hold-all domain is rectangular with dimensions 6×1. It is modelled using symmetry conditions and a uniform mesh with 4800 elements. The loading consists of forces $q_1 = (0.0, -2.0)$ and $q_2 = (-1.0, 0.0)$. In this case, the horizontal force can be seen as a pre-stress applied to the structure, with the objective of reducing the tensile stresses in the lower part of the structure. The required volume fraction is set as $M = 50\%$, while the parameter $\lambda_2 = 2.0$. Here we assume that the load scale factors are Gaussian variables with mean equal to $\mu_1 = \mu_2 = 1.0$ and standard deviation equal to $s_1 = s_2 = 0.2$, while $\beta = 2.0$.

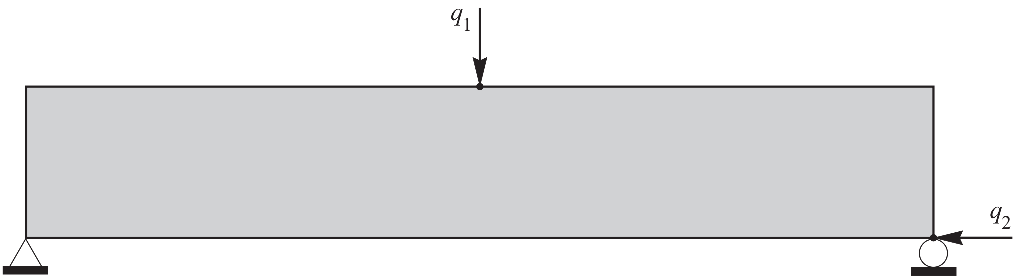


FIGURE 12. Example 3: initial guess and boundary conditions.

The solution of the deterministic problem obtained at iteration 45 is presented in Fig. 13. Note that the structure is supported by inclined bars, that rely on the horizontal force q_2 in order to be stable. The mean compliance and the compliance standard deviation of this structure are respectively given by 153.27 and 75.53, which results in a coefficient of variation of 0.49.

FIGURE 13. Example 3: result at iteration 45 for the deterministic case.

The solution of the robust problem obtained at iteration 77 is presented in Fig. 14. We note that the optimum topology is significantly different from the deterministic case, mainly because the horizontal force uncertainty results in a optimum structure without the simple inclined bars near the supports. The mean compliance and the compliance standard deviation of this structure were found to be 129.79 and 55.84, respectively. This results in a coefficient of variation of 0.43, which indicates that the proposed robust approach is indeed able to obtain better solutions than its deterministic counterpart.

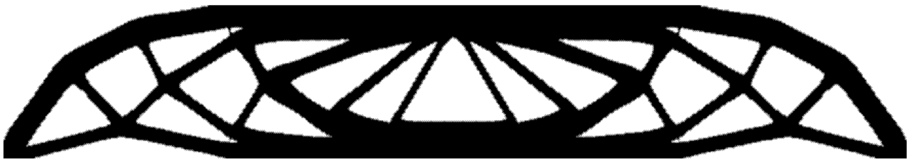
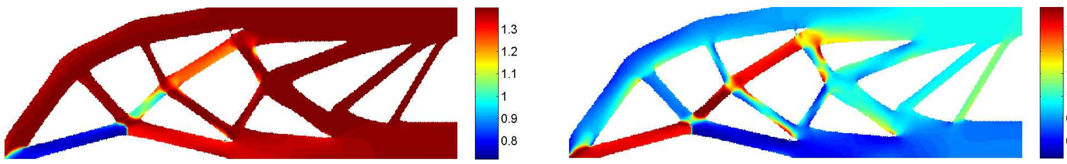


FIGURE 14. Example 3: result at iteration 77 for the robust case.

The distributions of the load scale factors in the worst case scenario x_1 and x_2 are shown in Fig. 15. We note that some parts of the structure were designed by reducing some applied load while the other one is increased. However, the worst case scenario at some parts of the structure are obtained by increasing both applied loads.



(a) index x_1 (b) index x_2

FIGURE 15. Example 3: load scale factors x_1 and x_2 for the robust case.

5. Concluding Remarks

In this paper an alternative approach for robust topology optimization has been presented. Thanks to the Sequential Optimization and Reliability Assessment approach, the problem can be rewritten as a deterministic optimization problem with modified parameters, that are obtained with inverse reliability analysis. This allows standard deterministic algorithms already developed to be used in the context of robust optimization. Here, the deterministic topology optimization problem has been solved using an efficient method based on the topological derivative concept together with a level-set domain representation method as proposed in [3]. We note that the resulting robust topology optimization algorithm converges slower than the standard deterministic algorithm, as can be seen by the increased number of iterations required for convergence. Besides, the computational effort required per step is also increased, since one FEA is required for each applied load (in order to apply linear superposition of the effects). However, this increasing on the computational effort is acceptable in the context of robust optimization and the problem is still tractable even with standard desktop computers.

