TOPOLOGICAL DERIVATIVE METHOD FOR ELECTRICAL IMPEDANCE TOMOGRAPHY PROBLEMS

A. D. FERREIRA, A. A. NOVOTNY, AND J. SOKOŁOWSKI

Abstract. In the field of shape and topology optimization the new concept is the topological derivative of a given shape functional. The asymptotic analysis is applied in order to determine the topological derivative of shape functionals for elliptic problems. The topological derivative (TD) is a tool to measure the influence on the specific shape functional of insertion of small defect into a geometrical domain for the elliptic boundary value problem (BVP) under considerations. The domain with the small defect stands for perturbed domain by topological variations. This means that given the topological derivative, we have in hand the first order approximation with respect to the small parameter which governs the volume of the defect for the shape functional evaluated in the perturbed domain. TD is a function defined in the original (unperturbed) domain which can be evaluated from the knowledge of solutions to BVP in such a domain. This means that we can evaluate TD by solving only the BVP in the intact domain. One can consider the first and the second order topological derivatives as well, which furnish the approximation of the shape functional with better precision compared to the first order TD expansion in perturbed domain. In this work the topological derivative is applied in the context of Electrical Impedance Tomography (EIT). In particular, we are interested in reconstructing a number of anomalies embedded within a medium subject to a set of current fluxes, from measurements of the corresponding electrical potentials on its boundary. The basic idea consists in minimize a functional measuring the misfit between the boundary measurements and the electrical potentials obtained from the model with respect to a set of ball-shaped anomalies. The first and second order topological derivatives are used, leading to a non-iterative second order reconstruction algorithm. Finally, a numerical experiment is presented, showing that the resulting reconstruction algorithm is very robust with respect to noisy data.

1. Introduction

Shape and topology optimization techniques are used in the wide domain of applications, in particular for solution of inverse problems. The modern theory of shape and topology optimization is a branch of calculus of variations, differential geometry, analysis of boundary value problems for partial differential equations, numerical methods in engineering and structural mechanics, among others. The mathematical analysis of such problems provides the existence of optimal shapes and optimal topologies, together with the necessary conditions for optimality and the numerical schemas for evaluation of approximate solutions as well as the convergence of the proposed schemas. Since the shape optimization problems are in general non-convex, the numerical results are obtained for local solutions only.

The class of inverse problems considered can be formulated as minimizations of shape functionals. Given a geometrical domain Ω with the boundary $\Gamma = \partial\Omega$ and a boundary value problem defined in Ω whose solution is denoted by u^*, we are able to observe the response of the system on the boundary Γ. For example we know the response to the Dirichlet boundary conditions given by the Dirichlet-to-Neumann map for the second

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order elliptic equation [18],

$$\Lambda_{\omega^*} \ : \ u^* = U \mapsto Q := \frac{\partial u^*}{\partial n} \quad \text{on } \Gamma.$$

Assuming that the couple (U,Q) is known however the real defect ω^* is unknown we have an inverse problem. Therefore, given (U,Q) we want to determine the size and the position of a small defect $\omega^* \subset \Omega$ inside of the hold-all domain. The mathematical model of the system furnishes the mapping $\omega \mapsto \Lambda_\omega$ for a family of defects ω. Thus, taking U we can generate the output of the model $\Lambda_\omega(U)$ and compare with the given function $Q = \Lambda_{\omega^*}(U)$. In this way a sequence of approximate solutions to the inverse problem is constructed. In general such a sequence converges to a local solution of the minimization procedure for the distance between the real data and the data obtained from the model.

Hence, using the mathematical model we can consider the associated shape-topological optimization problem based on the distance minimization between the observation (U,Q) and the model response $(U,\Lambda_\omega(U))$ over the family of admissible defects ω. This is a numerical method which uses the shape and topological derivatives of the specific shape functional defined for the inverse problem.

