SENSITIVITY OF A GENERAL CLASS OF SHAPE FUNCTIONALS TO TOPOLOGICAL CHANGES

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Abstract. The topological derivative represents the first term of the asymptotic expansion of a given shape functional with respect to the small parameter which measures the size of singular domain perturbations. The topological derivative has been successfully applied in the treatment of problems such as topology optimization, inverse analysis and image processing. In this paper, the calculation of the topological derivative for a general class of shape functionals is presented. In particular, we evaluate the topological derivative of a modified energy shape functional associated to the steady-state heat conduction problem, considering the nucleation of a small circular inclusion as the topological perturbation. Several methods were proposed to calculate the topological derivative. In this paper, the so-called topological-shape sensitivity method is extended to deal with a modified adjoint method, leading to an alternative approach to calculate the topological derivative based on shape sensitivity analysis together with a modified Lagrangian method. Since we are dealing with a general class of shape functionals, which are not necessarily associated to the energy, we will show that this new approach simplifies the most delicate step of the topological derivative calculation, namely, the asymptotic analysis of the adjoint state.

1. Introduction

The topological derivative represents the first term of the asymptotic expansion of a given shape functional with respect to the small parameter which measures the size of singular domain perturbations, such as holes, inclusions, source-terms and cracks. The topological asymptotic analysis was introduced in the fundamental paper by [14] and can be seen as a mathematical justification for the so-called bubble method [6]. The topological derivative has been successfully applied in the treatment of problems such as topology optimization [5], inverse analysis [10] and image processing [9]. More recently, it has also been applied in the multi-scale constitutive modeling context [4], fracture mechanics sensitivity analysis [16] and damage evolution modeling [1]. See also the book by Novotny & Sokolowski [13].

In order to introduce these ideas, let us consider an open and bounded domain $\Omega \subset \mathbb{R}^2$, which is subject to a non-smooth perturbation confined in a small region $\omega_\varepsilon(\hat{x}) = \hat{x} + \varepsilon\omega$ of size ε, as shown in fig. 1. Here, $\hat{x}$ is an arbitrary point of Ω and ω is a fixed domain of $\mathbb{R}^2$. We introduce a characteristic function $x \mapsto \chi(x)$, $x \in \mathbb{R}^2$, associated to the unperturbed domain, namely $\chi = \mathbb{1}_\Omega$. Then, we define a characteristic function associated to the topologically perturbed domain of the form $x \mapsto \chi_\varepsilon(\hat{x}; x)$, $x \in \mathbb{R}^2$. In the case of a hole, for example, $\chi_\varepsilon(\hat{x}) = \mathbb{1}_\Omega - \mathbb{1}_{\overline{\omega_\varepsilon(\hat{x})}}$ and the singular perturbed domain is given by $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$. Then, we assume that a given shape functional $\psi(\chi_\varepsilon(\hat{x}))$, associated to the topologically perturbed domain, admits the following topological asymptotic expansion

$$\psi(\chi_\varepsilon(\hat{x})) = \psi(\chi) + f(\varepsilon) D_T\psi(\hat{x}) + o(f(\varepsilon)) \ , \tag{1.1}$$

where $\psi(\chi)$ is the shape functional associated to the unperturbed domain and $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0$. The function $\hat{x} \mapsto D_T\psi(\hat{x})$ is called the topological derivative of ψ at $\hat{x}$. Therefore, the term $f(\varepsilon) D_T\psi(\hat{x})$ represents a first order correction of $\psi(\chi)$ to approximate $\psi(\chi_\varepsilon(\hat{x}))$. Since we are dealing with singular domain perturbations, the shape functionals $\psi(\chi_\varepsilon(\hat{x}))$ and $\psi(\chi)$ are associated to topologically different domains. Therefore, we

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need to perform an asymptotic analysis of the shape functional $\psi(\chi_\varepsilon(\hat{x}))$ with respect to the small parameter ε.

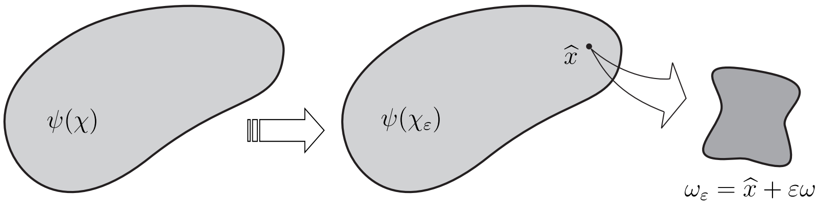


Figure 1. The topological derivative concept.

Several methods were proposed to calculate the topological derivative. In this paper, we extend the so-called topological-shape sensitivity method developed by [12] to deal with the modified adjoint method proposed by [3], leading to an alternative approach to calculate the topological derivative based on shape sensitivity analysis together with a modified Lagrangian method. The proposed approach is based on the following result [12]:

$$D_T\psi(\hat{x}) = \lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\frac{d}{d\varepsilon}\psi(\chi_\varepsilon(\hat{x})) \ . \tag{1.2}$$

The derivative of $\psi(\chi_\varepsilon(\hat{x}))$ with respect to ε can be seen as the sensitivity of $\psi(\chi_\varepsilon(\hat{x}))$, in the classical sense [15], to the domain variation produced by a uniform expansion of the perturbation ω_ε, namely, $\omega_{\varepsilon+t}(\hat{x}) = \omega_\varepsilon(\hat{x}) + t\omega$. In fact, we have

$$\frac{d}{d\varepsilon}\psi(\chi_\varepsilon(\hat{x})) = \lim_{t\to 0}\frac{\psi(\chi_{\varepsilon+t}(\hat{x})) - \psi(\chi_\varepsilon(\hat{x}))}{t} \ , \tag{1.3}$$

where $\psi(\chi_{\varepsilon+t}(\hat{x}))$ is the shape functional associated to the perturbed domain, whose perturbation is given by $\omega_{\varepsilon+t}$. Therefore, since $\psi(\chi_{\varepsilon+t}(\hat{x}))$ and $\psi(\chi_\varepsilon(\hat{x}))$ are now associated to topologically identical domains, we can use the concept of shape sensitivity analysis as an intermediate step in the topological derivative calculation.

In this paper the calculation of the topological derivative for a general class of shape functionals is presented. In particular, the topological derivative of a modified energy shape functional associated to the Laplace equation, considering the nucleation of a small inclusion as the topological perturbation, is derived. Since we are dealing with a general class of shape functionals, which are not necessarily associated to the energy, we will show later that the proposed new approach simplifies the most delicate step of the topological derivative calculation, namely, the asymptotic analysis of the adjoint state.

