STRAIN ENERGY CHANGE TO THE INSERTION OF INCLUSIONS ASSOCIATED TO A THERMO-MECHANICAL SEMI-COUPLED SYSTEM

S. M. GIUSTI, A. A. NOVOTNY, J.E. MUÑOZ RIVERA, AND J.E. ESPARTA RODRIGUEZ

Abstract. The topological derivative measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation. According to the literature, the topological derivative has been fully developed for a wide range of one single physical phenomenon modeled by partial differential equations. In addition, the topological asymptotic analysis associated to multi-physics problems has been reported in the literature only on the level of mathematical analysis of singularly perturbed geometrical domains. In this work, we present the topological derivative in its closed form for the total potential mechanical energy associated to a thermo-mechanical semi-coupled system, when a small circular inclusion is introduced at an arbitrary point of the domain. In particular, we consider the linear elasticity system (modeled by the Navier equation) coupled with the steady-state heat conduction problem (modeled by the Laplace equation). The mechanical coupling term comes out from the thermal stress induced by the temperature field. Since this term is non-local, we introduce a non-standard adjoint state, which allows to obtain a closed form for the topological derivative. Finally, we provide a full mathematical justification for the derived formulas and develop precise estimates for the remainders of the topological asymptotic expansion.

1. Introduction

The topological derivative represents a first order asymptotic correction term of a given shape functional with respect to a singular domain perturbation [18]. It has been applied in topology design optimization [2], inverse problems [12], image processing [11], multi-scale constitutive modeling [8], fracture mechanic sensitivity analysis [20] and damage evolution modeling [1]. See also the book by [16] and references therein.

For the sake of completeness, we recall the basic concepts on topological sensitivity analysis. Let us consider a bounded domain $\Omega \subset \mathbb{R}^2$, which is subject to a non-smooth perturbation confined in a small region $\omega_\varepsilon(\widehat{x}) = \widehat{x} + \varepsilon\omega$ of size ε, as shown in fig. 1. Here, $\widehat{x}$ is an arbitrary point of Ω and ω is a fixed bounded domain of $\mathbb{R}^2$. Associated to the domain Ω we introduce a characteristic function $x \mapsto \chi(x)$, $x \in \mathbb{R}^2$, namely $\chi = \mathbb{1}_\Omega$. Also, for the topologically perturbed domain we can define a characteristic function of the form $x \mapsto \chi_\varepsilon(\widehat{x}; x)$. If the perturbation is given by a perforation, the characteristic function can be written as $\chi_\varepsilon(\widehat{x}) = \mathbb{1}_\Omega - \mathbb{1}_{\overline{\omega_\varepsilon(\widehat{x})}}$ and the perforated domain is obtained now as $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$. Now, by assuming the following topological asymptotic expansion of a given shape functional $\psi(\chi_\varepsilon(\widehat{x}))$, associated to the topologically perturbed domain,

$$\psi(\chi_\varepsilon(\widehat{x})) = \psi(\chi) + f(\varepsilon) D_T\psi(\widehat{x}) + o(f(\varepsilon)) , \tag{1.1}$$

the function $\widehat{x} \mapsto D_T\psi(\widehat{x})$ is called the topological derivative of ψ at $\widehat{x}$. In (1.1), $\psi(\chi)$ is the shape functional associated to the original (unperturbed) domain and $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$, when $\varepsilon \to 0$. After rearranging (1.1) we have

$$\frac{\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi)}{f(\varepsilon)} = D_T\psi(\widehat{x}) + \frac{o(f(\varepsilon))}{f(\varepsilon)} . \tag{1.2}$$

The limit passage $\varepsilon \to 0$ in the above expression leads to

$$D_T\psi(\widehat{x}) = \lim_{\varepsilon\to 0} \frac{\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi)}{f(\varepsilon)} . \tag{1.3}$$

Key words and phrases. Topological derivative, thermo-mechanical semi-coupled system, multi-physic topology optimization, asymptotic analysis.

