ENERGY CHANGE TO INSERTION OF INCLUSIONS ASSOCIATED WITH THE REISSNER-MINDLIN PLATE BENDING MODEL

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Abstract. The topological derivative concept has been proved to be useful in many relevant applications such as topology optimization, inverse problems, image processing, multi-scale constitutive modeling, fracture mechanics and damage evolution modeling. The topological asymptotic analysis has been fully developed for a wide range of problems modeled by partial differential equations. On the other hand, the topological derivatives associated with coupled problems have been derived only in their abstract forms. In this paper, therefore, we deal with the Reissner-Mindlin plate bending model, which is written in the form of a coupled system of partial differential equations. In particular, the topological asymptotic analysis of the associated total potential energy is developed and the topological derivative with respect to the nucleation of a circular inclusion is derived in its closed form. Finally, we provide the estimates for the remainders of the topological asymptotic expansion and perform a complete mathematical justification for the derived formulas.

1. Introduction

The topological asymptotic analysis leads to the asymptotic expansion of a given shape functional with respect to a singular domain perturbation such as holes, inclusions, cracks, etc. The main term of such expansion is defined as the topological derivative [22], which has been proved to be useful in many relevant applications such as topology optimization [6], inverse problems [12], image processing [11], multi-scale constitutive modeling [4], fracture mechanics [23] and damage evolution modeling [2]. For a comprehensive account on the topological derivative concept see, for instance, the book by Novotny & Sokołowski [18].

In particular, the topological derivative for the Kirchhoff plate bending problem has been rigorously derived in [5], which involves a forth-order differential operator. For the topological asymptotic analysis associated with higher-order elliptic differential operators see [7]. On the other hand, the topological derivative associated with the Reissner-Mindlin plate bending problem has not been reported in the literature yet. This mechanical model leads to a coupled system of second-order partial differential equations. In fact, only a few works dealing with coupled problems can be found in the literature, whose derived formulas are presented only in their abstract forms [8].

Therefore, in this work the topological derivative for the total potential energy associated with the Reissner-Mindlin plate bending problem is derived. In particular, arguments on the existence of the topological derivative for this model are presented, together with precise estimates for the remainders of the associated topological asymptotic expansion. Finally, the topological derivative with respect to the nucleation of an infinitesimal circular inclusion is derived in its closed form. Since the Reissner-Mindlin plate bending model takes into account the shear effects, we believe that the derived formula shall be useful for practical applications, allowing for overcome some numerical difficulties associated with the Kirchhoff plate bending model reported in [17], for instance.

This paper is organized as follows. In Section 2 we introduce the topological derivative concept. The mechanical model which we are dealing with is presented in Section 3, together with the existence of the associated topological derivative. The explicit form of the topological derivative is derived in Section 4. The estimates for the remainders of the topological asymptotic expansion are presented in Section 5. Finally, the paper ends with some concluding remarks in Section 6.

Key words and phrases. Topological derivative, Reissner-Mindlin plates, asymptotic analysis.

2. The Topological Derivative Concept

Let us consider an open and bounded domain $\Omega \subset \mathbb{R}^2$, which is subject to a nonsmooth perturbation confined in a small region $\omega_\varepsilon(\widehat{x}) = \widehat{x} + \varepsilon\omega$ of size ε and center at $\widehat{x} \in \Omega$, as shown in Figure 1.

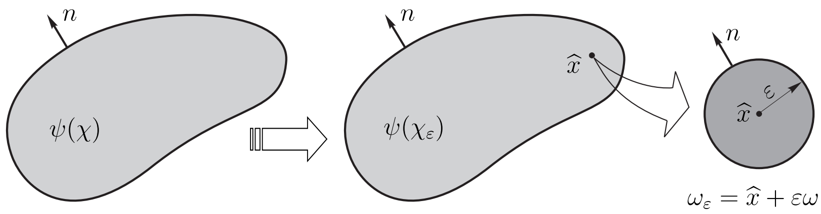

Figure 1. The topological derivative concept.

We introduce a characteristic function associated to the unperturbed domain $x \mapsto \chi(x)$, $x \in \mathbb{R}^2$, of the form $\chi = \mathbb{1}_\Omega$, so that:

$$|\Omega| = \int_{\mathbb{R}^2} \chi \, . \tag{2.1}$$

On the other hand, we define a piecewise constant function associated to the perturbed domain $x \mapsto \chi_\varepsilon(\widehat{x}, x)$, $x \in \mathbb{R}^2$, with ε and $\widehat{x} \in \Omega$ fixed, so that $\chi_\varepsilon(\widehat{x}) = \mathbb{1}_\Omega - (1-\gamma)\mathbb{1}_{\omega_\varepsilon(\widehat{x})}$. Therefore,

$$\chi_\varepsilon(\widehat{x}, x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \setminus \omega_\varepsilon(\widehat{x}) \, , \\ \gamma & \text{if} \quad x \in \omega_\varepsilon(\widehat{x}) \, , \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \setminus \Omega \, , \end{cases} \tag{2.2}$$

where $0 < \gamma < \infty$ is the contrast on the material properties.

Now, we assume that a given shape functional associated to the topological perturbed problem $\psi(\chi_\varepsilon(\widehat{x}))$ admits the following topological asymptotic expansion:

$$\psi(\chi_\varepsilon(\widehat{x})) = \psi(\chi) + f(\varepsilon)\mathcal{T}(\widehat{x}) + o(f(\varepsilon)) \, , \tag{2.3}$$

where $\psi(\chi)$ is the shape functional associated to the unperturbed problem, $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$, when $\varepsilon \to 0^+$. Function $\widehat{x} \mapsto \mathcal{T}(\widehat{x})$ is defined as the topological derivative of ψ at the point $\widehat{x}$. This derivative can be understood as a first order correction on $\psi(\chi)$ to approximate $\psi(\chi_\varepsilon(\widehat{x}))$. After rearranging the equation (2.3), the limit passage $\varepsilon \to 0^+$ leads to:

$$\mathcal{T}(\widehat{x}) = \lim_{\varepsilon \to 0^+} \frac{\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi)}{f(\varepsilon)} \, . \tag{2.4}$$

Note that in our particular case the geometrical domains remain fixed, while the topological perturbation is going to be driven by a contrast γ on the material properties. In the singular case associated with the nucleation of holes, e.g. $\gamma = 0$, the analysis becomes much more involved and can be seen in details in [16], for instance.

3. Thick Plate Bending Model

In this section we introduce a plate bending problem under the kinematic assumptions of Reissner-Mindlin [15, 20]. Thus, let us consider a plate represented by a two-dimensional domain $\Omega \subset \mathbb{R}^2$, with thickness $h > 0$ supposed to be constant for the sake of simplicity. We assume that the plate is submitted to bending and shear effects under the following kinematic assumptions:

> *The normal fibers to the middle plane of the plate remain straight during the deformation process, but not necessarily normal to the middle plane, and do not suffer variations in their length. Consequently, the transversal shear deformations are not negligible and the normal deformations are null.*

The Reissner-Mindlin plate bending problem leads to a coupled system of second-order partial differential equations, which is known to be strongly elliptic. See, for instance [21] and [13, Ch. 4]. This important property will be exhaustively used in the derivations to be presented in what follows.

3.1. **Unperturbed Problem.** Let us introduce the total potential energy associated with the unperturbed plate problem, namely:

$$\psi(\chi) := \mathcal{J}(\theta, w) = \frac{1}{2}\int_\Omega (\mathcal{M}(\theta)\cdot\nabla^s\theta + \mathcal{Q}(\theta,w)\cdot(\theta - \nabla w)) - \int_{\Gamma_{N_\theta}} \overline{m}\cdot\theta + \int_{\Gamma_{N_w}} \overline{q}w\ , \tag{3.1}$$

where $\mathcal{M}(\theta) = \mathbb{C}\nabla^s\theta$ is the generalized bending moment tensor and $\mathcal{Q}(\theta, w) = \mathbf{K}(\theta - \nabla w)$ is the generalized shear tensor. The constitutive fourth $\mathbb{C}$ and second $\mathbf{K}$ order tensors are assumed to be isotropic and homogeneous, which are respectively given by

$$\mathbb{C} = \frac{Eh^3}{12(1-\nu^2)}((1-\nu)\mathbb{I} + \nu(\mathbf{I}\otimes\mathbf{I}))\ , \tag{3.2}$$

$$\mathbf{K} = \frac{\kappa E h}{2(1+\nu)}\mathbf{I}\ , \tag{3.3}$$

where E is the Young modulus, ν is the Poisson ration, $\kappa = 5/6$ is the shear correction factor and h the plate thickness. In addition, $\mathbf{I}$ and $\mathbb{I}$ are the second and fourth order identity tensors, respectively. The rotation θ and the transversal displacement w are solutions to the following coupled variational problem: For all $(\eta_\theta, \eta_w) \in \mathcal{V}$, find the field $(\theta, w) \in \mathcal{U}$, such that

$$\left\{\begin{array}{rcl} \displaystyle\int_\Omega (\mathcal{M}(\theta)\cdot\nabla^s\eta_\theta + \mathcal{Q}(\theta,w)\cdot\eta_\theta) &=& \displaystyle\int_{\Gamma_{N_\theta}} \overline{m}\cdot\eta_\theta\ , \\ \displaystyle\int_\Omega \mathcal{Q}(\theta,w)\cdot\nabla\eta_w &=& \displaystyle\int_{\Gamma_{N_w}} \overline{q}\eta_w\ . \end{array}\right. \tag{3.4}$$