2. Problem Formulation

The shape functional in the unperturbed domain which we are dealing with is defined as

$$\psi(\chi) := \mathcal{J}_\chi(u) = -\frac{1}{2}\int_\Omega Bq(u)\cdot\nabla u \ , \tag{2.1}$$

where B is a given second order symmetric constant tensor and the scalar function u is the solution to the variational problem:

$$\left\{\begin{array}{l} \text{Find } u \in \mathcal{U}, \text{ such that} \\ \displaystyle\int_\Omega q(u)\cdot\nabla\eta = \int_{\Gamma_N}\overline{q}\eta \quad \forall\eta\in\mathcal{V} \ , \\ \qquad\qquad \text{with} \quad q(u) = -k\nabla u \ . \end{array}\right. \tag{2.2}$$

In the above equation, k is the thermal conductivity of the medium, assumed to be constant everywhere. The set $\mathcal{U}$ and the space $\mathcal{V}$ are respectively defined as

$$\mathcal{U}{:=}\{\varphi\in H^1(\Omega) : \varphi_{|\Gamma_D} = \overline{u}\} \quad \text{and} \quad \mathcal{V}{:=}\{\varphi\in H^1(\Omega) : \varphi_{|\Gamma_D} = 0\} \ . \tag{2.3}$$

In addition, $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ with $\Gamma_D \cap \Gamma_N = \varnothing$, where Γ_D and Γ_N are Dirichlet and Neumann boundaries, respectively. Thus $\overline{u}$ is a Dirichlet data on Γ_D and $\overline{q}$ is a Neumann data on Γ_N, both assumed to be smooth enough. See the details in fig. 2. The strong equation associated to the above variational problem (2.2) reads:

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{Find}\, u, \text{ such that}} \\ \operatorname{div} q(u) & = & 0 & \text{in} \quad \Omega \ , \\ q(u) & = & -k\nabla u & \\ u & = & \overline{u} & \text{on} \quad \Gamma_D \ , \\ q(u)\cdot n & = & \overline{q} & \text{on} \quad \Gamma_N \ . \end{array}\right. \tag{2.4}$$

Remark 1. *The functional (2.1) includes a large range of shape functions, which shall be useful for practical applications. In particular, when $B = I$, the functional (2.1) degenerates to the energy. In addition, when $B \neq I$, the analysis becomes much more involved, which justifies the introduction of a modified adjoint state.*

The domain is topologically perturbed by the nucleation of a small inclusion. More precisely, the perturbed domain is obtained when a circular hole $\omega_\varepsilon(\hat{x}) = \mathcal{B}_\varepsilon(\hat{x})$ is introduced inside Ω, where $\mathcal{B}_\varepsilon(\hat{x})$ is used to denote a ball of radius ε and center at $\hat{x} \in \Omega$. Next, this region is filled by an inclusion with different material property. In particular, we introduce a piecewise constant function γ_ε of the form

$$\gamma_\varepsilon := \left\{\begin{array}{lll} 1 & \text{in} & \Omega \setminus \overline{\mathcal{B}_\varepsilon} \\ \gamma & \text{in} & \mathcal{B}_\varepsilon \end{array}\right. , \tag{2.5}$$

where $\gamma \in \mathbb{R}^+$ is the contrast in the material property. In the case of a circular inclusion, we can construct a shape change velocity field $\mathfrak{V} \in C^\infty(\Omega)$ that represents a uniform expansion of $\mathcal{B}_\varepsilon(\hat{x})$. In fact, it is sufficient to define $\mathfrak{V}$ on the boundaries $\partial\Omega$ and $\partial\mathcal{B}_\varepsilon$ in the following way

$$\left\{\begin{array}{rcrll} \mathfrak{V} & = & 0 & \text{on} & \partial\Omega \\ \mathfrak{V} & = & -n & \text{on} & \partial\mathcal{B}_\varepsilon \end{array}\right. , \tag{2.6}$$

where $n = -(x-\hat{x})/\varepsilon$, with $x \in \partial\mathcal{B}_\varepsilon$, is the normal unit vector field pointing toward the center of the circular inclusion $\mathcal{B}_\varepsilon$. We will see later that this velocity field $\mathfrak{V}$ is the key point in using the result (1.2), leading to a simple and constructive method to calculate the topological derivative. Note that in this case the topologies of the original and perturbed domains are preserved. However, we are introducing a non-smooth perturbation in the coefficients of the differential operator through the contrast γ_ε, by changing the material property of the background in a small region $\mathcal{B}_\varepsilon \subset \Omega$. Therefore, the sensitivity of the shape functional with respect to the nucleation of an inclusion can also be studied through the topological asymptotic analysis concept, which is, in fact, the most appropriate approach for such a problem. Now, let us state the same problem in the perturbed domain. In this case, the shape functional reads

$$\psi(\chi_\varepsilon) := \mathcal{J}_{\chi_\varepsilon}(u_\varepsilon) = -\frac{1}{2}\int_\Omega Bq_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon \ , \tag{2.7}$$

where the scalar function u_ε solves the variational problem:

$$\left\{\begin{array}{l} \text{Find}\, u_\varepsilon \in \mathcal{U}_\varepsilon, \text{ such that} \\ \displaystyle\int_\Omega q_\varepsilon(u_\varepsilon)\cdot\nabla\eta = \int_{\Gamma_N} \overline{q}\eta \quad \forall \eta \in \mathcal{V}_\varepsilon \ , \\ \qquad\qquad \text{with} \quad q_\varepsilon(u_\varepsilon) = -\gamma_\varepsilon k \nabla u_\varepsilon \ . \end{array}\right. \tag{2.8}$$

with γ_ε defined by (2.5). The set $\mathcal{U}_\varepsilon$ and the space $\mathcal{V}_\varepsilon$ are defined as

$$\mathcal{U}_\varepsilon := \{\varphi \in \mathcal{U} : [\![\varphi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon\} \quad \text{and} \quad \mathcal{V}_\varepsilon := \{\varphi \in \mathcal{V} : [\![\varphi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon\} \ , \tag{2.9}$$

where the operator $[\![\varphi]\!]$ is used to denote the jump of function φ on the boundary of the inclusion $\partial\mathcal{B}_\varepsilon$, namely, $[\![\varphi]\!] := \varphi_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \varphi_{|_{\mathcal{B}_\varepsilon}}$ on $\partial\mathcal{B}_\varepsilon$. See the details in fig. 2. The strong equation associated to the variational problem (2.8) reads:

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } u_\varepsilon, \text{ such that}} \\ \mathrm{div} q_\varepsilon(u_\varepsilon) & = & 0 & \text{in} \quad \Omega \,, \\ q_\varepsilon(u_\varepsilon) & = & -\gamma_\varepsilon k \nabla u_\varepsilon & \\ u_\varepsilon & = & \overline{u} & \text{on} \quad \Gamma_D \,, \\ q_\varepsilon(u_\varepsilon)\cdot n & = & \overline{q} & \text{on} \quad \Gamma_N \,, \\ [\![u_\varepsilon]\!] & = & 0 & \multirow{2}{*}{$\left.\right\}$ on $\partial\mathcal{B}_\varepsilon$.} \\ [\![q_\varepsilon(u_\varepsilon)]\!]\cdot n & = & 0 & \end{array}\right. \tag{2.10}$$

Now, we need to introduce the adjoint state v_ε. In this particular case, v_ε is the solution to the adjoint equation of the form:

$$\left\{\begin{array}{rcl} \multicolumn{3}{l}{\text{Find } v_\varepsilon \in \mathcal{V}_\varepsilon, \text{ such that}} \\ -\displaystyle\int_\Omega q_\varepsilon(v_\varepsilon)\cdot\nabla\eta & = & -\langle D_u \mathcal{J}_{\chi_\varepsilon}(u), \eta\rangle \\ & = & \displaystyle\int_\Omega B q_\varepsilon(u)\cdot\nabla\eta \quad \forall \eta\in\mathcal{V}_\varepsilon \,, \\ & & \text{with} \quad q_\varepsilon(v_\varepsilon) = -\gamma_\varepsilon k\nabla v_\varepsilon \,. \end{array}\right. \tag{2.11}$$