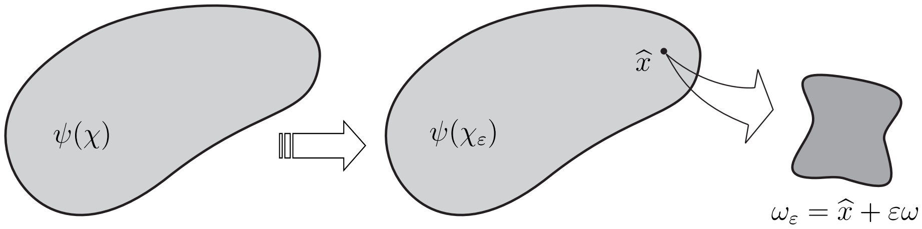


Figure 1. The topological derivative concept.

Since we are dealing with singular domain perturbations, the shape functionals $\psi(\chi_\varepsilon(\widehat{x}))$ and $\psi(\chi)$ are associated to topologically different domains. Therefore, the above limit is not trivial to be calculated. In particular, we need to perform an asymptotic analysis of the shape functional $\psi(\chi_\varepsilon(\widehat{x}))$ with respect to the small parameter ε. In order to calculate the topological derivative, in this work we will apply the approach fully developed in the book by [16]. The method is based on the following result, whose rigorous mathematical justification can be found in the paper by [14]:

$$D_T\psi(\widehat{x}) = \lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\frac{d}{d\varepsilon}\psi(\chi_\varepsilon(\widehat{x})) \ . \tag{1.4}$$

The derivative of $\psi(\chi_\varepsilon(\widehat{x}))$ with respect to ε can be seen as the sensitivity of $\psi(\chi_\varepsilon(\widehat{x}))$, in the classical sense of [5] and [19], to the domain variation produced by an uniform expansion of the perturbation ω_ε.

According to the literature, the topological derivative has been fully developed for a wide range of one single physical phenomenon modeled by partial differential equations. In addition, only a few works dealing with multi-physics problems have been reported in the literature, and, in general, the derived formulas are presented in their abstract forms (see, for instance, the paper by [4] on topological derivatives for piezoelectric materials. In this work, therefore, we derive the topological derivative in its closed form for the total potential mechanical energy associated to a thermo-mechanical semi-coupled system, when a small circular inclusion is introduced at an arbitrary point of the domain. In particular, we consider the linear elasticity system (modeled by the Navier equation) coupled with the steady-state heat conduction problem (modeled by the Laplace equation). The mechanical coupling term comes out from the thermal stress induced by the temperature field. Since this term is non-local, we introduce a non-standard adjoint state, which simplifies the analysis allowing to obtain a closed form for the topological derivative. Finally, we provide a full mathematical justification for the derived formula and develop precise estimates for the remainders of the topological asymptotic expansion. We note that this result can be applied in technological research areas such as multi-physic topology design of structures under mechanical and/or thermal loads.

This paper is organized as follows. Section 2 describes the model associated to a thermo-mechanical semi-coupled problem. The topological sensitivity analysis is presented in Section 3, where the main result of this work is derived: the topological derivative in its closed form for the total potential mechanical energy associated to a thermo-mechanical semi-coupled system. Also in this section, a computational framework designed to the numerical validation of the topological derivative formula is presented. The paper ends in Section 4 where concluding remarks are presented.

2. Formulation of the problem

In this work the topological derivative of the total potential energy associated to the mechanical problem submitted to thermal stresses is derived. The topologically perturbed domain is obtained when a small hole is introduced inside the geometrical domain. Then, the resulting void is filled by an inclusion with a contrast on the elastic, thermal and thermal-expansion material properties. Therefore, we need to formulate the problems associated to both original and topologically perturbed domains.

2.1. **Unperturbed problem.** Consider an open and bounded domain $\Omega \in \mathbb{R}^2$ representing an elastic solid body subject to a linear thermo-mechanical deformation process. Assuming small deformation and variations of temperatures, the functional that represents the total potential energy of the mechanical system for a given temperature field θ is written as:

$$\mathcal{J}_\chi(u,\theta) := \frac{1}{2}\int_\Omega \sigma(u)\cdot\nabla u^s - \int_\Omega Q(\theta)\cdot\nabla u^s - \int_{\Gamma_{N_u}} \bar{t}\cdot u, \tag{2.1}$$

where u represents the displacement field and $\bar{t}$ is a external traction acting on boundary Γ_{N_u}. The displacement field on the boundary Γ_{D_u} satisfies $u|_{\Gamma_{D_u}} = \bar{u}$, being $\bar{u}$ a prescribed displacement. Moreover, note that $\Gamma_{D_u}\cap\Gamma_{N_u} = \varnothing$ and $\overline{\Gamma_{D_u}}\cup\overline{\Gamma_{N_u}} = \partial\Omega$. The Cauchy stress tensor $\sigma(u)$ in (2.1) is defined as:

$$\sigma(u) := \mathbb{C}\nabla u^s\ , \tag{2.2}$$

where ∇u^s is used to denote the symmetric part of the gradient of the displacement field u, i.e.