In the variational problem (3.4), the set $\mathcal{U}$ of admissible functions and the space $\mathcal{V}$ of admissible variations are defined by:

$$\mathcal{U}{:=}\{(\varphi_\theta,\varphi_w), \varphi_\theta \in \mathbf{H}^1(\Omega) \text{ and } \varphi_w \in H^1(\Omega) : \varphi_w|_{\Gamma_{D_w}} = \overline{w}, \varphi_\theta|_{\Gamma_{D_\theta}} = \overline{\theta}\}\ , \tag{3.5}$$

$$\mathcal{V}{:=}\{(\varphi_\theta,\varphi_w), \varphi_\theta \in \mathbf{H}^1(\Omega) \text{ and } \varphi_w \in H^1(\Omega) : \varphi_w|_{\Gamma_{D_w}} = 0, \varphi_\theta|_{\Gamma_{D_\theta}} = 0\}\ , \tag{3.6}$$

with $\mathbf{H}^1(\Omega) := H^1(\Omega)\times H^1(\Omega)$, where Γ_{N_θ} and Γ_{N_w} are Neumann boundaries, while Γ_{D_θ} and Γ_{D_w} are Dirichlet boundaries. Then, $\overline{\theta} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_{D_\theta})$ is a Dirichlet data representing a rotation prescribed on Γ_{D_θ}, while $\overline{w} \in H^{\frac{1}{2}}(\Gamma_{D_w})$ is a Dirichlet data associated with a transversal displacement prescribed on Γ_{D_w}. In addition, the Neumann data are given by $\overline{m} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{N_\theta})$, representing a distributed moment prescribed on Γ_{N_θ}, and $\overline{q} \in H^{-\frac{1}{2}}(\Gamma_{N_w})$, representing a distributed shear on Γ_{N_w}. Finally, $\Gamma_{D_w}\cap\Gamma_{N_w} = \emptyset$ and $\Gamma_{D_\theta}\cap\Gamma_{N_\theta} = \emptyset$, with Γ_{D_θ} and Γ_{D_w} of nonzero measure. In order to guarantee the existence and uniqueness of a solution to (3.4), either $\Gamma_{D_w}\cap\Gamma_{D_\theta} \neq \emptyset$ or Γ_{D_w} is not straight (its unit normal is not constant). The strong form associated with the variational problem (3.4) reads: Find the field (θ, w), such that

$$\left\{\begin{array}{rcll} -\mathrm{div}(\mathcal{M}(\theta)) + \mathcal{Q}(\theta,w) &=& 0 & \text{in} \quad \Omega, \\ \mathrm{div}(\mathcal{Q}(\theta,w)) &=& 0 & \text{in} \quad \Omega, \\ \mathcal{M}(\theta) &=& \mathbb{C}\nabla^s\theta, & \\ \mathcal{Q}(\theta,w) &=& \mathbf{K}(\theta - \nabla w), & \\ w &=& \overline{w} & \text{on} \quad \Gamma_{D_w}, \\ \theta &=& \overline{\theta} & \text{on} \quad \Gamma_{D_\theta}, \\ \mathcal{M}(\theta)n &=& \overline{m} & \text{on} \quad \Gamma_{N_\theta}, \\ \mathcal{Q}(\theta,w)\cdot n &=& \overline{q} & \text{on} \quad \Gamma_{N_w}, \end{array}\right. \tag{3.7}$$

where n is the outward unit normal vector pointing toward the exterior of Ω on $\partial\Omega$.

3.2. **Perturbed Problem.** The total potential energy associated with the topologically perturbed plate problem reads:

$$\psi(\chi_\varepsilon(\widehat{x})) := \mathcal{J}_\varepsilon(\theta_\varepsilon, w_\varepsilon) = \frac{1}{2}\int_\Omega \gamma_{\omega_\varepsilon}\left(\mathcal{M}(\theta_\varepsilon)\cdot\nabla^s\theta_\varepsilon + \mathcal{Q}(\theta_\varepsilon, w_\varepsilon)\cdot(\theta_\varepsilon - \nabla w_\varepsilon)\right) - \int_{\Gamma_{N_\theta}} \overline{m}\cdot\theta_\varepsilon + \int_{\Gamma_{N_w}} \overline{q} w_\varepsilon \ , \quad (3.8)$$

where the parameter $\gamma_{\omega_\varepsilon}$ is given by:

$$\gamma_{\omega_\varepsilon} := \begin{cases} 1 & \text{in} \quad \Omega \setminus \overline{\omega}_\varepsilon \ , \\ \gamma & \text{in} \quad \omega_\varepsilon \ . \end{cases} \quad (3.9)$$

Therefore, the topological perturbation is given by the nucleation of a small inclusion assumed to be circular, namely, $\omega_\varepsilon(\widehat{x}) = B_\varepsilon(\widehat{x})$, where $B_\varepsilon(\widehat{x})$ is a ball of radius ε and center at $\widehat{x} \in \Omega$. We assume in addition that $\overline{B_\varepsilon(\widehat{x})} \Subset \Omega$. That is, the topological perturbation $B_\varepsilon(\widehat{x})$ doesn't touch the boundary $\partial\Omega$. The parameter $0 < \gamma < \infty$ represents a contrast on the material properties. The functions θ_ε and w_ε in (3.8) are solutions to the following variational problem: For all $(\eta_\theta, \eta_w) \in \mathcal{V}$, find the field $(\theta_\varepsilon, w_\varepsilon) \in \mathcal{U}$, such that

$$\begin{cases} \displaystyle\int_\Omega \gamma_{\omega_\varepsilon}\left(\mathcal{M}(\theta_\varepsilon)\cdot\nabla^s\eta_\theta + \mathcal{Q}(\theta_\varepsilon, w_\varepsilon)\cdot\eta_\theta\right) = \int_{\Gamma_{N_\theta}} \overline{m}\cdot\eta_\theta \ , \\ \displaystyle\int_\Omega \gamma_{\omega_\varepsilon}\mathcal{Q}(\theta_\varepsilon, w_\varepsilon)\cdot\nabla\eta_w = \int_{\Gamma_{N_w}} \overline{q}\eta_w \ . \end{cases} \quad (3.10)$$

The strong form associated with the variational problem (3.10) can be stated as: Find the field $(\theta_\varepsilon, w_\varepsilon)$, such that

$$\begin{cases} -\mathrm{div}(\gamma_{\omega_\varepsilon}\mathcal{M}(\theta_\varepsilon)) + \gamma_{\omega_\varepsilon}\mathcal{Q}(\theta_\varepsilon, w_\varepsilon) = 0 & \text{in} \quad \omega_\varepsilon \cup (\Omega\setminus\omega_\varepsilon) \ , \\ \mathrm{div}(\gamma_{\omega_\varepsilon}\mathcal{Q}(\theta_\varepsilon, w_\varepsilon)) = 0 & \text{in} \quad \omega_\varepsilon \cup (\Omega\setminus\omega_\varepsilon) \ , \\ \mathcal{M}(\theta_\varepsilon) = \mathbb{C}\nabla^s\theta_\varepsilon \ , & \\ \mathcal{Q}(\theta_\varepsilon, w_\varepsilon) = \mathbf{K}(\theta_\varepsilon - \nabla w_\varepsilon) \ , & \\ w_\varepsilon = \overline{w} & \text{on} \quad \Gamma_{D_w} \ , \\ \theta_\varepsilon = \overline{\theta} & \text{on} \quad \Gamma_{D_\theta} \ , \\ \gamma_{\omega_\varepsilon}\mathcal{M}(\theta_\varepsilon)n = \overline{m} & \text{on} \quad \Gamma_{N_w} \ , \\ \gamma_{\omega_\varepsilon}\mathcal{Q}(\theta_\varepsilon, w_\varepsilon)\cdot n = \overline{q} & \text{on} \quad \Gamma_{N_\theta} \ , \\ \left.\begin{aligned} [\![\theta_\varepsilon]\!] &= 0 \\ [\![w_\varepsilon]\!] &= 0 \\ [\![\gamma_{\omega_\varepsilon}\mathcal{M}(\theta_\varepsilon)]\!]n &= 0 \\ [\![\gamma_{\omega_\varepsilon}\mathcal{Q}(\theta_\varepsilon, w_\varepsilon)]\!]\cdot n &= 0 \end{aligned}\right\} & \text{on} \quad \partial\omega_\varepsilon \ . \end{cases} \quad (3.11)$$

where n is the outward unit normal vector pointing toward the exterior of Ω on $\partial\Omega$ and ω_ε on $\partial\omega_\varepsilon$, as shown in Fig. 1, and $[\![\varphi]\!] := \varphi\big|_{\Omega\setminus\omega_\varepsilon} - \varphi\big|_{\omega_\varepsilon}$ is used to denote the jump of the vector function φ on $\partial\omega_\varepsilon$.