The strong equation associated to the variational problem (2.11) reads:

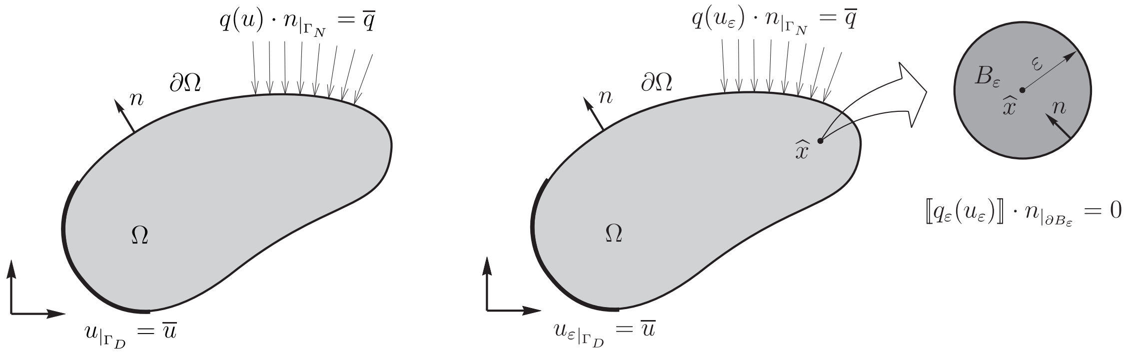
$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } v_\varepsilon, \text{ such that}} \\ \mathrm{div} q_\varepsilon(v_\varepsilon) & = & -\mathrm{div}(B q_\varepsilon(u)) & \text{in} \quad \Omega \,, \\ q_\varepsilon(v_\varepsilon) & = & -\gamma_\varepsilon k \nabla v_\varepsilon & \\ v_\varepsilon & = & 0 & \text{on} \quad \Gamma_D \,, \\ q_\varepsilon(v_\varepsilon)\cdot n & = & -B q_\varepsilon(u)\cdot n & \text{on} \quad \Gamma_N \,, \\ [\![v_\varepsilon]\!] & = & 0 & \multirow{2}{*}{$\left.\right\}$ on $\partial\mathcal{B}_\varepsilon$.} \\ [\![q_\varepsilon(v_\varepsilon)]\!]\cdot n & = & -[\![B q_\varepsilon(u)]\!]\cdot n & \end{array}\right. \tag{2.12}$$

Hence, with this construction the right hand side of the adjoint equation does not depend on the parameter ε through the function u_ε. This feature will simplify the asymptotic analysis of the adjoint state v_ε. Finally, the adjoint state associated to the unperturbed domain is given by taking $\varepsilon = 0$ in (2.11), namely, v is the solution to the adjoint equation of the form:

$$\left\{\begin{array}{rcl} \multicolumn{3}{l}{\text{Find } v \in \mathcal{V}, \text{ such that}} \\ -\displaystyle\int_\Omega q(v)\cdot\nabla\eta & = & -\langle D_u \mathcal{J}_{\chi}(u), \eta\rangle \\ & = & \displaystyle\int_\Omega B q(u)\cdot\nabla\eta \quad \forall \eta\in\mathcal{V} \,, \\ & & \text{with} \quad q(v) = -k\nabla v \,. \end{array}\right. \tag{2.13}$$

The strong equation associated to the variational problem (2.11) reads:

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } v, \text{ such that}} \\ \mathrm{div} q(v) & = & -\mathrm{div}(B q(u)) & \text{in} \quad \Omega \,, \\ q(v) & = & -k \nabla v & \\ v & = & 0 & \text{on} \quad \Gamma_D \,, \\ q(v)\cdot n & = & -B q(u)\cdot n & \text{on} \quad \Gamma_N \,, \end{array}\right. \tag{2.14}$$



(a) unperturbed domain (b) perturbed domain

FIGURE 2. The Laplace problem defined in the unperturbed and perturbed domains.

3. Shape Sensitivity Analysis

In order to apply the result (1.2), we firstly need to evaluate the shape derivative of functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon)$ with respect to a uniform expansion of the inclusion $\mathcal{B}_\varepsilon$. Before starting, we note that after considering the constitutive relation $q_\varepsilon(u_\varepsilon) = -\gamma_\varepsilon k\nabla u_\varepsilon$ in (2.7), with the contrast γ_ε given by (2.5), the shape functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon)$ can be written as follows

$$\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon) = -\frac{1}{2}\left(\int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} Bq(u_\varepsilon)\cdot\nabla u_\varepsilon + \int_{\mathcal{B}_\varepsilon}\gamma Bq(u_\varepsilon)\cdot\nabla u_\varepsilon\right) . \tag{3.1}$$

where $q(u_\varepsilon) = -k\nabla u_\varepsilon$. Thus, we have the explicit dependence with respect to parameter ε. Therefore, let us start by proving the following result:

Proposition 2. *Let $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon)$ be the shape functional defined by (2.7). Then, the derivative of this functional with respect to the small parameter ε is given by*

$$\frac{d}{d\varepsilon}\psi(\chi_\varepsilon) = \dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon) = -\int_{\partial\mathcal{B}_\varepsilon}[\![\Sigma_\varepsilon - \nabla u_\varepsilon\otimes Bq_\varepsilon(u_\varepsilon - u)]\!]n\cdot n - \int_\Omega Bq_\varepsilon(u_\varepsilon - u)\cdot\nabla u_\varepsilon' . \tag{3.2}$$

where $\mathfrak{V}$ stands for the shape change velocity field defined through (2.6), and Σ_ε is a generalization of the classical Eshelby energy-momentum tensor [7] *given by*

$$\Sigma_\varepsilon = -\frac{1}{2}(Bq_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon + 2q_\varepsilon(u_\varepsilon)\cdot\nabla v_\varepsilon)I + (\nabla u_\varepsilon\otimes Bq_\varepsilon(u_\varepsilon) + \nabla u_\varepsilon\otimes q_\varepsilon(v_\varepsilon) + \nabla v_\varepsilon\otimes q_\varepsilon(u_\varepsilon)) \tag{3.3}$$

Proof. Before starting, let us recall that the constitutive operator is defined as $q_\varepsilon(\varphi) = -\gamma_\varepsilon k\nabla\varphi$ and that the shape change velocity field vanishes on the exterior boundary, namely, $\mathfrak{V} = 0$ on $\partial\Omega$. Thus, by making use of Reynolds' transport theorem and the concept of material derivative of spatial fields together with the relation between material and spatial derivatives of scalar fields, namely, $\dot{\varphi} = \varphi' + \nabla\varphi\cdot\mathfrak{V}$ [8], the derivative with respect to ε of the shape functional (3.1) is given by

$$\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon) = -\int_\Omega Bq_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon' - \frac{1}{2}\int_{\partial\mathcal{B}_\varepsilon}[\![Bq_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon]\!]n\cdot\mathfrak{V} . \tag{3.4}$$