The topological derivative represents the first term of the asymptotic expansion of a given shape functional with respect to the small parameter which measures the size of singular domain perturbations, such as holes, inclusions, source-terms and cracks. This relatively new concept was introduced in the fundamental paper [56] and has been successfully applied to many relevant fields such as shape and topology optimization [1, 8, 11, 12, 15, 17, 29, 38, 40, 48, 49, 50, 59], inverse problems [10, 19, 20, 21, 23, 30, 32, 34, 36, 42], imaging processing [13, 14, 31, 33, 39], multiscale material design [9, 26, 27, 28, 52] and mechanical modeling including damage [2] and fracture [60] evolution phenomena. Regarding the theoretical development of the topological asymptotic analysis, see for instance [6, 7, 22, 24, 25, 35, 37, 41, 43, 44, 45, 46, 47, 57, 58]. For an account of new developments in this branch of shape optimization we refer to the book by Novotny & Sokołowski [51]. In this paper the topological derivative is applied in the context of Electrical Impedance Tomography.

In our frame the applications of topological derivatives is of twofold interest. First of all, for one defect and the associated shape functional which measures the discrepancy between unknown ω^* and the actual ω in the model we can define the first order asymptotic expansion for solutions u_ε of the model with small defect of the size $|\omega_\varepsilon| \to 0$ located at $\widehat{x} \in \Omega$,

$$J(\omega_\varepsilon, u_\varepsilon) = J(\varnothing, u_0) + |\omega_\varepsilon| \mathcal{T}(\widehat{x}) + o(|\omega_\varepsilon|),$$

where $u_0 = u_\varepsilon$ for $\varepsilon = 0$. If we minimize the shape functional for the purposes of inverse problem solution, the selection of small ω_ε uses for its centre $\widehat{x}$ the condition

$$\mathcal{T}(\widehat{x}) < 0.$$

In addition, the size of the defect $|\omega_\varepsilon|$ can be deduced from the second order expansion of the shape functional

$$J(\omega_\varepsilon, u_\varepsilon) = J(\varnothing, u_0) + |\omega_\varepsilon| \mathcal{T}(\widehat{x}) + |\omega_\varepsilon|^2 \mathcal{T}^2(\widehat{x}) + o(|\omega_\varepsilon|^2).$$

It is clear that the proposed procedure strongly depends on the choice of the shape functional which should be of energy type, if possible.

In the paper the tomography framework is considered for the purposes of numerical solution of inverse problems. The special attention is paid to the electrical impedance tomography which is a robust technique in the field of noninvasive detection of small defects.

The tomography techniques for solution of inverse problems are developed in Poland, see e.g., [54] on the impedance and optical tomography, [55] on industrial and biological tomography, as well as [53] on electrical capacitance tomography.

In the present paper, a new method for solution of inverse problems based on the topological derivative concept is proposed. The method is useful for identification of small defects and it is based on asymptotic analysis of associated PDEs with respect to the size of defects, for the size which tends to zero. The characteristics of defects are given by the shape functionals, and the numerical methods employ the asymptotic expansions of the functional with respect to the size of defects.

2. Problem Formulation

Let us consider a domain $\Omega \subset \mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, which represents a body endowed with the capability of conducting electricity. Its electrical conductivity coefficient is denoted by $k^*(x) \geq k_0 > 0$, with $x \in \Omega$ and $k_0 \in \mathbb{R}_+$. If the body Ω is subjected to a given electric flux Q on $\partial\Omega$, then the resulting electric potential in Ω is observed on a part of the boundary $\Gamma_m \subset \partial\Omega$. The objective is to reconstruct the electrical conductivity k^* over Ω from the obtained boundary measurement $U := u^*_{|\Gamma_m}$, solution of the following over-determined boundary value problem

$$\left\{ \begin{array}{rcll} \operatorname{div}[q(u^*)] & = & 0 & \text{in } \Omega, \\ q(u^*) & = & -k^*\nabla u^*, & \\ q(u^*)\cdot n & = & Q & \text{on } \partial\Omega, \\ u^* & = & U & \text{on } \Gamma_m. \end{array} \right. \tag{2.1}$$