3.3. **Existence of the Topological Derivative.** The original and perturbed shape functionals in which we are dealing with were introduced through equations (3.1) and (3.8), respectively. Now we are in position to state the following import result associated with the existence of the topological derivative for the problem under analysis:

Lemma 1. *Let $(\theta_\varepsilon, w_\varepsilon)$ and (θ, w) be solutions to the perturbed* (3.10) *and original* (3.4) *variational problems, respectively. Then, the following estimates hold true*

$$\|\theta_\varepsilon - \theta\|_{\mathbf{H}^1(\Omega)} \le C\varepsilon \ , \quad (3.12)$$

$$\|w_\varepsilon - w\|_{H^1(\Omega)} \le C\varepsilon \ , \quad (3.13)$$

where C is used to denote a generic constant independent of the control parameter ε.

Proof. By subtracting (3.4) from (3.10), we have for all $(\eta_\theta, \eta_w) \in \mathcal{V}$:

$$\left\{ \begin{array}{rcl} \displaystyle\int_\Omega (\gamma_{\omega_\varepsilon}\mathcal{M}(\theta_\varepsilon) - \mathcal{M}(\theta)) \cdot \nabla^s \eta_\theta + (\gamma_{\omega_\varepsilon}\mathcal{Q}(\theta_\varepsilon, w_\varepsilon) - \mathcal{Q}(\theta, w)) \cdot \eta_\theta & = & 0 \, , \\ \displaystyle\int_\Omega (\gamma_{\omega_\varepsilon}\mathcal{Q}(\theta_\varepsilon, w_\varepsilon) - \mathcal{Q}(\theta, w)) \cdot \nabla \eta_w & = & 0 \, . \end{array} \right. \tag{3.14}$$

From the definition for the contrast $\gamma_{\omega_\varepsilon}$ given by (3.9) and after introducing the notations $e_\theta = \theta_\varepsilon - \theta$ and $e_w = w_\varepsilon - w$, the above variational equation can be written as

$$\left\{ \begin{array}{rcl} \displaystyle\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(e_\theta) \cdot \nabla^s \eta_\theta + \mathcal{Q}(e_\theta, e_w) \cdot \eta_\theta) & = & \displaystyle(1-\gamma)\int_{B_\varepsilon} \mathcal{M}(\theta) \cdot \nabla^s \eta_\theta + \mathcal{Q}(\theta, w) \cdot \eta_\theta \, , \\ \displaystyle\int_\Omega \gamma_{\omega_\varepsilon}\mathcal{Q}(e_\theta, e_w) \cdot \nabla \eta_w & = & \displaystyle(1-\gamma)\int_{B_\varepsilon} \mathcal{Q}(\theta, w) \cdot \nabla \eta_w \, , \end{array} \right. \tag{3.15}$$

where the terms

$$\pm \int_{B_\varepsilon} \gamma \mathcal{M}(\theta) \cdot \nabla^s \eta_\theta + \gamma \mathcal{Q}(\theta, w) \cdot \eta_\theta \quad \text{and} \quad \pm \int_{B_\varepsilon} \gamma \mathcal{Q}(\theta, w) \cdot \nabla \eta_w \, ,$$

were added to $(3.14)_1$ and $(3.14)_2$, respectively. Now, let us take $\eta_\theta = e_\theta$ and $\eta_w = e_w$ as test functions in (3.15) to obtain the following equalities:

$$\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(e_\theta) \cdot \nabla^s e_\theta + \mathcal{Q}(e_\theta, e_w) \cdot e_\theta) = (1-\gamma)\int_{B_\varepsilon} \mathcal{M}(\theta) \cdot \nabla^s e_\theta + \mathcal{Q}(\theta, w) \cdot e_\theta \, , \tag{3.16}$$

$$\int_\Omega \gamma_{\omega_\varepsilon}\mathcal{Q}(e_\theta, e_w) \cdot \nabla e_w = (1-\gamma)\int_{B_\varepsilon} \mathcal{Q}(\theta, w) \cdot \nabla e_w \, . \tag{3.17}$$

Let us subtracting the second equality from the first one to obtain

$$\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(e_\theta) \cdot \nabla^s e_\theta + \mathcal{Q}(e_\theta, e_w) \cdot (e_\theta - \nabla e_w)) = (1-\gamma)\int_{B_\varepsilon} \mathcal{M}(\theta) \cdot \nabla^s e_\theta + \mathcal{Q}(\theta, w) \cdot (e_\theta - \nabla e_w) \, . \tag{3.18}$$

Taking into account the Cauchy-Schwarz inequality together with the triangular inequality, there is:

$$\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(e_\theta) \cdot \nabla^s e_\theta + \mathcal{Q}(e_\theta, e_w) \cdot (e_\theta - \nabla e_w)) \le C_1 \varepsilon \left(\|e_\theta\|_{\mathbf{H}^1(B_\varepsilon)} + \|e_w\|_{H^1(B_\varepsilon)} \right) \, , \tag{3.19}$$

where we have used the elliptic regularity of θ and w. Now, from the coercivity of the bilinear form on the left hand side of the above inequality it follows:

$$\alpha \left(\|e_\theta\|^2_{\mathbf{H}^1(\Omega)} + \|e_w\|^2_{H^1(\Omega)} \right) \le \int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(e_\theta) \cdot \nabla^s e_\theta + \mathcal{Q}(e_\theta, e_w) \cdot (e_\theta - \nabla e_w)) \, . \tag{3.20}$$

Therefore,

$$C_1 \varepsilon \left(\|e_\theta\|_{\mathbf{H}^1(\Omega)} + \|e_w\|_{H^1(\Omega)} \right) \ge \alpha \left(\|e_\theta\|^2_{\mathbf{H}^1(\Omega)} + \|e_w\|^2_{H^1(\Omega)} \right) \tag{3.21}$$

$$\ge \frac{\alpha}{2} \left(\|e_\theta\|_{\mathbf{H}^1(\Omega)} + \|e_w\|_{H^1(\Omega)} \right)^2 \, . \tag{3.22}$$

Finally, we have

$$\|e_\theta\|_{\mathbf{H}^1(\Omega)} + \|e_w\|_{H^1(\Omega)} \le C \varepsilon \, , \tag{3.23}$$

which leads to the result with the constant $C = \dfrac{2C_1}{\alpha}$ independent of the small parameter ε. □

4. Derivation of an Explicit Form for the Topological Derivative

Let us choose $\eta_w = w_\varepsilon - w$ and $\eta_\theta = \theta_\varepsilon - \theta$ as test functions in (3.4). Then, after subtracting the second result from the first one, we have

$$\int_\Omega (\mathcal{M}(\theta)\cdot\nabla^s\theta + \mathcal{Q}(\theta,w)\cdot(\theta-\nabla w)) = \int_\Omega (\mathcal{M}(\theta)\cdot\nabla^s\theta_\varepsilon + \mathcal{Q}(\theta,w)\cdot(\theta_\varepsilon-\nabla w_\varepsilon)) - \int_{\Gamma_{N_\theta}} \overline{m}\cdot(\theta_\varepsilon-\theta) + \int_{\Gamma_{N_w}} \overline{q}(w_\varepsilon - w)\ . \quad (4.1)$$

From the equality (4.1), the shape functional (3.1) can be re-written as

$$\psi(\chi) = \frac{1}{2}\int_\Omega (\mathcal{M}(\theta)\cdot\nabla^s\theta_\varepsilon + \mathcal{Q}(\theta,w)\cdot(\theta_\varepsilon-\nabla w_\varepsilon)) - \frac{1}{2}\int_{\Gamma_{N_\theta}} \overline{m}\cdot(\theta_\varepsilon+\theta) + \frac{1}{2}\int_{\Gamma_{N_w}} \overline{q}(w_\varepsilon + w)\ . \quad (4.2)$$

Similarly, let us set $\eta_w = w_\varepsilon - w$ and $\eta_\theta = \theta_\varepsilon - \theta$ as test functions in (3.10), which leads to

$$\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\theta_\varepsilon)\cdot\nabla^s\theta_\varepsilon + \mathcal{Q}(\theta_\varepsilon,w_\varepsilon)\cdot(\theta_\varepsilon-\nabla w_\varepsilon)) = \int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\theta_\varepsilon)\cdot\nabla^s\theta + \mathcal{Q}(\theta_\varepsilon,w_\varepsilon)\cdot(\theta-\nabla w)) + \int_{\Gamma_{N_\theta}} \overline{m}\cdot(\theta_\varepsilon-\theta) - \int_{\Gamma_{N_w}} \overline{q}(w_\varepsilon - w)\ . \quad (4.3)$$

After replacing (4.3) in the shape functional (3.8), we obtain

$$\psi(\chi_\varepsilon(\widehat{x})) = \frac{1}{2}\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\theta_\varepsilon)\cdot\nabla^s\theta + \mathcal{Q}(\theta_\varepsilon,w_\varepsilon)\cdot(\theta-\nabla w)) - \frac{1}{2}\int_{\Gamma_{N_\theta}} \overline{m}\cdot(\theta_\varepsilon+\theta) + \frac{1}{2}\int_{\Gamma_{N_w}} \overline{q}(w_\varepsilon + w)\ . \quad (4.4)$$

Now, we can compute the difference between (4.4) and (4.2) to obtain the following important result

$$\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi) = -\frac{1-\gamma}{2}\int_{B_\varepsilon} (\mathcal{M}(\theta)\cdot\nabla^s\theta_\varepsilon + \mathcal{Q}(\theta,w)\cdot(\theta_\varepsilon-\nabla w_\varepsilon))\ , \quad (4.5)$$

where we have used the definition (3.9) for the contrast $\gamma_{\omega_\varepsilon}$. Note that the above difference leads to an integral concentrated over the ball B_ε. Therefore, we need to know the asymptotic behavior of functions θ_ε and w_ε with respect to $\varepsilon \to 0$ in the neighborhood of the topological perturbation $B_\varepsilon(\widehat{x})$.