In addition, we have

$$\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon) = -\frac{1}{2}\int_{\partial\mathcal{B}_\varepsilon}[\![Bq_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon]\!]n\cdot\mathfrak{V} - \int_\Omega Bq_\varepsilon(u_\varepsilon)\cdot\nabla\dot{u}_\varepsilon + \int_\Omega Bq_\varepsilon(u_\varepsilon)\cdot\nabla(\nabla u_\varepsilon\cdot\mathfrak{V}) . \tag{3.5}$$

From integration by parts, we obtain

$$\begin{aligned}\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon) \quad &= \quad -\frac{1}{2}\int_{\partial\mathcal{B}_\varepsilon}[\![Bq_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon]\!]n\cdot\mathfrak{V} + \int_{\partial\mathcal{B}_\varepsilon}[\![(\nabla u_\varepsilon\cdot\mathfrak{V})Bq_\varepsilon(u_\varepsilon)]\!]\cdot n \\ &\quad -\int_\Omega \operatorname{div}(Bq_\varepsilon(u_\varepsilon))\nabla u_\varepsilon\cdot\mathfrak{V} - \int_\Omega Bq_\varepsilon(u)\cdot\nabla\dot{u}_\varepsilon - \int_\Omega Bq_\varepsilon(u_\varepsilon - u)\cdot\nabla\dot{u}_\varepsilon ,\end{aligned} \tag{3.6}$$

and after some rearrangements

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon) &= -\frac{1}{2}\int_{\partial\mathcal{B}_\varepsilon} [\![Bq_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon I - 2\nabla u_\varepsilon \otimes Bq_\varepsilon(u_\varepsilon)]\!] n\cdot\mathfrak{V} \\
&\quad - \int_\Omega \operatorname{div}(Bq_\varepsilon(u_\varepsilon - u))\nabla u_\varepsilon\cdot\mathfrak{V} - \int_\Omega \operatorname{div}(Bq_\varepsilon(u))\nabla u_\varepsilon\cdot\mathfrak{V} \\
&\quad - \int_\Omega Bq_\varepsilon(u)\cdot\nabla\dot{u}_\varepsilon - \int_\Omega Bq_\varepsilon(u_\varepsilon - u)\cdot\nabla\dot{u}_\varepsilon\ . \qquad (3.7)
\end{aligned}
$$

Now, let us differentiate both sides of the state equation (2.8) with respect to ε, which leads to

$$
-\int_\Omega (q_\varepsilon(u_\varepsilon)\cdot\nabla\eta)' = \int_{\partial\mathcal{B}_\varepsilon} [\![q_\varepsilon(u_\varepsilon)\cdot\nabla\eta]\!] n\cdot\mathfrak{V}\ , \qquad (3.8)
$$

which can be rewritten as

$$
\begin{aligned}
\int_\Omega (q_\varepsilon(u_\varepsilon)\cdot\nabla\eta)' &= \int_\Omega (q_\varepsilon(u_\varepsilon)\cdot\nabla\eta)^{\cdot} - \int_\Omega \nabla(q_\varepsilon(u_\varepsilon)\cdot\nabla\eta)\cdot\mathfrak{V} \\
&= \int_\Omega q_\varepsilon(\dot{u}_\varepsilon)\cdot\nabla\eta - \int_\Omega (\nabla\mathfrak{V}^\top\nabla u_\varepsilon\cdot q_\varepsilon(\eta) + \nabla\mathfrak{V}^\top\nabla\eta\cdot q_\varepsilon(u_\varepsilon)) \\
&\quad - \int_\Omega ((\nabla\nabla u_\varepsilon)^\top q_\varepsilon(\eta)\cdot\mathfrak{V} + (\nabla\nabla\eta)^\top q_\varepsilon(u_\varepsilon)\cdot\mathfrak{V})\ . \qquad (3.9)
\end{aligned}
$$

After some rearrangements, we obtain

$$
\int_\Omega (q_\varepsilon(u_\varepsilon)\cdot\nabla\eta)' = \int_\Omega q_\varepsilon(\dot{u}_\varepsilon)\cdot\nabla\eta - \int_\Omega \nabla(\nabla u_\varepsilon\cdot\mathfrak{V})\cdot q_\varepsilon(\eta) - \int_\Omega \nabla(\nabla\eta\cdot\mathfrak{V})\cdot q_\varepsilon(u_\varepsilon)\ , \qquad (3.10)
$$

where we have taken into account that $\nabla\nabla(\cdot) = (\nabla\nabla(\cdot))^\top$. Therefore, we have

$$
-\int_\Omega q_\varepsilon(\dot{u}_\varepsilon)\cdot\nabla\eta = \int_{\partial\mathcal{B}_\varepsilon} [\![q_\varepsilon(u_\varepsilon)\cdot\nabla\eta]\!] n\cdot\mathfrak{V} - \int_\Omega q_\varepsilon(\eta)\cdot\nabla(\nabla u_\varepsilon\cdot\mathfrak{V}) - \int_\Omega q_\varepsilon(u_\varepsilon)\cdot\nabla(\nabla\eta\cdot\mathfrak{V}) \qquad (3.11)
$$

Since $\dot{u}_\varepsilon \in \mathcal{V}_\varepsilon$ and the modified adjoint state $v_\varepsilon \in \mathcal{V}_\varepsilon$, then we can take $\eta = v_\varepsilon$ in the above equation and $\eta = \dot{u}_\varepsilon$ in the modified adjoint equation (2.11), leading to

$$
\begin{aligned}
-\int_\Omega q_\varepsilon(v_\varepsilon)\cdot\nabla\dot{u}_\varepsilon &= \int_\Omega Bq_\varepsilon(u)\cdot\nabla\dot{u}_\varepsilon \qquad (3.12) \\
-\int_\Omega q_\varepsilon(\dot{u}_\varepsilon)\cdot\nabla v_\varepsilon &= \int_{\partial\mathcal{B}_\varepsilon} [\![q_\varepsilon(u_\varepsilon)\cdot\nabla v_\varepsilon]\!] n\cdot\mathfrak{V} - \int_\Omega q_\varepsilon(v_\varepsilon)\cdot\nabla(\nabla u_\varepsilon\cdot\mathfrak{V}) - \int_\Omega q_\varepsilon(u_\varepsilon)\cdot\nabla(\nabla v_\varepsilon\cdot\mathfrak{V}) \qquad (3.13)
\end{aligned}
$$

By symmetry of the above bilinear forms, we have

$$
\int_\Omega Bq_\varepsilon(u)\cdot\nabla\dot{u}_\varepsilon = \int_{\partial\mathcal{B}_\varepsilon} [\![q_\varepsilon(u_\varepsilon)\cdot\nabla v_\varepsilon]\!] n\cdot\mathfrak{V} - \int_\Omega q_\varepsilon(v_\varepsilon)\cdot\nabla(\nabla u_\varepsilon\cdot\mathfrak{V}) - \int_\Omega q_\varepsilon(u_\varepsilon)\cdot\nabla(\nabla v_\varepsilon\cdot\mathfrak{V}) \qquad (3.14)
$$