Without loss of generality, we are considering only one boundary measurement U on Γ_m. The extension to several boundary measurements is trivial. Furthermore, we assume that the unknown electrical conductivity k^* we are looking for belongs to the following set

$$C_\gamma(\Omega) := \left\{ \varphi \in L^\infty(\Omega) : \varphi = k\left(\mathbb{1}_\Omega - \sum_{i=1}^{N} (1-\gamma_i)\mathbb{1}_{\omega_i} \right) \right\}, \tag{2.2}$$

where $k \in \mathbb{R}_+$ is the electrical conductivity of the background. The sets $\omega_i \subset \Omega$, with $i = 1, \cdots, N$, are such that $\omega_i \cap \omega_j = \emptyset$, for $i \neq j$. In addition, $\mathbb{1}_\Omega$ and $\mathbb{1}_{\omega_i}$ are used to denote the characteristics functions of Ω and ω_i, respectively. Finally, $\gamma_i \in \mathbb{R}_+$ are the contrasts with respect to the electrical conductivity of the background k. We assume that the electrical conductivity of the background k and the associated contrasts γ_i are known. Therefore, the inverse problem we are dealing with can be written in the form of a topology optimization problem with respect to the sets $\omega^* = \bigcup_{i=1}^N \omega_i$. See sketch in Figure 1. Let us introduce the following auxiliary Neumann boundary value problem:

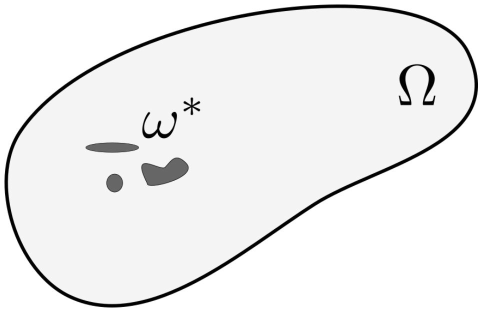


Figure 1. Body with anomalies.

Find u, such that

$$\left\{\begin{array}{rcll} \operatorname{div}[q(u)] & = & 0 & \text{in } \Omega \\ q(u) & = & -k\nabla u & \\ q(u)\cdot n & = & Q & \text{on } \partial\Omega \\ \int_{\partial\Omega} Q & = & 0 & \\ \int_{\Gamma_m} u & = & \int_{\Gamma_m} U, & \end{array}\right. \tag{2.3}$$

where Q and U are the boundary excitation and boundary measurement, respectively. Finally, we introduce the following shape functional measuring the misfit between the boundary measurement U and the solution u of (2.3) evaluated on Γ_m, namely

$$\underset{\omega^*\subset\Omega}{\text{Minimize}}\ \mathcal{J}(u) = \int_{\Gamma_m} (u-U)^2, \tag{2.4}$$

which will be solved by using the first and second order topological derivatives concepts. See related works [3, 4, 5, 16, 34].

3. Topological Asymptotic Expansion

Let us consider that the domain Ω is perturbed by the nucleation of N ball-shaped inclusions $B_{\varepsilon_i}(x_i)$ with contrast γ_i, $i = 1, \cdots, N$. We assume that $B_{\varepsilon_i}(x_i) \subset \Omega$ is a ball with center at $x_i \in \Omega$ and radius ε_i, such that $B_{\varepsilon_i}(x_i) \cap B_{\varepsilon_j}(x_j) = \emptyset$ for $i \neq j$. We introduce the notations $\xi = (x_1, \cdots, x_N)$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_N)$. The topologically perturbed counterpart of the shape functional (2.4) is given by

$$\mathcal{J}(u_\varepsilon) = \int_{\Gamma_m} (u_\varepsilon - U)^2, \tag{3.1}$$

where u_ε is solution of the following boundary value problem

$$\left\{\begin{array}{rcll} \operatorname{div}[q_\varepsilon(u_\varepsilon)] & = & 0 & \text{in } \Omega \\ q_\varepsilon(u_\varepsilon) & = & -\gamma_\varepsilon k\nabla u_\varepsilon & \\ q_\varepsilon(u_\varepsilon)\cdot n & = & Q & \text{on } \partial\Omega \\ \int_{\partial\Omega} Q & = & 0 & \\ \int_{\Gamma_m} u_\varepsilon & = & \int_{\Gamma_m} U & \\ [\![u_\varepsilon]\!] & = & 0 & \text{on } \bigcup_{i=1}^N \partial B_{\varepsilon_i}(x_i) \\ [\![q_\varepsilon(u_\varepsilon)]\!]\cdot n & = & 0 & \text{on } \bigcup_{i=1}^N \partial B_{\varepsilon_i}(x_i) \end{array}\right. \tag{3.2}$$