Let us introduce the following ansätze for the solutions $(\theta_\varepsilon, w_\varepsilon)$ to the perturbed boundary value problem (3.11)

$$\theta_\varepsilon(x) = \theta(x) + \varepsilon\phi((x-\widehat{x})/\varepsilon) + \widetilde{\theta}_\varepsilon(x)\ , \quad (4.6)$$

$$w_\varepsilon(x) = w(x) + \varepsilon z((x-\widehat{x})/\varepsilon) + \widetilde{w}_\varepsilon(x)\ . \quad (4.7)$$

Some terms in the above expansions require explanations. Functions θ and w are solutions to the unperturbed boundary value problem (3.7), while functions ϕ and z are solutions to exterior boundary value problems and $\widetilde{\theta}_\varepsilon$ and $\widetilde{w}_\varepsilon$ are the remainders. In particular, after introducing the above ansätze in (3.11) we obtain

$$-\varepsilon\,\mathrm{div}(\gamma_{\omega_\varepsilon}\mathbb{C}\nabla^s\phi) - \mathrm{div}(\gamma_{\omega_\varepsilon}\mathcal{M}(\widetilde{\theta}_\varepsilon)) + \gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon) + \varepsilon\,\gamma_{\omega_\varepsilon}\mathcal{Q}(\phi,z) = 0\ , \quad (4.8)$$

$$-\varepsilon\,\mathrm{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\nabla z) + \varepsilon\,\mathrm{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\phi) + \mathrm{div}(\gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon)) = 0\ , \quad (4.9)$$

since (θ, w) is solution to (3.7). Now, let us consider a change of variables in the form $(x - \widehat{x}) = \varepsilon y$, which implies $\nabla_y z(y) = \varepsilon \nabla z((x-\widehat{x})/\varepsilon)$ and $\nabla^s_y \phi(y) = \varepsilon \nabla^s \phi((x-\widehat{x})/\varepsilon)$. Therefore, in the fast variable y the first terms of both equations above have order $O(\varepsilon^{-1})$, allowing us to choose ϕ and z such that

$$\operatorname{div}_y(\gamma_\omega \mathbb{C} \nabla^s_y \phi) = 0 \qquad \text{and} \qquad \operatorname{div}_y(\gamma_\omega \mathbf{K} \nabla^s_y z) = 0 \ , \tag{4.10}$$

where $\omega = B_1$, with B_1 used to denote a ball of unitary radius and

$$\gamma_\omega = \begin{cases} 1 & \text{in} \quad \mathbb{R}^2 \setminus \omega \ , \\ \gamma & \text{in} \quad \omega \ . \end{cases} \tag{4.11}$$

Now, let us consider the transmission conditions on $\partial\omega_\varepsilon = \partial B_\varepsilon$ that appear in (3.11). In particular, taking into account that $n = (x-\widehat{x})/\varepsilon$ on the boundary ∂B_ε, we have

$$\begin{aligned} &(1-\gamma)\mathcal{M}(\theta)(\widehat{x})n + [\![\gamma_\omega \mathbb{C}\nabla^s_y \phi(y)]\!]n + \\ &\qquad \varepsilon(1-\gamma)[(\nabla\mathcal{M}(\theta)(\xi))n]n + [\![\gamma_{\omega_\varepsilon}\mathcal{M}(\widetilde{\theta}_\varepsilon(x))]\!]n = 0 \ , \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &(1-\gamma)\mathcal{Q}(\theta,w)(\widehat{x})\cdot n - [\![\gamma_\omega \mathbf{K}\nabla_y z(y)]\!]\cdot n + \\ &\varepsilon(1-\gamma)\mathbf{K}\phi((x-\widehat{x})/\varepsilon)\cdot n + \varepsilon(1-\gamma)(\nabla\mathcal{Q}(\theta,w)(\zeta))n\cdot n + [\![\gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon)(x)]\!]\cdot n = 0 \ , \end{aligned} \tag{4.13}$$

where $\mathcal{M}(\theta)(x)$ and $\mathcal{Q}(\theta,w)(x)$ have been expanded in Taylor series around $\widehat{x}$, so that ξ and ζ are used to denote intermediate points between x and $\widehat{x}$. After collecting the terms of the same power of ε, we obtain the following exterior problems for $\varepsilon \to 0$ defined in the new variable $y = (x-\widehat{x})/\varepsilon$

$$\left\{ \begin{array}{rcll} \operatorname{div}_y(\gamma_\omega \mathbb{C}\nabla^s_y\phi) &=& 0 & \text{in} \quad \omega \cup (\mathbb{R}^2\setminus\omega) \ , \\ \phi &\to& 0 & \|y\| \to \infty \ , \\ [\![\phi]\!] &=& 0 & \\ [\![\gamma_\omega \mathbb{C}\nabla^s_y\phi]\!]n &=& g_1 & \end{array} \right\} \text{on} \quad \partial\omega \ . \tag{4.14}$$

where $g_1 = -(1-\gamma)\mathcal{M}(\theta)(\widehat{x})n$, and

$$\left\{ \begin{array}{rcll} \operatorname{div}_y(\gamma_\omega \mathbf{K}\nabla_y z) &=& 0 & \text{in} \quad \omega \cup (\mathbb{R}^2\setminus\omega) \ , \\ z &\to& 0 & \|y\| \to \infty \ , \\ [\![z]\!] &=& 0 & \\ [\![\gamma_\omega \mathbf{K}\nabla_y z]\!]\cdot n &=& g_2 & \end{array} \right\} \text{on} \quad \partial\omega \ . \tag{4.15}$$

with $g_2 = (1-\gamma)\mathcal{Q}(\theta,w)(\widehat{x})\cdot n$. Finally, the remainder $(\widetilde{\theta}_\varepsilon, \widetilde{w}_\varepsilon)$ is solution to a boundary value problem that compensates for the discrepancies introduced by the boundary layers ϕ and z and by the higher order terms of the Taylor series expansion of $\mathcal{M}(\theta)(x)$ and $\mathcal{Q}(\theta,w)(x)$ around the point $\widehat{x} \in \Omega$, namely

$$\left\{ \begin{array}{rcll} \operatorname{div}(\gamma_{\omega_\varepsilon}\mathcal{M}(\widetilde{\theta}_\varepsilon)) - \gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon) &=& \varepsilon\gamma_{\omega_\varepsilon}\mathcal{Q}(\phi,z) & \text{in} \quad \omega_\varepsilon \cup (\Omega\setminus\omega_\varepsilon) \ , \\ -\operatorname{div}(\gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon)) &=& \varepsilon\operatorname{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\phi) & \text{in} \quad \omega_\varepsilon \cup (\Omega\setminus\omega_\varepsilon) \ , \\ \mathcal{M}(\widetilde{\theta}_\varepsilon) &=& \mathbb{C}\nabla^s\widetilde{\theta}_\varepsilon \ , & \\ \mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon) &=& \mathbf{K}(\widetilde{\theta}_\varepsilon - \nabla\widetilde{w}_\varepsilon) \ , & \\ \widetilde{\theta}_\varepsilon &=& \varepsilon^2\theta_0 & \text{on} \quad \Gamma_{D_\theta} \ , \\ \widetilde{w}_\varepsilon &=& \varepsilon^2 w_0 & \text{on} \quad \Gamma_{D_w} \ , \\ \mathcal{M}(\widetilde{\theta}_\varepsilon)n &=& \varepsilon^2\mathcal{M}(\theta_0)n & \text{on} \quad \Gamma_{N_\theta} \ , \\ \mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon)\cdot n &=& \varepsilon^2\mathcal{Q}(\theta_0,w_0)\cdot n & \text{on} \quad \Gamma_{N_w} \ , \\ [\![\gamma_{\omega_\varepsilon}\mathcal{M}(\widetilde{\theta}_\varepsilon)]\!]n &=& \varepsilon\widetilde{g}_1 & \\ [\![\gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}_\varepsilon,\widetilde{w}_\varepsilon)]\!]\cdot n &=& \varepsilon\widetilde{g}_2 & \\ [\![\widetilde{\theta}_\varepsilon]\!] &=& 0 & \\ [\![\widetilde{w}_\varepsilon]\!] &=& 0 & \end{array} \right. \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} \text{on} \quad \partial\omega_\varepsilon \ , \tag{4.16}$$

where $w_0 := -\varepsilon^{-1} z\big|_{\partial\Omega}$, $\theta_0 := -\varepsilon^{-1}\phi\big|_{\partial\Omega}$, $\widetilde{g}_1 = -(1-\gamma)[(\nabla\mathcal{M}(\theta)(\xi))n]n$ and $\widetilde{g}_2 = \widetilde{g}_h + \widetilde{g}_p$, with $\widetilde{g}_h = -(1-\gamma)[(\nabla\mathcal{Q}(\theta,w)(\zeta))n]\cdot n$ and $\widetilde{g}_p = -(1-\gamma)\mathbf{K}\phi(n)\cdot n$, where ξ and ζ are used to denote intermediate points between x and $\widehat{x}$.