$$\nabla u^s := \frac{1}{2}(\nabla u + (\nabla u)^\top)\ . \tag{2.3}$$

The induced thermal stress tensor $Q(\theta)$ in (2.1) is defined as:

$$Q(\theta) := \mathbb{C}B\theta\ . \tag{2.4}$$

Therefore the total stress, i.e. the contribution of the mechanical and thermal stresses, is defined as

$$S(u,\theta) = \sigma(u) - Q(\theta). \tag{2.5}$$

In addition, $\mathbb{C}$ denotes the four-order elastic tensor and B denotes the second-order thermo-elastic tensor. In the case of isotropic elastic body, theses tensors are given by:

$$\mathbb{C} = 2\mu I\!I + \lambda(I\otimes I) \quad \text{and} \quad B = \alpha I \quad \Rightarrow \quad \mathbb{C}B = 2\alpha(\lambda+\mu)I, \tag{2.6}$$

with μ and λ denoting the Lame's coefficients, and α the thermal expansion coefficient. In terms of the engineering constant E (Young's modulus) and ν (Poisson's ratio) the above constitutive response can be written as:

$$\mathbb{C} = \frac{E}{1-\nu^2}[(1-\nu)I\!I + \nu(I\otimes I)] \quad \text{and} \quad \mathbb{C}B = \frac{\alpha E}{1-\nu}I\,. \tag{2.7}$$

Considering the previous definitions, we have that the field u is the solution of the following variational problem: given the temperature field θ, find $u\in\mathcal{U}^M$, such that

$$\int_\Omega \sigma(u)\cdot\nabla\eta^s = \int_\Omega Q(\theta)\cdot\nabla\eta^s + \int_{\Gamma_{N_u}} \bar{t}\cdot\eta \quad \forall\eta\in\mathcal{V}^M. \tag{2.8}$$

In the variational problem (2.8), the set $\mathcal{U}^M$ and the space $\mathcal{V}^M$ are defined as

$$\mathcal{U}^M := \left\{\phi\in H^1(\Omega;\mathbb{R}^2) : \phi = \bar{u} \text{ on } \Gamma_{D_u}\right\}, \tag{2.9}$$

$$\mathcal{V}^M := \left\{\phi\in H^1(\Omega;\mathbb{R}^2) : \phi = 0 \text{ on } \Gamma_{D_u}\right\}. \tag{2.10}$$

Moreover, the temperature field θ must satisfy the following variational problem: find $\theta\in\mathcal{U}^T$, such that

$$\int_\Omega q(\theta)\cdot\nabla\eta = \int_{\Gamma_{N_\theta}} \bar{q}\eta \quad \forall\eta\in\mathcal{V}^T, \tag{2.11}$$

where $\bar{q}$ is a prescribed heat flux on the Neumann boundary Γ_{N_θ}. In the Dirichlet boundary Γ_{D_θ} there is a prescribed temperature denoted as $\bar{\theta}$. Then, $\Gamma_{D_\theta}\cap\Gamma_{N_\theta} = \varnothing$ and $\overline{\Gamma_{D_\theta}}\cup\overline{\Gamma_{N_\theta}} = \partial\Omega$. The heat flux operator $q(\theta)$ is defined as

$$q(\theta) = -K\nabla\theta, \tag{2.12}$$

where K is an second order tensor representing the thermal conductivity of the medium. In the isotropic case, the tensor K can be written as

$$K = kI, \tag{2.13}$$

being k the thermal conductivity coefficient. In the variational problem (2.11), the set $\mathcal{U}^T$ and the space $\mathcal{V}^T$ are defined as:

$$\mathcal{U}^T := \left\{\phi \in H^1(\Omega) : \phi = \bar{\theta} \text{ on } \Gamma_{D_\theta}\right\}, \tag{2.14}$$

$$\mathcal{V}^T := \left\{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \Gamma_{D_\theta}\right\}. \tag{2.15}$$

Remark 1. *In the case of a general thermo-elasticity model, the strain rate induces a change in the temperature of the body, leading to a fully coupled system. In our simplified setting, the temperature is completely independent of the mechanical strains, which leads to the so-called thermo-mechanical semi-coupled system. For the mathematical analysis of a fully coupled piezo-electric system in the context of singularly perturbed geometrical domains, see* [4].