with the contrast defined as

$$\gamma_\varepsilon = \gamma_\varepsilon(x) = \left\{\begin{array}{l} 1, \text{ if } x \in \Omega\setminus\bigcup_{i=1}^N B_{\varepsilon_i}(x_i) \\ \gamma_i, \text{ if } x \in B_{\varepsilon_i}(x_i). \end{array}\right. \tag{3.3}$$

From these elements, the topological asymptotic expansion the shape functional $\mathcal{J}(u_\varepsilon)$ is given by

$$\mathcal{J}(u_\varepsilon) = \mathcal{J}(u) + d(\xi)\cdot\alpha + \frac{1}{2}H(\xi)\alpha\cdot\alpha + \mathcal{E}(\varepsilon), \tag{3.4}$$

where $d(\xi)$ and $H(\xi)$ is the first and second order topological derivatives, respectively. In addition, $\alpha = (\varepsilon_1^2, \cdots, \varepsilon_N^2)$ and $\mathcal{E}(\varepsilon)$ is the remainder. Some terms in the above expression still require explanations. The vector $d(\xi)$ and the matrix $H(\xi)$ are defined as

$$d(\xi) := \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \quad \text{and} \quad H(\xi) := \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NN}, \end{pmatrix} \tag{3.5}$$

where each component d_i is given by

$$d_i = -2\int_{\Gamma_m} \rho_i (u-U)(g_i + \tilde{u}_i), \tag{3.6}$$

while each entry h_{ij} is defined as

$$h_{ii} = 4\int_{\Gamma_m} (u-U)(\rho_i h_i + \rho_i \tilde{g}_i + \tilde{\tilde{u}}_i) + 2\int_{\Gamma_m} (\rho_i g_i + \tilde{u}_i)^2, \tag{3.7}$$

$$h_{ij} = 2\int_{\Gamma_m} (u-U)(\rho_j \theta_i^j + \rho_i \theta_j^i + u_i^j + u_j^i) + 2\int_{\Gamma_m} (\rho_i g_i + \tilde{u}_i)(\rho_j g_j + \tilde{u}_j),\ j \neq i. \tag{3.8}$$

In addition,

$$\rho_i = \frac{1-\gamma_i}{1+\gamma_i}, \tag{3.9}$$

and the functions $g_i(x)$, $h_i(x)$, $\tilde{g}_i(x)$ and $\theta_i^j(x)$ are respectively given by

$$g_i(x) = \frac{1}{\|x-x_i\|^2}\nabla u(x_i)\cdot(x-x_i), \tag{3.10}$$

$$h_i(x) = \frac{1}{2}\frac{1}{\|x-x_i\|^4}\nabla^2 u(x_i)(x-x_i)^2, \tag{3.11}$$

$$\tilde{g}_i(x) = \frac{1}{\|x-x_i\|^2}\nabla \tilde{u}_i(x_i)\cdot(x-x_i), \tag{3.12}$$

$$\theta_i^j(x) = \frac{1}{\|x-x_j\|^2}A(x_j)\nabla u(x_i)\cdot(x-x_j). \tag{3.13}$$

where the second order tensor $A(x)$ is written as

$$A(x) = \frac{1}{\|x-x_i\|^2}\left[I - 2\frac{(x-x_i)\otimes(x-x_i)}{\|x-x_i\|^2}\right]. \tag{3.14}$$