Lemma 2. *Let $(\widetilde{\theta}_\varepsilon, \widetilde{w}_\varepsilon)$ be solution to (4.16). Then, the estimates $\|\widetilde{\theta}_\varepsilon\|_{\mathbf{H}^1(\Omega)} = o(\varepsilon)$ and $\|\widetilde{w}_\varepsilon\|_{H^1(\Omega)} = o(\varepsilon)$ hold true.*

Proof. The proof is left to Section 5.1 □

We replace (4.6) and (4.7) into (4.5) to obtain the following result,

$$\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi) = -\frac{1-\gamma}{2}\int_{B_\varepsilon} ((\mathcal{M}(\theta)\cdot\nabla^s\theta + \mathcal{Q}(\theta,w)\cdot(\theta-\nabla w))(\widehat{x}) + \varepsilon\mathcal{M}(\theta)(\widehat{x})\cdot\nabla^s\phi - \varepsilon\mathcal{Q}(\theta,w)(\widehat{x})\cdot\nabla z) + \mathcal{E}(\varepsilon)\ , \tag{4.17}$$

where $\mathcal{E}(\varepsilon) = \sum_{i=1}^{6}\mathcal{E}_i(\varepsilon) = o(\varepsilon^2)$ as can be seen in Section 5.2, with:

$$\mathcal{E}_1(\varepsilon) = -\frac{1-\gamma}{2}\int_{B_\varepsilon} (\mathcal{M}(\theta)\cdot\nabla^s\widetilde{\theta}_\varepsilon + \mathcal{Q}(\theta,w)\cdot(\widetilde{\theta}_\varepsilon - \nabla\widetilde{w}_\varepsilon))\ , \tag{4.18}$$

$$\mathcal{E}_2(\varepsilon) = -\frac{1-\gamma}{2}\int_{B_\varepsilon} (\mathcal{M}(\theta)\cdot\nabla^s\theta - (\mathcal{M}(\theta)\cdot\nabla^s\theta)(\widehat{x}))\ , \tag{4.19}$$

$$\mathcal{E}_3(\varepsilon) = -\frac{1-\gamma}{2}\int_{B_\varepsilon} (\mathcal{Q}(\theta,w)\cdot(\theta-\nabla w) - (\mathcal{Q}(\theta,w)\cdot(\theta-\nabla w))(\widehat{x}))\ , \tag{4.20}$$

$$\mathcal{E}_4(\varepsilon) = -\varepsilon\frac{1-\gamma}{2}\int_{B_\varepsilon} (\nabla^s\phi\cdot(\mathcal{M}(\theta) - \mathcal{M}(\theta)(\widehat{x})))\ , \tag{4.21}$$

$$\mathcal{E}_5(\varepsilon) = -\varepsilon\frac{1-\gamma}{2}\int_{B_\varepsilon} (\phi-\nabla z)\cdot(\mathcal{Q}(\theta,w) - \mathcal{Q}(\theta,w)(\widehat{x}))\ , \tag{4.22}$$

$$\mathcal{E}_6(\varepsilon) = -\varepsilon\frac{1-\gamma}{2}\int_{B_\varepsilon} \phi\cdot\mathcal{Q}(\theta,w)(\widehat{x})\ . \tag{4.23}$$

Let us consider again a change of variable in the form $(x-\widehat{x}) = \varepsilon y$. Then, the difference (4.17) can be written as:

$$\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi) = -\varepsilon^2\frac{1-\gamma}{2}\int_{B_1} ((\mathcal{M}(\theta)\cdot\nabla^s\theta + \mathcal{Q}(\theta,w)\cdot(\theta-\nabla w))(\widehat{x}) + \nabla^s\theta(\widehat{x})\cdot\mathbb{C}\nabla^s_y\phi(y) - (\theta-\nabla w)(\widehat{x})\cdot\mathbf{K}\nabla_y z(y) + \mathcal{E}(\varepsilon)\ . \tag{4.24}$$

The solutions to the exterior problems (4.14) and (4.15) are known in the literature since they have exactly the same structure as the Navier and Laplace boundary value problems, respectively. In addition, for the particular case associated with circular inclusions such solutions are explicitly known. See for instance [18, Ch. 5, pp. 144 and 156]. Therefore, from the Eshelby theorem [9, 10] we have the following representations for the solution to (4.14)

$$\mathbb{C}\nabla^s_y\phi(y)\Big|_{B_1} = \mathbb{T}\mathcal{M}(\theta)(\widehat{x})\ , \tag{4.25}$$

where $\mathbb{T}$ is a fourth order isotropic tensor given by:

$$\mathbb{T} = \frac{1}{2}\frac{1-\gamma}{1+\gamma\alpha_2}\left(2\alpha_2\mathbb{I} + \frac{\alpha_1-\alpha_2}{1+\gamma\alpha_1}\mathbf{I}\otimes\mathbf{I}\right)\ , \tag{4.26}$$

and for the solution to (4.15)

$$\mathbf{K}\nabla_y z(y)\Big|_{B_1} = \mathbf{T}\mathcal{Q}(\theta,w)(\widehat{x})\ , \tag{4.27}$$

where $\mathbf{T}$ is a second order isotropic tensor written as:

$$\mathbf{T} = -\frac{1-\gamma}{1+\gamma}\mathbf{I} \,. \tag{4.28}$$

Now, let us consider these last results in (4.24), which allow us to evaluate the integral over B_1 explicitly, leading to

$$\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi) \;=\; \pi\varepsilon^2(\mathbb{P}\mathcal{M}(\theta(\widehat{x}))\cdot\nabla^s\theta(\widehat{x}) + \mathbf{P}\mathcal{Q}(\theta(\widehat{x}),w(\widehat{x}))\cdot(\theta(\widehat{x}) - \nabla w(\widehat{x}))) + \mathcal{E}(\varepsilon) \;. \tag{4.29}$$

Finally, we have all necessary elements to state the main result of the paper, which is:

Theorem 3. *Let $\mathcal{J}_\varepsilon(\theta_\varepsilon, w_\varepsilon)$ be the topologically perturbed energy shape functional given by (3.8). Then, it admits the topological asymptotic expansion of the form*

$$\mathcal{J}_\varepsilon(\theta_\varepsilon, w_\varepsilon) \;=\; \mathcal{J}(\theta,w) + \pi\varepsilon^2(\mathbb{P}\mathcal{M}(\theta)\cdot\nabla^s\theta + \mathbf{P}\mathcal{Q}(\theta,w)\cdot(\theta - \nabla w)) + \mathcal{E}(\varepsilon) \quad , \tag{4.30}$$

with the function $f(\varepsilon) = \pi\varepsilon^2$ and the remainder $\mathcal{E}(\varepsilon) = o(\varepsilon^2)$ according to Section 5.2. So that the topological derivative is given by

$$\mathcal{T}(\widehat{x}) = \mathbb{P}\mathcal{M}(\theta(\widehat{x}))\cdot\nabla^s\theta(\widehat{x}) + \mathbf{P}\mathcal{Q}(\theta(\widehat{x}),w(\widehat{x}))\cdot(\theta(\widehat{x}) - \nabla w(\widehat{x})) \,. \tag{4.31}$$

Some terms in the above expression require explanation. The polarization matrix $\mathbb{P}$, associated with the bending effects, is given by the following fourth order isotropic tensor

$$\mathbb{P} = -\frac{1-\gamma}{2}(\mathbb{I} + \mathbb{T}) = -\frac{1}{2}\frac{1-\gamma}{1+\gamma\alpha_2}\left((1+\alpha_2)\mathbb{I} + \frac{1}{2}(\alpha_1 - \alpha_2)\frac{1-\gamma}{1+\gamma\alpha_1}\mathbf{I}\otimes\mathbf{I}\right) \,, \tag{4.32}$$

while the polarization matrix $\mathbf{P}$, associated with the shear effects, is given by a second order isotropic tensor defined as

$$\mathbf{P} = -\frac{1-\gamma}{2}(\mathbf{I} - \mathbf{T}) = -\frac{1-\gamma}{1+\gamma}\mathbf{I} \,, \tag{4.33}$$

where $\mathbf{I}$ and $\mathbb{I}$ are the second and fourth order identity tensors, respectively. Finally, the coefficients α_1 and α_2 are written as:

$$\alpha_1 = \frac{1+\nu}{1-\nu} \qquad \text{and} \qquad \alpha_2 = \frac{3-\nu}{1+\nu} \,. \tag{4.34}$$

5. Estimation for the Remainders

In this section, the proof of Lemma 2 and the estimation for the remainder $\mathcal{E}(\varepsilon)$ left in the asymptotic expansion (4.17) are presented. We assume that the topological perturbation $B_\varepsilon(\widehat{x})$ doesn't touch the boundary $\partial\Omega$, namely, $\overline{B_\varepsilon(\widehat{x})} \Subset \Omega$.