Finally, in order to simplify further analysis, the following auxiliary problem is introduced: find $\varphi \in \mathcal{V}^T$, such that:

$$\int_\Omega q(\varphi) \cdot \nabla \eta = \int_\Omega Q(\eta) \cdot \nabla u^s \quad \forall \eta \in \mathcal{V}^T. \tag{2.16}$$

Note that φ can be seen as the adjoint state associated to the thermal stress induced by the temperature θ (see, for instance, [19].

2.2. **Perturbed problem.** Considering the introduction of a circular inclusion, denoted as $\omega_\varepsilon(\widehat{x}) := \mathcal{B}_\varepsilon(\widehat{x})$, with radius ε and centered at point $\widehat{x}$ in Ω, the total potential energy functional associated to the perturbed domain of the mechanical system for a given temperature field θ_ε can be written as:

$$\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon) := \frac{1}{2}\int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon^s - \int_\Omega Q_\varepsilon(\theta_\varepsilon) \cdot \nabla u_\varepsilon^s - \int_{\Gamma_{N_u}} \bar{t} \cdot u_\varepsilon, \tag{2.17}$$

where u_ε and θ_ε denotes, respectively, the displacement and temperature fields, both associated to the perturbed system. In addition, $\sigma_\varepsilon(u_\varepsilon)$ and $Q_\varepsilon(\theta_\varepsilon)$ are used to denote the mechanical and the induced thermal stresses tensors associated to the perturbed problem. These tensors are defined as:

$$\sigma_\varepsilon(u_\varepsilon) := \gamma_\varepsilon^M \mathbb{C} \nabla u_\varepsilon^s \quad \text{and} \quad Q_\varepsilon(\theta_\varepsilon) := \gamma_\varepsilon^M \gamma_\varepsilon^C \mathbb{C} B \theta_\varepsilon, \tag{2.18}$$

and the corresponding total stress operator $S_\varepsilon(u_\varepsilon, \theta_\varepsilon)$ associated to the perturbed problem is given by

$$S_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \sigma_\varepsilon(u_\varepsilon) - Q_\varepsilon(\theta_\varepsilon). \tag{2.19}$$

The contrast parameters in the material properties γ_ε^M and γ_ε^C are defined as

$$\gamma_\varepsilon^M := \begin{cases} 1 & \text{in } \Omega \setminus \overline{\mathcal{B}_\varepsilon} \\ \gamma^M & \text{in } \mathcal{B}_\varepsilon \end{cases} \quad \text{and} \quad \gamma_\varepsilon^C := \begin{cases} 1 & \text{in } \Omega \setminus \overline{\mathcal{B}_\varepsilon} \\ \gamma^C & \text{in } \mathcal{B}_\varepsilon \end{cases}. \tag{2.20}$$

with γ^M and γ^C used to denote the values of the contrast on the Young modulus and thermal-expansion coefficient, respectively. In the perturbed configuration, the displacement field satisfies the variational problem: given the temperature field θ_ε, find $u_\varepsilon \in \mathcal{U}_\varepsilon^M$, such that

$$\int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla \eta^s = \int_\Omega Q_\varepsilon(\theta_\varepsilon) \cdot \nabla \eta^s + \int_{\Gamma_{N_u}} \bar{t} \cdot \eta \quad \forall \eta \in \mathcal{V}_\varepsilon^M. \tag{2.21}$$

The set $\mathcal{U}_\varepsilon^M$ and the space $\mathcal{V}_\varepsilon^M$ in the variational problem (2.21) are defined as

$$\mathcal{U}_\varepsilon^M := \left\{\phi \in \mathcal{U}^M : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon\right\}, \tag{2.22}$$

$$\mathcal{V}_\varepsilon^M := \left\{\phi \in \mathcal{V}^M : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon\right\}, \tag{2.23}$$

where the operator $[\![(\cdot)]\!]$ is introduced to denote the jump of $(\cdot)$ across the boundary of the perturbation.