From integration by parts and some rearrangements, we obtain

$$
\begin{aligned}
\int_\Omega Bq_\varepsilon(u)\cdot\nabla\dot{u}_\varepsilon &= \int_{\partial\mathcal{B}_\varepsilon} [\![q_\varepsilon(u_\varepsilon)\cdot\nabla v_\varepsilon]\!] n\cdot\mathfrak{V} - \int_{\partial\mathcal{B}_\varepsilon} [\![\nabla u_\varepsilon\otimes q_\varepsilon(v_\varepsilon)]\!] n\cdot\mathfrak{V} - \int_{\partial\mathcal{B}_\varepsilon} [\![\nabla v_\varepsilon\otimes q_\varepsilon(u_\varepsilon)]\!] n\cdot\mathfrak{V} \\
&\quad + \int_\Omega \operatorname{div}(q_\varepsilon(v_\varepsilon))\nabla u_\varepsilon\cdot\mathfrak{V} + \int_\Omega \operatorname{div}(q_\varepsilon(u_\varepsilon))\nabla v_\varepsilon\cdot\mathfrak{V}\ . \qquad (3.15)
\end{aligned}
$$

After considering this last result, we obtain

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon) &= \int_{\partial\mathcal{B}_\varepsilon} [\![\Sigma_\varepsilon]\!] n\cdot\mathfrak{V} - \int_\Omega \operatorname{div}(Bq_\varepsilon(u_\varepsilon - u))\nabla u_\varepsilon\cdot\mathfrak{V} - \int_\Omega Bq_\varepsilon(u_\varepsilon - u)\cdot\nabla\dot{u}_\varepsilon \\
&\quad - \int_\Omega [\operatorname{div}(q_\varepsilon(v_\varepsilon)) + \operatorname{div}(Bq_\varepsilon(u))]\nabla u_\varepsilon\cdot\mathfrak{V} - \int_\Omega \operatorname{div}(q_\varepsilon(u_\varepsilon))\nabla v_\varepsilon\cdot\mathfrak{V}\ . \qquad (3.16)
\end{aligned}
$$

By taking into account that u_ε is the solution to the state equation (2.10) and that v_ε is the solution to the modified adjoint equation (2.12), namely, $\operatorname{div}q_\varepsilon(u_\varepsilon) = 0$ and $\operatorname{div}q_\varepsilon(v_\varepsilon) = -\operatorname{div}(Bq_\varepsilon(u))$, respectively, the last two terms in the above equation vanish. From the relation between material and spatial derivatives of scalar fields and after integration by parts, we have that

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon) &= \int_{\partial\mathcal{B}_\varepsilon} [\![\Sigma_\varepsilon]\!] n \cdot \mathfrak{V} - \int_\Omega \operatorname{div}(Bq_\varepsilon(u_\varepsilon - u))\nabla u_\varepsilon \cdot \mathfrak{V} \\
&- \int_\Omega Bq_\varepsilon(u_\varepsilon - u) \cdot \nabla u'_\varepsilon - \int_\Omega Bq_\varepsilon(u_\varepsilon - u) \cdot \nabla(\nabla u_\varepsilon \cdot \mathfrak{V}) \\
&= \int_{\partial\mathcal{B}_\varepsilon} [\![\Sigma_\varepsilon]\!] n \cdot \mathfrak{V} - \int_{\partial\mathcal{B}_\varepsilon} [\![(\nabla u_\varepsilon \cdot \mathfrak{V}) Bq_\varepsilon(u_\varepsilon - u)]\!] \cdot n - \int_\Omega Bq_\varepsilon(u_\varepsilon - u) \cdot \nabla u'_\varepsilon , \quad (3.17)
\end{aligned}
$$

which leads to the result □

4. Asymptotic Analysis of the Solutions

The shape derivative of functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon)$ is given in terms of an integral over the boundary of the inclusion $\partial\mathcal{B}_\varepsilon$ and also in terms of a domain integral associated to u'_ε (3.2). This domain integral comes out from the introduction of the modified adjoint state, solution to (2.11). Therefore, in order to apply the result (1.2), we need to know the behavior of the functions u_ε and v_ε with respect to ε. In particular, once we know these behaviors explicitly, we can identify function $f(\varepsilon)$ and perform the limit passage $\varepsilon \to 0$ in (1.2) to obtain the final formula for the topological derivative $D_T\psi(\hat{x})$ of the shape functional ψ. However, in general this is not an easy task. In fact, we need to perform an asymptotic analysis of u_ε and v_ε with respect to ε. In this section we present the formal calculation of the expansions for the solutions associated to the transmission condition on the inclusion. The rigorous justification for the asymptotic expansions of u_ε and v_ε is given in A.

4.1. Asymptotic expansion of the direct state.

Let us propose an ansatz for the expansion of u_ε in the form [11]:

$$
\begin{aligned}
u_\varepsilon(x) &= u(x) + w_\varepsilon(x) + \widetilde{u}_\varepsilon(x) \\
&= u(\hat{x}) + \nabla u(\hat{x}) \cdot (x - \hat{x}) + \frac{1}{2}\nabla\nabla u(\xi)(x - \hat{x}) \cdot (x - \hat{x}) + w_\varepsilon(x) + \widetilde{u}_\varepsilon(x) , \quad (4.1)
\end{aligned}
$$

where ξ is an intermediate point between x and $\hat{x}$. On the boundary of the inclusion $\partial\mathcal{B}_\varepsilon$ we have

$$
[\![q_\varepsilon(u_\varepsilon)]\!] \cdot n = 0 \quad \Rightarrow \quad \partial_n u_{\varepsilon}{}_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma \partial_n u_{\varepsilon}{}_{|_{\mathcal{B}_\varepsilon}} = 0 , \quad (4.2)
$$

with $q_\varepsilon(\varphi) = -\gamma_\varepsilon k \nabla\varphi$. Thus, the normal derivative of the above expansion, evaluated on $\partial\mathcal{B}_\varepsilon$, leads to

$$
(1-\gamma)\nabla u(\hat{x})\cdot n - \varepsilon(1-\gamma)\nabla\nabla u(\xi) n\cdot n + \partial_n w_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n w_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} + \partial_n \widetilde{u}_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n \widetilde{u}_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} = 0 . \quad (4.3)
$$

Thus, we can choose w_ε such that

$$
\partial_n w_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n w_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} = -(1-\gamma)\nabla u(\hat{x}) \cdot n \quad \text{on} \quad \partial\mathcal{B}_\varepsilon . \quad (4.4)
$$

Now, the following exterior problem is considered, and formally obtained with $\varepsilon \to 0$:

$$
\left\{
\begin{array}{rcll}
\multicolumn{4}{l}{\text{Find } w_\varepsilon, \text{ such that}} \\
\operatorname{div} q_\varepsilon(w_\varepsilon) &=& 0 & \text{in} \quad \mathbb{R}^2 , \\
q_\varepsilon(w_\varepsilon) &=& -\gamma_\varepsilon k \nabla w_\varepsilon & \\
w_\varepsilon &\to& 0 & \text{at} \quad \infty , \\
w_{\varepsilon}{}_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - w_{\varepsilon}{}_{|_{\mathcal{B}_\varepsilon}} &=& 0 & \multirow{2}{*}{\text{on} \quad \partial\mathcal{B}_\varepsilon ,} \\
\partial_n w_{\varepsilon}{}_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n w_{\varepsilon}{}_{|_{\mathcal{B}_\varepsilon}} &=& \hat{u} &
\end{array}
\right. \quad (4.5)
$$

with $\hat{u} = -(1-\gamma)\nabla u(\hat{x}) \cdot n$. The above boundary-value problem admits an explicit solution, namely

$$
\begin{aligned}
w_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} &= \frac{1-\gamma}{1+\gamma}\frac{\varepsilon^2}{\|x-\hat{x}\|^2}\nabla u(\hat{x})\cdot(x-\hat{x}) \ , && (4.6)\\
w_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} &= \frac{1-\gamma}{1+\gamma}\nabla u(\hat{x})\cdot(x-\hat{x}) \ . && (4.7)
\end{aligned}
$$