5.1. **Proof of Lemma 2 .** For the sake of completeness, we introduce the explicit solution to the scalar exterior problem (4.15), which can be found in many references (see for instance [18, Ch. 5, pp. 144]). Namely,

$$z((x-\widehat{x})/\varepsilon)\Big|_{\Omega\setminus\overline{B_\varepsilon}} \;=\; \frac{\varepsilon}{\|x-\widehat{x}\|^2}\beta\cdot(x-\widehat{x}) \,, \tag{5.1}$$

$$z((x-\widehat{x})/\varepsilon)\Big|_{B_\varepsilon} \;=\; \varepsilon^{-1}\beta\cdot(x-\widehat{x}) \,, \tag{5.2}$$

where $\beta\in\mathbb{R}^2$ is a constant vector. From the above formulas, we observe that $z\big|_{\partial\Omega} = -\varepsilon w_0$, with function w_0 independent of the small parameter ε. In addition, from a simple calculation there are $\|z\|_{L^2(\partial B_\varepsilon)} = O(\sqrt{\varepsilon})$ and $\|z\|_{L^2(\Omega)} = O(\varepsilon\sqrt{|\log\varepsilon|}) = o(\varepsilon^\delta)$, with $\delta < 1$. The vector case is completely analogous to the scalar one [3]. In fact, using the same arguments, the explicit solution to the vector exterior problem (4.14) provides $\phi\big|_{\partial\Omega} = -\varepsilon\theta_0$, with function θ_0 independent of the small parameter ε [19, Ch. 4, pp. 105]. Finally, $\|\phi\|_{\mathbf{L}^2(\partial B_\varepsilon)} = O(\sqrt{\varepsilon})$ and $\|\phi\|_{\mathbf{L}^2(\Omega)} = o(\varepsilon^\delta)$.

Before proceeding, let us state the following important result, which can be found in many references (see, for instance, [1, 14]).

Theorem 4. *Let us consider an open and bounded domain $D \subset \mathbb{R}^2$, with Lipschitz boundary ∂D. Then,*

$$\|\varphi\|_{L^2(\partial D)} \leq C\|\varphi\|_{H^1(D)} , \tag{5.3}$$

for each $\varphi \in H^1(D)$, where the constant C is independent of φ. The same result holds true for vector functions in $H^1(D) \times H^1(D)$.

Now, we have all elements to prove Lemma 2. We start by decomposing the solution to (4.16) as $(\widetilde{\theta}_\varepsilon, \widetilde{w}_\varepsilon) = (\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) + (\widetilde{\theta}^p_\varepsilon, \widetilde{w}^p_\varepsilon)$. Therefore:

Lemma 5. *Let $(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon)$ be solution to the following variational problem: Find $(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \in \widetilde{\mathcal{U}}_\varepsilon$, for all $(\eta_\theta, \eta_w) \in \mathcal{V}$, such that:*

$$\left\{ \begin{array}{rcl} \displaystyle\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^h_\varepsilon) \cdot \nabla^s \eta_\theta + \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot \eta_\theta) &=& \displaystyle\varepsilon \int_{\partial B_\varepsilon} \widetilde{g}_1 \cdot \eta_\theta + \varepsilon^2 \int_{\Gamma_{N_\theta}} \mathcal{M}(\theta_0) n \cdot \eta_\theta \ , \\ \displaystyle\int_\Omega \gamma_{\omega_\varepsilon} \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot \nabla \eta_w &=& \displaystyle\varepsilon \int_{\partial B_\varepsilon} \widetilde{g}_h \, \eta_w + \varepsilon^2 \int_{\Gamma_{N_w}} (\mathcal{Q}(\theta_0, w_0) \cdot n) \eta_w \ , \end{array} \right. \tag{5.4}$$

with the set $\widetilde{\mathcal{U}}_\varepsilon$ defined as $\widetilde{\mathcal{U}}_\varepsilon := \mathcal{V} + \varepsilon^2\{(\varphi_\theta, \varphi_w)\}$, where $(\varphi_\theta, \varphi_w) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$ is a lifting of the Dirichlet boundary data (θ_0, w_0), endowed with the additional properties $\varphi_\theta = 0$ on $\partial B_\varepsilon \cup (\partial\Omega \setminus \Gamma_{D_\theta})$ and $\varphi_w = 0$ on $\partial B_\varepsilon \cup (\partial\Omega \setminus \Gamma_{D_w})$. In addition $\widetilde{g}_1 = -(1-\gamma)[(\nabla \mathcal{M}(\theta)(\xi))n]n$ and $\widetilde{g}_h = -(1-\gamma)[(\nabla \mathcal{Q}(\theta, w)(\zeta))n] \cdot n$, with ξ and ζ used to denote intermediate points between x and $\widehat{x}$. Then, the estimates $\|\widetilde{\theta}^h_\varepsilon\|_{\mathbf{H}^1(\Omega)} = O(\varepsilon^2)$ and $\|\widetilde{w}^h_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^2)$ hold true.

Proof. By taking $\eta_\theta = \widetilde{\theta}^h_\varepsilon - \varepsilon^2 \varphi_\theta$ and $\eta_w = \widetilde{w}^h_\varepsilon - \varepsilon^2 \varphi_w$ in (5.4), we have the equalities

$$\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^h_\varepsilon) \cdot \nabla^s \widetilde{\theta}^h_\varepsilon + \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot \widetilde{\theta}^h_\varepsilon) = \varepsilon \int_{\partial B_\varepsilon} \widetilde{g}_1 \cdot \widetilde{\theta}^h_\varepsilon + \\ \varepsilon^2 \int_{\Gamma_{N_\theta}} \mathcal{M}(\theta_0) n \cdot \widetilde{\theta}^h_\varepsilon + \varepsilon^2 \int_{\Gamma_{D_\theta}} \mathcal{M}(\widetilde{\theta}^h_\varepsilon) n \cdot \theta_0 \ , \tag{5.5}$$

and

$$\int_\Omega \gamma_{\omega_\varepsilon} \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot \nabla \widetilde{w}^h_\varepsilon = \varepsilon \int_{\partial B_\varepsilon} \widetilde{g}_h \, \widetilde{w}^h_\varepsilon + \\ \varepsilon^2 \int_{\Gamma_{N_w}} (\mathcal{Q}(\theta_0, w_0) \cdot n) \widetilde{w}^h_\varepsilon + \varepsilon^2 \int_{\Gamma_{D_w}} (\mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot n) w_0 \ . \tag{5.6}$$

The last two terms in the above equalities can be respectively replaced by

$$\begin{aligned} \int_{\Gamma_{D_\theta}} \mathcal{M}(\widetilde{\theta}^h_\varepsilon) n \cdot \theta_0 &= \int_{\partial\Omega} \gamma_{\omega_\varepsilon} \mathcal{M}(\widetilde{\theta}^h_\varepsilon) n \cdot \varphi_\theta \\ &= \int_\Omega \gamma_{\omega_\varepsilon} \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot \varphi_\theta + \int_\Omega \gamma_{\omega_\varepsilon} \mathcal{M}(\widetilde{\theta}^h_\varepsilon) \cdot \nabla^s \varphi_\theta \ , \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \int_{\Gamma_{D_w}} (\mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot n) w_0 &= \int_{\partial\Omega} \gamma_{\omega_\varepsilon} (\mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot n) \varphi_w \\ &= \int_\Omega \gamma_{\omega_\varepsilon} \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon) \cdot \nabla \varphi_w \ , \end{aligned} \tag{5.8}$$

since $\varphi_\theta = 0$ on $\partial B_\varepsilon \cup (\partial\Omega \setminus \Gamma_{D_\theta})$, $\varphi_w = 0$ on $\partial B_\varepsilon \cup (\partial\Omega \setminus \Gamma_{D_w})$, $\gamma_{\omega_\varepsilon} = 1$ on $\partial\Omega$, $\operatorname{div}(\gamma_{\omega_\varepsilon} \mathcal{M}(\widetilde{\theta}^h_\varepsilon)) = \gamma_{\omega_\varepsilon} \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon)$ and $\operatorname{div}(\gamma_{\omega_\varepsilon} \mathcal{Q}(\widetilde{\theta}^h_\varepsilon, \widetilde{w}^h_\varepsilon)) = 0$ in $\omega_\varepsilon \cup (\Omega \setminus \omega_\varepsilon)$. After subtracting (5.6) from (5.5) and taking into account these last two results, we have

$$\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^h_\varepsilon)\cdot\nabla^s\widetilde{\theta}^h_\varepsilon + \mathcal{Q}(\widetilde{\theta}^h_\varepsilon,\widetilde{w}^h_\varepsilon)\cdot(\widetilde{\theta}^h_\varepsilon - \nabla\widetilde{w}^h_\varepsilon)) = \varepsilon\int_{\partial B_\varepsilon}\widetilde{g}_1\cdot\widetilde{\theta}^h_\varepsilon - \varepsilon\int_{\partial B_\varepsilon}\widetilde{g}_h\ \widetilde{w}^h_\varepsilon +$$
$$\varepsilon^2\int_\Omega \gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}^h_\varepsilon,\widetilde{w}^h_\varepsilon)\cdot\varphi_\theta + \varepsilon^2\int_\Omega\gamma_{\omega_\varepsilon}\mathcal{M}(\widetilde{\theta}^h_\varepsilon)\cdot\nabla^s\varphi_\theta - \varepsilon^2\int_\Omega\gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}^h_\varepsilon,\widetilde{w}^h_\varepsilon)\cdot\nabla\varphi_w +$$
$$\varepsilon^2\int_{\Gamma_{N_\theta}}\mathcal{M}(\theta_0)n\cdot\widetilde{\theta}^h_\varepsilon - \varepsilon^2\int_{\Gamma_{N_w}}(\mathcal{Q}(\theta_0,w_0)\cdot n)\widetilde{w}^h_\varepsilon\ . \quad (5.9)$$