The examples presented here considered only the intensities of the applied load as random variables. In this case, the stress and strain tensors can be written as a linear combination of the load scale factors and the canonical stress and strain tensors, reducing drastically the computational effort. The examples presented show that the approach is able to obtain optimum solutions that take into account uncertainties of some parameters.

We point out that the structural compliance (the objective function of the problem) does not depend linearly on the random variables (the load scale factors). Therefore, the mean value of the structural compliance is not equal to the compliance obtained with the mean value of the random variables (as would be the case if the objective function depends linearly on the random variables). Consequently, the use of deterministic approach with the mean value of the random variables has little meaning from the robust point of view. For this reason, the deterministic approach gives poor results in comparison with the proposed robust approach, both in terms of mean values and standard deviations.

Since the proposed approach is simple and general, more realistic problems could be considered, such as structural topology optimization under point-wise stress constraints [5]. In addition, our approach could be adapted to deal with more general loading scenarios and distributed parameters as random variables, for instance. Finally, more refined probabilistic modeling of the random variables shall be considered. These topics are now under investigation.

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Appendix A. Advanced Mean Value algorithm

In this appendix we describe the Advanced Mean Value (AMV) algorithm [36, 37], used to solve the problem from (2.7) point-wisely. In the AMV algorithm we assume that the worst case scenario $\mathbf{x}^*$ satisfies $\|T(\mathbf{x}^*)\| = \beta$, i.e. the worst case scenario occurs at the boundary of the set of possible outcomes. Note that this is a valid assumption in the context of this work, since the strain energy (the objective function of the problem from (2.7)) depends quadratically on the load scale factors (the random variables). In this case, the problem of finding the worst case scenario can be written in the general form:

$$\text{Find}\,\mathbf{x}^* \in \mathbb{R}^m\text{, such that:} \quad \begin{cases} \textbf{Maximize} & g(\mathbf{x})\ , \\ \textbf{Subject to} & \|\mathbf{u}\| = \beta\ , \end{cases} \tag{A.1}$$

where $g(\mathbf{x})$ is the performance function (the strain energy $\sigma_\Omega(\mathbf{x}) \cdot \varepsilon_\Omega(\mathbf{x})$ in the context of this work) and $\mathbf{u} = T(\mathbf{x})$, as defined in (2.2), is the vector of random variables in the normal space. The Lagrangian of this problem is written as

$$L(\mathbf{x}, \mu) = -g(\mathbf{x}) + \mu\left(\|\mathbf{u}\| - \beta\right), \tag{A.2}$$

where μ is the Lagrange multiplier. The first order optimality conditions are then

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \mu) = -\nabla_{\mathbf{x}} g(\mathbf{x}) + \mu \nabla_{\mathbf{x}} \|\mathbf{u}\| = \mathbf{0}, \tag{A.3}$$

$$\partial\mu L(\mathbf{x}, \mu) = \|\mathbf{u}\| - \beta = 0, \tag{A.4}$$

where $\nabla_{\mathbf{x}}$ represents the gradient with respect to $\mathbf{x}$. Since $\nabla_{\mathbf{x}}\|\mathbf{u}\| = \mathbf{u}/\|\mathbf{u}\|$ (as can be verified by the reader), the first optimality condition gives

$$\mathbf{u} = \frac{\|\mathbf{u}\|}{\mu} \nabla_{\mathbf{x}} g(\mathbf{x}). \tag{A.5}$$

Finally, from the equality constraint $\|\mathbf{u}\| = \beta$ we conclude that the solution of the problem must satisfy the fixed point condition

$$\mathbf{u} = \beta \frac{\nabla_{\mathbf{x}} g(\mathbf{x})}{\|\nabla_{\mathbf{x}} g(\mathbf{x})\|}. \tag{A.6}$$

In the AMV algorithm we solve the problem iteratively, by taking an initial guess $\mathbf{x}^{(0)}$ and using the above fixed point condition as an update rule. The resulting algorithm is given by

$$\mathbf{u}^{(k+1)} = \beta \frac{\nabla_{\mathbf{x}} g(\mathbf{x}^{(k)})}{\|\nabla_{\mathbf{x}} g(\mathbf{x}^{(k)})\|}, \tag{A.7}$$

$$\mathbf{x}^{(k+1)} = T^{-1}(\mathbf{u}^{(k+1)}), \tag{A.8}$$

where the superscripts $(k), (k+1)$ represent the values of the approximation in successive iterations. The iterations are repeated until some convergence criterion is satisfied. In this work we use the convergence criterion $\|\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}\| \leq \epsilon$, where ϵ is a small tolerance, taken here as $\epsilon = 10^{-9}$.

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