In addition, the thermal equilibrium problem can be written in the variational form as: find $\theta_\varepsilon \in \mathcal{U}_\varepsilon^T$, such that

$$\int_\Omega q_\varepsilon(\theta_\varepsilon) \cdot \nabla \eta = \int_{\Gamma_{N_\theta}} \bar{q} \eta \quad \forall \eta \in \mathcal{V}_\varepsilon^T, \tag{2.24}$$

with the thermal flux in the perturbed domain being defined as:

$$q_\varepsilon(\theta_\varepsilon) := -\gamma_\varepsilon^T K \nabla \theta_\varepsilon \,, \tag{2.25}$$

where γ_ε^T is the parameter that define the contrast between the thermal (constitutive) properties of the matrix and the inclusion, and is defined by:

$$\gamma_\varepsilon^T := \left\{ \begin{array}{ll} 1 & \text{in} \quad \Omega \setminus \overline{\mathcal{B}_\varepsilon} \\ \gamma^T & \text{in} \quad \mathcal{B}_\varepsilon \end{array} \right. \,, \tag{2.26}$$

being γ^T the value of the contrast on the thermal conductivity coefficient. In the variational problem (2.24) the set $\mathcal{U}_\varepsilon^T$ and the space $\mathcal{V}_\varepsilon^T$ are defined as:

$$\mathcal{U}_\varepsilon^T \;:=\; \left\{ \phi \in \mathcal{U}^T : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon \right\} , \tag{2.27}$$

$$\mathcal{V}_\varepsilon^T \;:=\; \left\{ \phi \in \mathcal{V}^T : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon \right\} . \tag{2.28}$$

Finally, the auxiliary problem associated to the topologically perturbed domain is written as: find $\varphi_\varepsilon \in \mathcal{V}_\varepsilon^T$, such that:

$$\int_\Omega q_\varepsilon(\varphi_\varepsilon) \cdot \nabla \eta = \int_\Omega Q_\varepsilon(\eta) \cdot \nabla u^s \quad \forall \eta \in \mathcal{V}_\varepsilon^T, \tag{2.29}$$

where φ_ε can be interpreted as the adjoint state associated to the thermal stress induced by the perturbed temperature θ_ε (see, for instance, [19].

3. Topological Sensitivity Analysis

In order to proceed, it is convenient to introduce an analogy to classical continuum mechanics [9] where by the shape change velocity field V is identified with the classical velocity field of a deforming continuum and ε is identified as a time parameter. Since we are dealing with an uniform expansion of the inclusion $\mathcal{B}_\varepsilon$, the shape velocity field V satisfies: $V|_{\partial\Omega} = 0$ and $V|_{\partial\mathcal{B}_\varepsilon} = -n$. Then, the shape derivative of the functional (2.17) can be written as:

$$\begin{aligned} \dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon) &= \left(\frac{1}{2} \int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon^s - \int_\Omega Q_\varepsilon(\theta_\varepsilon) \cdot \nabla u_\varepsilon^s - \int_{\Gamma_{N_u}} \bar{t} \cdot u_\varepsilon \right)^{\cdot} \\ &= \frac{1}{2} \left(\int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon^s \right)^{\cdot} - \left(\int_\Omega Q_\varepsilon(\theta_\varepsilon) \cdot \nabla u_\varepsilon^s \right)^{\cdot} \\ &- \int_{\Gamma_{N_u}} \bar{t} \cdot \dot{u}_\varepsilon \,, \end{aligned} \tag{3.1}$$

where we use both notations $(\cdot)^{\cdot}$ and $\dot{(\cdot)}$ to represent the total derivative with respect to the parameter ε. Therefore, we can state the following propositions:

Proposition 1. *Let $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon)$ be the functional defined by (2.17). Then, its derivative with respect to the small parameter ε is given by*

$$\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon) = \int_\Omega \Sigma_\varepsilon \cdot \nabla V - \int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon) \cdot \nabla (u_\varepsilon - u)^s \,, \tag{3.2}$$

where V is the shape change velocity field defined in Ω that satisfies $V|_{\partial\Omega} = 0$ and $V|_{\partial\mathcal{B}_\varepsilon} = -n$; $\dot{\theta}_\varepsilon$ is the material derivative of the temperature field and Σ_ε is a generalization of the classical Eshelby momentum-energy tensor [6]*, given - for this particular case - by*