From the Cauchy-Schwarz inequality, the Trace Theorem 4 and the triangular inequality, we obtain

$$\int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^h_\varepsilon)\cdot\nabla^s\widetilde{\theta}^h_\varepsilon + \mathcal{Q}(\widetilde{\theta}^h_\varepsilon,\widetilde{w}^h_\varepsilon)\cdot(\widetilde{\theta}^h_\varepsilon - \nabla\widetilde{w}^h_\varepsilon)) \leq \varepsilon^2 C_1(\|\widetilde{\theta}^h_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^h_\varepsilon\|_{H^1(\Omega)})\ , \quad (5.10)$$

where we have used the interior elliptic regularity of function θ and w. Finally, from the coercivity of the bilinear form on the left hand side of (5.10), namely,

$$\alpha\left(\|\widetilde{\theta}^h_\varepsilon\|^2_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^h_\varepsilon\|^2_{H^1(\Omega)}\right) \leq \int_\Omega \gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^h_\varepsilon)\cdot\nabla^s\widetilde{\theta}^h_\varepsilon + \mathcal{Q}(\widetilde{\theta}^h_\varepsilon,\widetilde{w}^h_\varepsilon)\cdot(\widetilde{\theta}^h_\varepsilon - \nabla\widetilde{w}^h_\varepsilon))\ , \quad (5.11)$$

we have,

$$\begin{aligned} C_1\varepsilon^2\left(\|\widetilde{\theta}^h_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^h_\varepsilon\|_{H^1(\Omega)}\right) &\geq \alpha\left(\|\widetilde{\theta}^h_\varepsilon\|^2_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^h_\varepsilon\|^2_{H^1(\Omega)}\right) \\ &\geq \frac{\alpha}{2}\left(\|\widetilde{\theta}^h_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^h_\varepsilon\|_{H^1(\Omega)}\right)^2\ . \end{aligned} \quad (5.12)$$

Then we obtain

$$\|\widetilde{\theta}^h_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^h_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^2\ , \quad (5.13)$$

which leads to the result, with $C = \dfrac{2C_1}{\alpha}$ independent of ε. □

Lemma 6. *Let $(\widetilde{\theta}^p_\varepsilon,\widetilde{w}^p_\varepsilon)$ be solution to the following variational problem: Find $(\widetilde{\theta}^p_\varepsilon,\widetilde{w}^p_\varepsilon)\in\mathcal{V}$, for all $(\eta_\theta,\eta_w)\in\mathcal{V}$, such that:*

$$\left\{\begin{aligned} \int_\Omega\gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^p_\varepsilon)\cdot\nabla^s\eta_\theta + \mathcal{Q}(\widetilde{\theta}^p_\varepsilon,\widetilde{w}^p_\varepsilon)\cdot\eta_\theta) &= \varepsilon\int_\Omega\gamma_{\omega_\varepsilon}\mathbf{K}\nabla z\cdot\eta_\theta - \varepsilon\int_\Omega\gamma_{\omega_\varepsilon}\mathbf{K}\phi\cdot\eta_\theta\ , \\ \int_\Omega\gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}^p_\varepsilon,\widetilde{w}^p_\varepsilon)\cdot\nabla\eta_w &= \varepsilon\int_\Omega \operatorname{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\phi)\eta_w + \varepsilon\int_{\partial B_\varepsilon}\widetilde{g}_p\eta_w\ , \end{aligned}\right. \quad (5.14)$$

where $\widetilde{g}_p = -(1-\gamma)\mathbf{K}\phi(n)\cdot n$ and since $\mathcal{Q}(\phi,z) = \mathbf{K}(\phi-\nabla z)$. Then, the estimates $\|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)} = o(\varepsilon)$ and $\|\widetilde{w}^p_\varepsilon\|_{H^1(\Omega)} = o(\varepsilon)$ hold true.

Proof. By setting $\eta_\theta = \widetilde{\theta}^p_\varepsilon$ and $\eta_w = \widetilde{w}^p_\varepsilon$ in (5.14), we have the equalities

$$\int_\Omega\gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^p_\varepsilon)\cdot\nabla^s\widetilde{\theta}^p_\varepsilon + \mathcal{Q}(\widetilde{\theta}^p_\varepsilon,\widetilde{w}^p_\varepsilon)\cdot\widetilde{\theta}^p_\varepsilon) = \varepsilon\int_\Omega\gamma_{\omega_\varepsilon}\mathbf{K}\nabla z\cdot\widetilde{\theta}^p_\varepsilon - \varepsilon\int_\Omega\gamma_{\omega_\varepsilon}\mathbf{K}\phi\cdot\widetilde{\theta}^p_\varepsilon\ , \quad (5.15)$$

$$\int_\Omega\gamma_{\omega_\varepsilon}\mathcal{Q}(\widetilde{\theta}^p_\varepsilon,\widetilde{w}^p_\varepsilon)\cdot\nabla\widetilde{w}_\varepsilon = \varepsilon\int_\Omega\operatorname{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\phi)\widetilde{w}^p_\varepsilon + \varepsilon\int_{\partial B_\varepsilon}\widetilde{g}_p\widetilde{w}^p_\varepsilon\ . \quad (5.16)$$

After subtracting the second equality from the first one, there is

$$\int_\Omega\gamma_{\omega_\varepsilon}(\mathcal{M}(\widetilde{\theta}^p_\varepsilon)\cdot\nabla^s\widetilde{\theta}^p_\varepsilon + \mathcal{Q}(\widetilde{\theta}^p_\varepsilon,\widetilde{w}^p_\varepsilon)\cdot(\widetilde{\theta}^p_\varepsilon - \nabla\widetilde{w}^p_\varepsilon)) = -\varepsilon\int_{\partial B_\varepsilon}\widetilde{g}_p\widetilde{w}^p_\varepsilon$$
$$\varepsilon\int_\Omega\gamma_{\omega_\varepsilon}\mathbf{K}\nabla z\cdot\widetilde{\theta}^p_\varepsilon - \varepsilon\int_\Omega\operatorname{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\phi)\widetilde{w}^p_\varepsilon - \varepsilon\int_\Omega\gamma_{\omega_\varepsilon}\mathbf{K}\phi\cdot\widetilde{\theta}^p_\varepsilon\ . \quad (5.17)$$

From the Cauchy-Schwarz inequality and the Trace Theorem 4, the first term on the right hand side of (5.17) can be bounded as follows

$$\int_{\partial B_\varepsilon}\widetilde{g}_p\widetilde{w}^p_\varepsilon \leq C_0\|\phi\|_{\mathbf{L}^2(\partial B_\varepsilon)}\|\widetilde{w}^p_\varepsilon\|_{H^1(B_\varepsilon)} \leq C_1\varepsilon^{1/2}\|\widetilde{w}^p_\varepsilon\|_{H^1(\Omega)}\ , \quad (5.18)$$

since $\|\phi\|_{\mathbf{L}^2(\partial B_\varepsilon)} = O(\sqrt{\varepsilon})$. Now, let us consider the second term on the right hand side of (5.17). Integration by parts yields

$$\int_\Omega \gamma_{\omega_\varepsilon} \mathbf{K}\nabla z \cdot \widetilde{\theta}^p_\varepsilon = (1-\gamma)\int_{\partial B_\varepsilon} \mathbf{K}\widetilde{\theta}^p_\varepsilon \cdot nz - \int_\Omega \operatorname{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\widetilde{\theta}^p_\varepsilon) z\,, \tag{5.19}$$

since $\widetilde{\theta}^p_\varepsilon = 0$ on $\partial\Omega$. From the Cauchy-Schwarz inequality and the Trace Theorem 4, there is

$$\int_\Omega \gamma_{\omega_\varepsilon} \mathbf{K}\nabla z \cdot \widetilde{\theta}^p_\varepsilon \le C_2 \|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(B_\varepsilon)} \|z\|_{L^2(\partial B_\varepsilon)} + C_3 \|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)} \|z\|_{L^2(\Omega)} \le C_4 \varepsilon^{1/2} \|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)}\,, \tag{5.20}$$

since $\|z\|_{L^2(\partial B_\varepsilon)} = O(\sqrt{\varepsilon})$ and $\|z\|_{L^2(\Omega)} = o(\varepsilon^\delta)$, with $\delta < 1$. Analogously, for the third term on the right hand side of (5.17), we have

$$\int_\Omega \operatorname{div}(\gamma_{\omega_\varepsilon}\mathbf{K}\phi)\widetilde{w}^p_\varepsilon \le C_5 \varepsilon^{1/2} \|\widetilde{w}^p_\varepsilon\|_{H^1(\Omega)}\,. \tag{5.21}$$

Using similar arguments for the fourth term on the right hand side of (5.17), there is

$$\int_\Omega \gamma_{\omega_\varepsilon} \mathbf{K}\phi \cdot \widetilde{\theta}^p_\varepsilon \le C_6 \varepsilon^\delta \|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)}\,. \tag{5.22}$$