Finally, the auxiliary function $\tilde{u}_i$ is solution to: Find $\tilde{u}_i$, such that

$$\left\{\begin{array}{rcll} \mathrm{div}[q(\tilde{u}_i)] &=& 0, & \text{in } \Omega,\\ q(\tilde{u}_i) &=& -k\nabla\tilde{u}_i, & \text{in } \Omega,\\ q(\tilde{u}_i)\cdot n &=& -\rho_i q(g_i)\cdot n, & \text{on } \partial\Omega\\ \int_{\Gamma_m}\tilde{u}_i &=& -\rho_i\int_{\Gamma_m} g_i, & \end{array}\right. \tag{3.15}$$

while the auxiliary function $\tilde{\tilde{u}}_i$ solves: Find $\tilde{\tilde{u}}_i$, such that

$$\left\{\begin{array}{rcll} \mathrm{div}[q(\tilde{\tilde{u}}_i)] &=& 0 & \text{in } \Omega,\\ q(\tilde{\tilde{u}}_i) &=& -k\nabla\tilde{\tilde{u}}_i & \text{in } \Omega,\\ q(\tilde{\tilde{u}}_i)\cdot n &=& -\rho_i q(h_i+\tilde{g}_i)\cdot n, & \text{on } \partial\Omega\\ \int_{\Gamma_m}\tilde{\tilde{u}}_i &=& -\rho_i\int_{\Gamma_m} h_i+\tilde{g}_i, & \end{array}\right. \tag{3.16}$$

and the auxiliary function u_i^j is solution to: Find u_i^j, such that

$$\left\{\begin{array}{rcll} \mathrm{div}[q(u_i^j)] &=& 0 & \text{in } \Omega,\\ q(u_i^j) &=& -k\nabla u_i^j & \text{in } \Omega,\\ q(u_i^j)\cdot n &=& -\rho_j q(\theta_i^j)\cdot n, & \text{on } \partial\Omega\\ \int_{\Gamma_m} u_i^j &=& -\rho_j\int_{\Gamma_m}\theta_i^j. & \end{array}\right. \tag{3.17}$$

The derivation of the above equations follows the same steps as presented in [34], for instance.

4. A Numerical Experiment

In this section we present the resulting non-interactive reconstruction algorithm based on the expansion (3.4). Let us introduce the quantity

$$\Psi(\xi,\alpha) = d(\xi)\cdot\alpha + \frac{1}{2}H(\xi)\alpha\cdot\alpha. \tag{4.1}$$

After minimize (4.1) with respect to α we obtain the following linear system

$$\alpha = \alpha(\xi) = -(H(\xi))^{-1}d(\xi). \tag{4.2}$$

Let us replace $\alpha(\xi)$ solution of (4.2) in (4.1), to obtain

$$\Psi(\xi,\alpha(\xi)) = -\frac{1}{2}d(\xi)\cdot\alpha(\xi). \tag{4.3}$$

Therefore, the pair of vectors $(\xi^\star,\alpha^\star)$ which minimize (4.1) is given by

$$\xi^\star := \arg\min_{\xi\in X}\left\{-\frac{1}{2}d(\xi)\cdot\alpha(\xi)\right\} \text{ and } \alpha^\star := \alpha(\xi^\star), \tag{4.4}$$

where X is the set of admissible locations of the inclusions. From these elements the Algorithm 1 is devised. Its input data are:

- The number of anomalies that are going to find;
- The first d and second H order topological derivatives;
- The size of the grid where we are seeking the inclusions, denoted by n_g;
- The index i_g of the grid.

As a result, the algorithm provides the location and optimum size of the anomalies $(\xi^\star,\alpha^\star)$, and the minimum value of the functional given by (4.3) denoted by by $S^\star$.

Finally, let us present a numerical example. We consider a disk of unitary radius. Its boundary is subdivided into 16 disjoint pieces. Each pair of such a pieces are used for injecting and draining the current. Therefore, the excitation Q is given by a pair $Q_{in} = 1$ of injection and $Q_{out} = -1$ of draining. The remainder part of the boundary becomes insulated. The associated potential U is measured only on these disjoint pieces, representing Γ_m. See sketch in Figure 2. The target consists of three ball-shaped anomalies, which

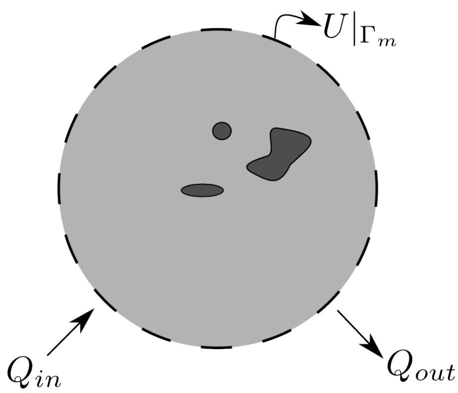


Figure 2. Model problem.