Now we can construct $\widetilde{u}_\varepsilon$ in such a way that it compensates for the discrepancies introduced by the higher-order terms in ε as well as by the boundary-layer w_ε on the exterior boundary $\partial\Omega$. It means that the remainder $\widetilde{u}_\varepsilon$ must be the solution to the following boundary-value problem:

$$
\left\{
\begin{array}{rcll}
\multicolumn{4}{l}{\text{Find}\, \widetilde{u}_\varepsilon, \text{ such that}}\\
\mathrm{div} q_\varepsilon(\widetilde{u}_\varepsilon) &=& 0 & \text{in} \quad \Omega \ ,\\
q_\varepsilon(\widetilde{u}_\varepsilon) &=& -\gamma_\varepsilon k \nabla \widetilde{u}_\varepsilon & \\
\widetilde{u}_\varepsilon &=& -\varepsilon^2 g & \text{on} \quad \Gamma_D \ ,\\
q_\varepsilon(\widetilde{u}_\varepsilon)\cdot n &=& -\varepsilon^2 q(g)\cdot n & \text{on} \quad \Gamma_N \ ,\\
[\![\widetilde{u}_\varepsilon]\!] &=& 0 & \multirow{2}{*}{$\Big\}$ on $\partial\mathcal{B}_\varepsilon$,}\\
[\![q_\varepsilon(\widetilde{u}_\varepsilon)]\!]\cdot n &=& \varepsilon h &
\end{array}
\right. \qquad (4.8)
$$

where $g = \varepsilon^{-2} w_\varepsilon$ and $h = k(1-\gamma)\nabla\nabla u(\xi) n \cdot n$. Clearly $\widetilde{u}_\varepsilon = O(\varepsilon)$, since $w_\varepsilon = O(\varepsilon^2)$ on the exterior boundary $\partial\Omega$. However, this estimate can be improved. In fact, according to Lemma 7 in A, with $\delta_\varepsilon = \widetilde{u}_\varepsilon$, we have $\|\widetilde{u}_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^2)$. Finally, the expansion for u_ε reads

$$
\begin{aligned}
u_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} &= u(x) + \frac{1-\gamma}{1+\gamma}\frac{\varepsilon^2}{\|x-\hat{x}\|^2}\nabla u(\hat{x})\cdot(x-\hat{x}) + O(\varepsilon^2) \ , && (4.9)\\
u_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} &= u(x) + \frac{1-\gamma}{1+\gamma}\nabla u(\hat{x})\cdot(x-\hat{x}) + O(\varepsilon^2) \ . && (4.10)
\end{aligned}
$$

4.2. **Asymptotic expansion of the adjoint state.** Let us propose again an ansatz for the expansion of v_ε in the form [11]:

$$
\begin{aligned}
v_\varepsilon(x) &= v(x) + w_\varepsilon(x) + \widetilde{v}_\varepsilon(x)\\
&= v(\hat{x}) + \nabla v(\hat{x})\cdot(x-\hat{x}) + \frac{1}{2}\nabla\nabla v(\xi)(x-\hat{x})\cdot(x-\hat{x}) + w_\varepsilon(x) + \widetilde{v}_\varepsilon(x) \ , \qquad (4.11)
\end{aligned}
$$

where ξ is an intermediate point between x and $\hat{x}$. On the boundary of the inclusion $\partial\mathcal{B}_\varepsilon$ we have

$$
[\![q_\varepsilon(v_\varepsilon)]\!]\cdot n \;=\; -[\![Bq_\varepsilon(u)]\!]\cdot n \Rightarrow \partial_n v_{\varepsilon|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n v_{\varepsilon|_{\mathcal{B}_\varepsilon}} = -(1-\gamma)B\nabla u\cdot n \ , \qquad (4.12)
$$

with $q_\varepsilon(\varphi) = -\gamma_\varepsilon k\nabla\varphi$. Thus, the normal derivative of the above expansion, evaluated on $\partial\mathcal{B}_\varepsilon$, leads to

$$
\begin{aligned}
&(1-\gamma)\nabla v(\hat{x})\cdot n - \varepsilon(1-\gamma)\nabla\nabla v(\xi)n\cdot n + \partial_n w_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n w_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} +\\
&\qquad\qquad \partial_n\widetilde{v}_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n\widetilde{v}_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} = -(1-\gamma)B\nabla u\cdot n \ . \qquad (4.13)
\end{aligned}
$$

Thus, we can choose w_ε such that

$$
\partial_n w_\varepsilon(x)_{|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n w_\varepsilon(x)_{|_{\mathcal{B}_\varepsilon}} = -(1-\gamma)(\nabla v(\hat{x}) + B\nabla u(\hat{x}))\cdot n \quad \text{on} \quad \partial\mathcal{B}_\varepsilon \ , \qquad (4.14)
$$

where we have expanded $B\nabla u(x)$ in Taylor's series around the center $\hat{x}$ of the inclusion. Note that from the construction of the adjoint state we have

$$
-\Delta v_\varepsilon = \mathrm{div}(B\nabla u) \quad \text{in} \quad \Omega \quad \text{and} \quad -\Delta v = \mathrm{div}(B\nabla u) \quad \text{in} \quad \Omega \ , \qquad (4.15)
$$

recalling that $q_\varepsilon(\varphi) = -\gamma_\varepsilon k\nabla\varphi$ and $q(\varphi) = -k\nabla\varphi$. The solution v_ε of the first equation should be understood together with the transmission conditions on the interface $\partial\mathcal{B}_\varepsilon$. In addition, $v_\varepsilon = v = 0$ on Γ_D and $\partial_n v_\varepsilon = \partial_n v = -B\nabla u\cdot n$ on Γ_N. It means that both boundary-value problems associated to v_ε and v have the same source-terms, except, of course, on the boundary of the inclusion $\partial\mathcal{B}_\varepsilon$. In particular, the transmission condition, namely $\partial_n v_{\varepsilon|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}}} - \gamma\partial_n v_{\varepsilon|_{\mathcal{B}_\varepsilon}} =$