Then, from the above results together with the triangular inequality we obtain

$$\int_\Omega \gamma_{\omega_\varepsilon} (\mathcal{M}(\widetilde{\theta}^p_\varepsilon) \cdot \nabla^s \widetilde{\theta}^p_\varepsilon + \mathcal{Q}(\widetilde{\theta}^p_\varepsilon, \widetilde{w}^p_\varepsilon) \cdot (\widetilde{\theta}^p_\varepsilon - \nabla \widetilde{w}^p_\varepsilon)) \le C_7 \varepsilon^{3/2} \left(\|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^p_\varepsilon\|_{H^1(\Omega)} \right)\,. \tag{5.23}$$

From the coercivity of the bilinear form on the left hand side of (5.23), namely,

$$\alpha \left(\|\widetilde{\theta}^p_\varepsilon\|^2_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^p_\varepsilon\|^2_{H^1(\Omega)} \right) \le \int_\Omega \gamma_{\omega_\varepsilon} (\mathcal{M}(\widetilde{\theta}^p_\varepsilon) \cdot \nabla^s \widetilde{\theta}^p_\varepsilon + \mathcal{Q}(\widetilde{\theta}^p_\varepsilon, \widetilde{w}^p_\varepsilon) \cdot (\widetilde{\theta}^p_\varepsilon - \nabla \widetilde{w}^p_\varepsilon))\,, \tag{5.24}$$

there is,

$$\begin{aligned} C_7 \varepsilon^{3/2} \left(\|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^p_\varepsilon\|_{H^1(\Omega)} \right) &\ge \alpha \left(\|\widetilde{\theta}^p_\varepsilon\|^2_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^p_\varepsilon\|^2_{H^1(\Omega)} \right) \\ &\ge \frac{\alpha}{2} \left(\|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^p_\varepsilon\|_{H^1(\Omega)} \right)^2\,. \end{aligned} \tag{5.25}$$

Then, we finally obtain

$$\|\widetilde{\theta}^p_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}^p_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^{3/2}\,, \tag{5.26}$$

which leads to the result, with $C = \dfrac{2C_7}{\alpha}$ independent of ε. □

Finally, the proof of Lemma 2 follows immediately from the results of Lemma 5 and Lemma 6.

5.2. **Estimation for the remainder** $\mathcal{E}(\varepsilon)$ **.** Let us start by considering the remainder $\mathcal{E}_1(\varepsilon)$ given by (4.18), namely

$$\mathcal{E}_1(\varepsilon) = -\frac{1-\gamma}{2} \int_{B_\varepsilon} (M(x) \pm M(\widehat{x})) \cdot \nabla^s \widetilde{\theta}_\varepsilon + (Q(x) \pm Q(\widehat{x})) \cdot (\widetilde{\theta}_\varepsilon - \nabla \widetilde{w}_\varepsilon)\,, \tag{5.27}$$

where the notations $M = \mathcal{M}(\theta)$ and $Q = \mathcal{Q}(\theta, w)$ have been introduced. From the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \mathcal{E}_1(\varepsilon) \le C_0 \Big(& \|M(x) - M(\widehat{x})\|_{\mathbf{L}^2(B_\varepsilon)} \|\nabla^s \widetilde{\theta}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \|M(\widehat{x})\|_{\mathbf{L}^2(B_\varepsilon)} \|\nabla^s \widetilde{\theta}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \\ & \|Q(x) - Q(\widehat{x})\|_{\mathbf{L}^2(B_\varepsilon)} \|\widetilde{\theta}_\varepsilon - \nabla \widetilde{w}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \|Q(\widehat{x})\|_{\mathbf{L}^2(B_\varepsilon)} \|\widetilde{\theta}_\varepsilon - \nabla \widetilde{w}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} \Big)\,. \end{aligned} \tag{5.28}$$

From the interior elliptic regularity of functions θ and w, there are $\|M(x) - M(\widehat{x})\| \le c_0 \|x - \widehat{x}\|$ and $\|Q(x) - Q(\widehat{x})\| \le c_1 \|x - \widehat{x}\|$ in $B_\varepsilon(\widehat{x})$, where c_0 and c_1 are constants independent of ε. Then

$$\mathcal{E}_1(\varepsilon) \le C_1 \varepsilon \left(\|\nabla^s \widetilde{\theta}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \|\widetilde{\theta}_\varepsilon - \nabla \widetilde{w}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} \right)\,, \tag{5.29}$$

where we have used the fact that $\|Q(\widehat{x})\|_{\mathbf{L}^2(B_\varepsilon)} = O(\varepsilon)$, $\|M(\widehat{x})\|_{\mathbf{L}^2(B_\varepsilon)} = O(\varepsilon)$ and $\|x-\widehat{x}\|_{\mathbf{L}^2(B_\varepsilon)} = O(\varepsilon^2)$. From the triangular inequality together with Lemma 2, we finally obtain

$$
\begin{aligned}
\mathcal{E}_1(\varepsilon) &\leq C_1\varepsilon \left(\|\nabla^s\widetilde{\theta}_\varepsilon\|_{\mathbf{L}^2(\Omega)} + \|\widetilde{\theta}_\varepsilon\|_{\mathbf{L}^2(\Omega)} + \|\nabla\widetilde{w}_\varepsilon\|_{L^2(\Omega)} \right) \\
&\leq C_1\varepsilon \left(\|\widetilde{\theta}_\varepsilon\|_{\mathbf{H}^1(\Omega)} + \|\widetilde{w}_\varepsilon\|_{H^1(\Omega)} \right) = o(\varepsilon^2) \ . \qquad (5.30)
\end{aligned}
$$

Regarding the remainders $\mathcal{E}_2(\varepsilon)$ and $\mathcal{E}_3(\varepsilon)$ given respectively by (4.19) and (4.20), let us introduce the notations $h_2 = \mathcal{M}(\theta)\cdot\nabla^s\theta$ and $h_3 = \mathcal{Q}(\theta, w)\cdot(\theta - \nabla w)$. From the interior elliptic regularity of the functions θ and w, we have $\|h_i(x) - h_i(\widehat{x})\| \leq c_i\|x - \widehat{x}\|$ for $i = 2, 3$ in $B_\varepsilon(\widehat{x})$, where c_i are constants independent of ε. Therefore,

$$
\begin{aligned}
\mathcal{E}_i(\varepsilon) &= -\frac{1-\gamma}{2}\int_{B_\varepsilon}(h_i(x) - h_i(\widehat{x})) \\
&\leq C_i\int_{B_\varepsilon}\|x - \widehat{x}\| = o(\varepsilon^2) \ . \qquad (5.31)
\end{aligned}
$$

We introduce the notations $G_4 = \nabla^s\phi$, $H_4 = \mathcal{M}(\theta)$ and $G_5 = \phi - \nabla z$, $H_5 = \mathcal{Q}(\theta, w)$. Once again, from the interior elliptic regularity of the functions θ and w, there are $\|H_i(x) - H_i(\widehat{x})\| \leq c_i\|x - \widehat{x}\|$ for $i = 4, 5$ in $B_\varepsilon(\widehat{x})$, where c_i are constants independent of ε. Thus

$$
\begin{aligned}
\mathcal{E}_i(\varepsilon) &= -\varepsilon\frac{1-\gamma}{2}\int_{B_\varepsilon}G_i\cdot(H_i(x) - H_i(\widehat{x})) \\
&\leq \varepsilon C_0\|G_i\|_{\mathbf{L}^2(B_\varepsilon)}\|H_i(x) - H_i(\widehat{x})\|_{\mathbf{L}^2(B_\varepsilon)} \\
&\leq \varepsilon C_i\|G_i\|_{\mathbf{L}^2(B_\varepsilon)}\|x - \widehat{x}\|_{\mathbf{L}^2(B_\varepsilon)} = o(\varepsilon^2) \ , \qquad (5.32)
\end{aligned}
$$

where we have also used the interior elliptic regularity of functions ϕ and z.

Finally, from a change of variable of the form $(x - \widehat{x}) = \varepsilon y$, the remainder $\mathcal{E}_6(\varepsilon)$ given by (4.23) can be written as follows

$$
\mathcal{E}_6(\varepsilon) = -\varepsilon^3\frac{1-\gamma}{2}\int_{B_1}\phi(y)\cdot\mathcal{Q}(\theta, w)(\widehat{x}) = o(\varepsilon^2) \ , \qquad (5.33)
$$

where B_1 is a ball of unitary radius.

6. Conclusions

In this paper the topological derivative for the total potential energy associated with the Reissner-Mindlin plate bending problem has been derived. Since it is modeled by a coupled system of partial differential equations, an appropriated ansätz has been introduced, leading to two exterior uncoupled problems. For the particular case associated with circular inclusions, these problems have explicit solutions. Therefore, a closed formula for the topological derivative has been derived with help of such solutions together with the Eshelby theorem. In addition, the existence of the topological derivative has been proved and precise estimates for the remainders have been derived. The obtained result can be used in many applications such as topology optimization of structures under inplane bending and transversal shear effects, allowing for overcome some numerical difficulties associated with the Kirchhoff plate bending model reported, for instance, in [17].

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