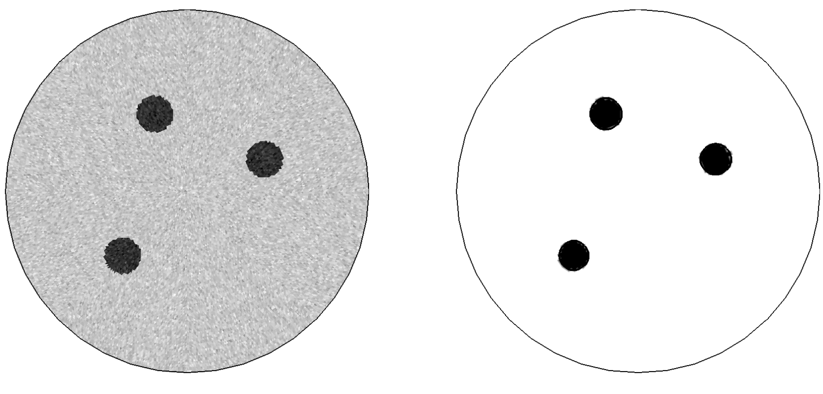
is corrupted with 10% of White Gaussian Noise, as shown in Figure 3(a). The obtained reconstruction with 64 partial boundary measurements is shown in Figure 3(b).

From an inspection of Figure 3 we observe that Algorithm 1 is actually very robust with respect to noisy data. It comes out from the fact that the proposed second-order reconstruction algorithm is non-iterative.

Algorithm 1: Reconstruction Algorithm

Data: N, n_g, $d_i(ig)$, $H_{ij}(ig)$
Result: $S^\star$, $\alpha^\star$, $\xi^\star$

Initialization: $S^\star \leftarrow \infty$; $\alpha^\star \leftarrow 0$; $\xi^\star \leftarrow 0$;
for $i_1 \leftarrow 1$ **to** n **do**
  for $i_2 \leftarrow i_1 + 1$ **to** n **do**
    $\vdots$
    for $i_N \leftarrow i_{N-1} + 1$ **to** n **do**
      $d \leftarrow \begin{pmatrix} d_1(i_1) \\ d_2(i_2) \\ \vdots \\ d_N(i_N)]^T \end{pmatrix}; \quad H \leftarrow \begin{pmatrix} H_{11}(i_1) & H_{12}(i_2) & \cdots & H_{1N}(i_N) \\ H_{21}(i_1) & H_{22}(i_2) & \cdots & H_{2N}(i_N) \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1}(i_1) & H_{N2}(i_2) & \cdots & H_{NN}(i_N) \end{pmatrix};$
      $\alpha \leftarrow -H^{-1}d$;
      if $\alpha_k > 0 \quad \forall k \in \{1, \ldots, N\}$ **then**
        $S \leftarrow -\frac{1}{2} d \cdot \alpha$;
        if $S < S^\star$ **then**
          $S^\star \leftarrow S$;
          $\alpha^\star \leftarrow \alpha$;
          $\xi^\star \leftarrow [i_1, i_2, ..., i_N]$;
        end if
      end if
    end for
  end for
end for
return $S^\star$, $\alpha^\star$, $\xi^\star$



(a) target (b) result

FIGURE 3. Target corrupted with 10% of White Gaussian Noise (left) and obtained result with 64 partial boundary measurements (right).

5. Concluding Remarks

In the paper new methods of numerical solutions for a class of electrical impedance tomography problems is proposed. The method is based on the topological derivatives of shape functionals associated with the inverse problems. It is assumed that there is a finite number of small defects within the domain (body) and that the influence of the defects on the Dirichlet-to-Neumann map is observed using the mathematical model in the form of linear elliptic boundary value problem. The noisy boundary measurements are compared with the mathematical model in order to identify the number, size and locations of the hidden imperfections.

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