$-(1-\gamma)B\nabla u \cdot n$ on $\partial\mathcal{B}_\varepsilon$, does not depend on the parameter ε through the solution u_ε. Now, the following exterior problem is considered, and formally obtained with $\varepsilon \to 0$:

$$\left\{\begin{array}{rcl@{\quad}l} \multicolumn{4}{l}{\text{Find } w_\varepsilon, \text{ such that}} \\ \operatorname{div} q_\varepsilon(w_\varepsilon) & = & 0 & \text{in} \quad \mathbb{R}^2 , \\ q_\varepsilon(w_\varepsilon) & = & -\gamma_\varepsilon k \nabla w_\varepsilon & \\ w_\varepsilon & \to & 0 & \text{at} \quad \infty , \\ w_\varepsilon|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} - w_\varepsilon|_{\mathcal{B}_\varepsilon} & = & 0 & \multirow{2}{*}{\text{on} \quad $\partial\mathcal{B}_\varepsilon$,} \\ \partial_n w_\varepsilon|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} - \gamma \partial_n w_\varepsilon|_{\mathcal{B}_\varepsilon} & = & \hat{v} & \end{array}\right. \tag{4.16}$$

with $\hat{v} = -(1-\gamma)(\nabla v(\hat{x}) + B\nabla u(\hat{x})) \cdot n$. The above boundary-value problem admits an explicit solution, namely

$$w_\varepsilon(x)|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} = \frac{1-\gamma}{1+\gamma}\frac{\varepsilon^2}{\|x-\hat{x}\|^2}(\nabla v(\hat{x}) + B\nabla u(\hat{x})) \cdot (x-\hat{x}) , \tag{4.17}$$

$$w_\varepsilon(x)|_{\mathcal{B}_\varepsilon} = \frac{1-\gamma}{1+\gamma}(\nabla v(\hat{x}) + B\nabla u(\hat{x})) \cdot (x-\hat{x}) . \tag{4.18}$$

Now we can construct $\widetilde{v}_\varepsilon$ in such a way that it compensates for the discrepancies introduced by the higher-order terms in ε as well as by the boundary-layer w_ε on the exterior boundary $\partial\Omega$. It means that the remainder $\widetilde{v}_\varepsilon$ must be the solution to the following boundary-value problem:

$$\left\{\begin{array}{rcl@{\quad}l} \multicolumn{4}{l}{\text{Find } \widetilde{v}_\varepsilon, \text{ such that}} \\ \operatorname{div} q_\varepsilon(\widetilde{v}_\varepsilon) & = & 0 & \text{in} \quad \Omega , \\ q_\varepsilon(\widetilde{v}_\varepsilon) & = & -\gamma_\varepsilon k \nabla \widetilde{v}_\varepsilon & \\ \widetilde{v}_\varepsilon & = & -\varepsilon^2 g & \text{on} \quad \Gamma_D , \\ q_\varepsilon(\widetilde{v}_\varepsilon) \cdot n & = & -\varepsilon^2 q(g) \cdot n & \text{on} \quad \Gamma_N , \\ [\![\widetilde{v}_\varepsilon]\!] & = & 0 & \multirow{2}{*}{\text{on} \quad $\partial\mathcal{B}_\varepsilon$,} \\ [\![q_\varepsilon(\widetilde{v}_\varepsilon)]\!] \cdot n & = & \varepsilon h & \end{array}\right. \tag{4.19}$$

with $g = \varepsilon^{-2} w_\varepsilon$ and $h = k(1-\gamma)(\nabla\nabla v(\xi)n + B(\nabla\nabla u(\zeta)n) \cdot n$, where ζ is an intermediate point between x and $\hat{x}$. Once again, we clearly have $\widetilde{v}_\varepsilon = O(\varepsilon)$, since $w_\varepsilon = O(\varepsilon^2)$ on the exterior boundary $\partial\Omega$. However, this estimate can be improved. In fact, according to Lemma 7 in A, with $\delta_\varepsilon = \widetilde{v}_\varepsilon$, we have $\|\widetilde{v}_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^2)$. Finally, the expansion for v_ε reads

$$v_\varepsilon(x)|_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} = v(x) + \frac{1-\gamma}{1+\gamma}\frac{\varepsilon^2}{\|x-\hat{x}\|^2}(\nabla u(\hat{x}) + B\nabla u(\hat{x})) \cdot (x-\hat{x}) + O(\varepsilon^2) , \tag{4.20}$$

$$v_\varepsilon(x)|_{\mathcal{B}_\varepsilon} = v(x) + \frac{1-\gamma}{1+\gamma}(\nabla u(\hat{x}) + B\nabla u(\hat{x})) \cdot (x-\hat{x}) + O(\varepsilon^2) . \tag{4.21}$$

5. Topological Derivative Evaluation

Now, we need to evaluate the integrals in formula (3.2) to collect the terms in powers of ε and recognize the function $f(\varepsilon)$. With these results, we can perform the limit passage $\varepsilon \to 0$. The integrals in (3.2) can be evaluated explicitly by using the expansions for the direct u_ε and adjoint v_ε states, respectively given by (4.9,4.10) and (4.20,4.21). The idea is to introduce a polar coordinate system (r,θ) with center at $\hat{x}$. Then, we can write u_ε, v_ε and also the tensor B in this coordinate system to evaluate the integrals explicitly. In particular, the first integral in (3.2) leads to

$$\begin{aligned} -\int_{\partial\mathcal{B}_\varepsilon} [\![\Sigma_\varepsilon - \nabla u_\varepsilon \otimes B q_\varepsilon(u_\varepsilon - u)]\!] n \cdot n = \\ 2\pi\varepsilon \Big(2\alpha q(u(\hat{x})) \cdot \nabla v(\hat{x}) + \frac{1}{2} B q(u(\hat{x})) \cdot \nabla u(\hat{x}) + \alpha B q(u(\hat{x})) \cdot \nabla u(\hat{x}) - \\ \frac{1}{2}\gamma(1+\alpha)^2 B q(u(\hat{x})) \cdot \nabla u(\hat{x}) + \frac{1}{4}\alpha^2 \operatorname{tr}(B) q(u(\hat{x})) \cdot \nabla u(\hat{x}) \Big) + o(\varepsilon) , \end{aligned} \tag{5.1}$$

with the parameter α given by $\alpha = (1-\gamma)/(1+\gamma)$. The second integral in (3.2) becomes

$$-\int_\Omega Bq_\varepsilon(u_\varepsilon - u)\cdot\nabla u'_\varepsilon = -2\pi\varepsilon\left(\frac{1}{2}\alpha^2 \mathrm{tr}(B) q(u(\hat{x}))\cdot\nabla u(\hat{x})\right) + o(\varepsilon)\ , \tag{5.2}$$

where u'_ε is obtained simply by calculating the derivative of u_ε in (4.9,4.10) with respect to ε. Finally, the topological derivative given by (1.2) leads to

$$\begin{aligned} D_T\psi(\hat{x}) = \lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\Bigg[2\pi\varepsilon\Bigg(&2\alpha q(u(\hat{x}))\cdot\nabla v(\hat{x}) + \frac{1}{2}Bq(u(\hat{x}))\cdot\nabla u(\hat{x}) + \alpha Bq(u(\hat{x}))\cdot\nabla u(\hat{x}) - \\ &\frac{1}{2}\gamma(1+\alpha)^2 Bq(u(\hat{x}))\cdot\nabla u(\hat{x}) - \frac{1}{4}\alpha^2\mathrm{tr}(B)q(u(\hat{x}))\cdot\nabla u(\hat{x})\Bigg) + o(\varepsilon)\Bigg]\ . \end{aligned} \tag{5.3}$$

where the remainder $o(\varepsilon)$ comes out from the estimates derived in A and from the elliptic regularity of u and v. Now, in order to extract the main term of the above expansion, we choose $f(\varepsilon) = \pi\varepsilon^2$, which leads to the following theorem:

Theorem 3. *The topological derivative of the shape functional (2.1) is given by*

$$\begin{aligned} D_T\psi(\hat{x}) = 2\alpha q(u(\hat{x}))\cdot\nabla v(\hat{x}) &+ \frac{1}{2}Bq(u(\hat{x}))\cdot\nabla u(\hat{x}) + \alpha Bq(u(\hat{x}))\cdot\nabla u(\hat{x}) - \\ &\frac{1}{2}\gamma(1+\alpha)^2 Bq(u(\hat{x}))\cdot\nabla u(\hat{x}) - \frac{1}{4}\alpha^2\mathrm{tr}(B)q(u(\hat{x}))\cdot\nabla u(\hat{x}) \quad \forall \hat{x}\in\Omega\ , \end{aligned} \tag{5.4}$$

recalling that u and v are solutions to the direct (2.2) and adjoint (2.13) problems, respectively, and $\alpha = (1-\gamma)/(1+\gamma)$.