$$\begin{aligned} \Sigma_\varepsilon &:= \frac{1}{2}((S_\varepsilon(u_\varepsilon, \theta_\varepsilon) - Q_\varepsilon(\theta_\varepsilon)) \cdot \nabla u_\varepsilon^s) I - (\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon, \theta_\varepsilon) \\ &+ \left[(q_\varepsilon(\theta_\varepsilon) \cdot \nabla \varphi_\varepsilon) I - 2 q_\varepsilon(\theta_\varepsilon) \otimes_s \nabla \varphi_\varepsilon \right] , \end{aligned} \tag{3.3}$$

with u_ε, θ_ε and φ_ε denoting the solutions to (2.21), (2.24) and to the auxiliary problem (2.29).

Proof. By making use of Reynolds' Transport Theorem [9, 19] we obtain the identities

$$\begin{aligned} \left(\int_\Omega \sigma_\varepsilon(u_\varepsilon)\cdot\nabla u^s_\varepsilon\right)^{\cdot} &= \int_\Omega (2\sigma_\varepsilon(u_\varepsilon)\cdot\nabla \dot{u}^s_\varepsilon - 2\sigma_\varepsilon(u_\varepsilon)\cdot(\nabla u_\varepsilon \nabla V)^s \\ &+ \int_\Omega (\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u^s_\varepsilon)\mathrm{div}V)\,, \qquad (3.4) \\ \left(\int_\Omega Q_\varepsilon(\theta_\varepsilon)\cdot\nabla u^s_\varepsilon\right)^{\cdot} &= \int_\Omega (Q_\varepsilon(\theta_\varepsilon)\cdot\nabla \dot{u}^s_\varepsilon - Q_\varepsilon(\theta_\varepsilon)\cdot(\nabla u_\varepsilon \nabla V)^s \\ &+ \int_\Omega (Q_\varepsilon(\theta_\varepsilon)\cdot\nabla u^s_\varepsilon)\mathrm{div}V + Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s_\varepsilon\,. \qquad (3.5) \end{aligned}$$

Then, by considering the above results in (3.1), the shape derivative of the functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)$ is given by

$$\begin{aligned} \dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) &= \int_\Omega \left(\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta_\varepsilon) - Q_\varepsilon(\theta_\varepsilon))\cdot\nabla u^s_\varepsilon)I - (\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta_\varepsilon)\right)\cdot\nabla V \\ &- \int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s_\varepsilon + \int_\Omega S_\varepsilon(u_\varepsilon,\theta_\varepsilon)\cdot\nabla\dot{u}^s_\varepsilon - \int_{\Gamma_{N_u}} \bar{t}\cdot\dot{u}_\varepsilon\,. \qquad (3.6) \end{aligned}$$

Since $\dot{u}_\varepsilon \in \mathcal{U}^M_\varepsilon$, see the work made by [19], the terms in $\dot{u}_\varepsilon$ satisfy the state equation (2.21), then

$$\begin{aligned} \dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) &= \int_\Omega \left(\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta_\varepsilon) - Q_\varepsilon(\theta_\varepsilon))\cdot\nabla u^s_\varepsilon)I - (\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta_\varepsilon)\right)\cdot\nabla V \\ &- \int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s_\varepsilon. \qquad (3.7) \end{aligned}$$

Now, adding the term $\pm\int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s$ in the above result, the derivative $\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)$ can be written alternatively as

$$\begin{aligned} \dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) &= \int_\Omega \left(\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta_\varepsilon) - Q_\varepsilon(\theta_\varepsilon))\cdot\nabla u^s_\varepsilon)I - (\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta_\varepsilon)\right)\cdot\nabla V \\ &- \int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla(u_\varepsilon - u)^s - \int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s\,. \qquad (3.8) \end{aligned}$$

On the other hand, the derivative of the state equation (2.24) with respect to the parameter ε is given by

$$\int_\Omega q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla\eta = -\int_\Omega [(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\eta)I - 2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\eta]\cdot\nabla V \quad \forall\eta\in\mathcal{V}^T_\varepsilon\,. \qquad (3.9)$$