Remark 4. *We observe that for $B = I$ we have the energy shape functional. In this case, the adjoint state reads $v = -(u+\varphi)$, where φ is the lifting of the Dirichlet boundary data $\overline{u}$ on Γ_D. Since we can construct φ such that $\hat{x}\notin \mathrm{supp}(\varphi)$, then the topological derivative becomes*

$$D_T\psi(\hat{x}) = -P_\gamma q(u(\hat{x}))\cdot\nabla u(\hat{x})\ , \tag{5.5}$$

where the polarization tensor P_γ is given by the following second order isotropic tensor

$$P_\gamma = \frac{1-\gamma}{1+\gamma} I\ . \tag{5.6}$$

Remark 5. *We note that the obtained polarization tensor is isotropic because we are dealing with circular inclusions. For the polarization tensor associated to arbitrary shaped inclusions the reader may refer to [2], for instance.*

Remark 6. *Formally, we can take the limit cases $\gamma\to 0$ and $\gamma\to\infty$. For $\gamma\to 0$, the inclusion leads to an ideal insulator and the transmission condition on the boundary of the inclusion degenerates to homogeneous Neumann boundary condition. In fact, in this case the polarization tensor is given by*

$$P_0 = I\ . \tag{5.7}$$

In addition, for $\gamma\to\infty$, the inclusion leads to an ideal conductor and the polarization tensor is given by

$$P_\infty = -I\ . \tag{5.8}$$

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Appendix A. Estimates for the Remainders

In this Section we proceed with the estimation of the remainders in the topological asymptotic expansion used in the derivation of the topological derivative expression (5.4). In particular, we study the asymptotic behavior of the remainders $\widetilde{u}_\varepsilon$ in (4.8) and $\widetilde{v}_\varepsilon$ in (4.19). Before proceeding, let us state the following result:

Lemma 7. *Let δ_ε be solution to the following variational problem:*

$$\begin{cases} \text{Find } \delta_\varepsilon \in \widetilde{\mathcal{U}}_\varepsilon, \text{ such that} \\ -\displaystyle\int_\Omega q_\varepsilon(\delta_\varepsilon)\cdot\nabla\eta = \varepsilon^2 \int_{\Gamma_N} q(g)\cdot n\,\eta + \varepsilon \int_{\partial\mathcal{B}_\varepsilon} h\,\eta \quad \forall \eta \in \widetilde{\mathcal{V}}_\varepsilon \ , \\ \qquad\qquad\qquad\qquad \text{with} \quad q_\varepsilon(\delta_\varepsilon) = -\gamma_\varepsilon k \nabla \delta_\varepsilon \ , \end{cases} \tag{A.1}$$

where the set $\widetilde{\mathcal{U}}_\varepsilon$ and the space $\widetilde{\mathcal{V}}_\varepsilon$ are defined as

$$\begin{aligned} \widetilde{\mathcal{U}}_\varepsilon &:= \{\varphi \in H^1(\Omega) : [\![\varphi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \varphi_{|\Gamma_D} = -\varepsilon^2 g\} \ , \\ \widetilde{\mathcal{V}}_\varepsilon &:= \{\varphi \in H^1(\Omega) : [\![\varphi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \varphi_{|\Gamma_D} = 0\} \ , \end{aligned}$$

with functions g and h independent of the small parameter ε. Then, we have the estimate $\|\delta_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^2)$ for the remainder.

Proof. By taking $\eta = \delta_\varepsilon - \varphi_\varepsilon$ in (A.1), where φ_ε is the lifting of the Dirichlet boundary data $\varepsilon^2 g$ on Γ_D, we have

$$-\int_\Omega q_\varepsilon(\delta_\varepsilon)\cdot\nabla\delta_\varepsilon = \varepsilon^2 \int_{\Gamma_N} q(g)\cdot n\,\delta_\varepsilon + \varepsilon^2 \int_{\Gamma_D} g\,q(\delta_\varepsilon)\cdot n + \varepsilon \int_{\partial\mathcal{B}_\varepsilon} h\,\delta_\varepsilon \ . \tag{A.2}$$

From the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} -\int_\Omega q_\varepsilon(\delta_\varepsilon)\cdot\nabla\delta_\varepsilon &\le \varepsilon^2 \|q(g)\cdot n\|_{H^{-1/2}(\Gamma_N)} \|\delta_\varepsilon\|_{H^{1/2}(\Gamma_N)} \\ &+ \varepsilon^2 \|g\|_{H^{1/2}(\Gamma_D)} \|q(\delta_\varepsilon)\cdot n\|_{H^{-1/2}(\Gamma_D)} \\ &+ \varepsilon \|h\|_{H^{-1/2}(\partial\mathcal{B}_\varepsilon)} \|\delta_\varepsilon\|_{H^{1/2}(\partial\mathcal{B}_\varepsilon)} \ . \end{aligned} \tag{A.3}$$

Taking into account the trace theorem, we have

$$\begin{aligned} -\int_\Omega q_\varepsilon(\delta_\varepsilon)\cdot\nabla\delta_\varepsilon &\le \varepsilon^2 C_1 \|\delta_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_2 \|\nabla\delta_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|h\|_{L^2(\mathcal{B}_\varepsilon)} \|\delta_\varepsilon\|_{H^1(\mathcal{B}_\varepsilon)} \\ &\le \varepsilon^2 C_1 \|\delta_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_3 \|\delta_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_4 \|\delta_\varepsilon\|_{H^1(\Omega)} \\ &\le \varepsilon^2 C_5 \|\delta_\varepsilon\|_{H^1(\Omega)} \ , \end{aligned} \tag{A.4}$$

where we have used the interior elliptic regularity of function δ. Finally, from the coercivity of the bilinear form on the left-hand side of (A.1), namely,

$$c\|\delta_\varepsilon\|^2_{H^1(\Omega)} \le -\int_\Omega q_\varepsilon(\delta_\varepsilon)\cdot\nabla\delta_\varepsilon \ , \tag{A.5}$$

we obtain

$$\|\delta_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^2 \ , \tag{A.6}$$

which leads to the result, with $C = C_5/c$ independent of the small parameter ε □

Corollary 8. *By setting $\delta_\varepsilon = \widetilde{u}_\varepsilon$ and $\delta_\varepsilon = \widetilde{v}_\varepsilon$ in Lemma 7, we get the estimates for both remainders, namely*

$$\|\widetilde{u}_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^2) \qquad \text{and} \qquad \|\widetilde{v}_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^2) \ . \tag{A.7}$$

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