Next, taking $\eta = \varphi_\varepsilon$ in the above expression, we obtain

$$\int_\Omega q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla\varphi_\varepsilon = -\int_\Omega [(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I - 2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot\nabla V, \qquad (3.10)$$

and tacking $\eta = \dot{\theta}_\varepsilon$ in the auxiliary problem (2.29), we obtain

$$\int_\Omega q_\varepsilon(\varphi_\varepsilon)\cdot\nabla\dot{\theta}_\varepsilon = \int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s\,. \qquad (3.11)$$

By using the definition of the heat flux operator (2.25) and comparing the two last expressions, the following identity holds

$$\int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s = -\int_\Omega [(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I - 2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot\nabla V\,. \qquad (3.12)$$

From the above result, the derivative of the shape functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)$ can be written equivalently in the following form:

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) &= \int_\Omega \left(\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta_\varepsilon) - Q_\varepsilon(\theta_\varepsilon))\cdot\nabla u_\varepsilon^s)I - (\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta_\varepsilon)\right)\cdot\nabla V \\
&+ \int_\Omega [(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I - 2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot\nabla V \\
&- \int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla(u_\varepsilon - u)^s\,, \qquad (3.13)
\end{aligned}
$$

which leads to the result with Σ_ε given by (3.3). □

Proposition 2. *Let $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)$ be the functional defined by (2.17). Then, its derivative with respect to the small parameter ε is given by*

$$
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) = -\int_{\partial\mathcal{B}_\varepsilon} [\![\Sigma_\varepsilon]\!] n\cdot n - \int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon - u)^s\,, \qquad (3.14)
$$

where V is the shape change velocity field defined in Ω that satisfies $V|_{\partial\Omega} = 0$ and $V|_{\partial\mathcal{B}_\varepsilon} = -n$; θ'_ε is the spatial derivative of the temperature field and Σ_ε is a generalization of the classical Eshelby momentum-energy tensor presented in (3.3).

Proof. By making use of the Reynolds' Transport Theorem [9, 19], we obtain the following identities:

$$
\begin{aligned}
\left(\int_\Omega \sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s\right)^{\cdot} &= \int_\Omega 2(\sigma_\varepsilon(u_\varepsilon)\cdot\nabla\dot{u}_\varepsilon^s + \operatorname{div}(\sigma_\varepsilon(u_\varepsilon))\cdot(\nabla u_\varepsilon)V) \\
&+ \int_{\partial\Omega}\left[(\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s)I - 2(\nabla u_\varepsilon)^\top\sigma_\varepsilon(u_\varepsilon)\right] n\cdot V \\
&+ \int_{\partial\mathcal{B}_\varepsilon} [\![(\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s)I - 2(\nabla u_\varepsilon)^\top\sigma_\varepsilon(u_\varepsilon)]\!] n\cdot V, \qquad (3.15) \\
\left(\int_\Omega Q_\varepsilon(\theta_\varepsilon)\cdot\nabla u_\varepsilon^s\right)^{\cdot} &= \int_\Omega (Q_\varepsilon(\theta_\varepsilon)\cdot\nabla\dot{u}_\varepsilon^s + Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla u_\varepsilon^s) \\
&+ \int_\Omega \operatorname{div}(Q_\varepsilon(\theta_\varepsilon))\cdot(\nabla u_\varepsilon)V \\
&- \int_{\partial\Omega}\left[(\nabla u_\varepsilon)^\top Q_\varepsilon(\theta_\varepsilon) - (Q_\varepsilon(\theta_\varepsilon)\cdot\nabla u_\varepsilon^s)I\right] n\cdot V \\
&- \int_{\partial\mathcal{B}_\varepsilon} [\![(\nabla u_\varepsilon)^\top Q_\varepsilon(\theta_\varepsilon) - (Q_\varepsilon(\theta_\varepsilon)\cdot\nabla u_\varepsilon^s)I]\!] n\cdot V\,. \qquad (3.16)
\end{aligned}
$$

Introducing the above expressions in the definitions of the shape derivative (3.1) and taking into account that: (i) $\dot{u}_\varepsilon\in\mathcal{U}_\varepsilon^M$, see the work made by [19], the terms in $\dot{u}_\varepsilon$ satisfy the state equation (2.21); (ii) $\operatorname{div}S_\varepsilon(u_\varepsilon,\theta_\varepsilon) = 0$ in Ω ; (iii) adding the term $\pm\int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla u^s$; and (iv) the shape change velocity field V defined in Ω satisfies $V|_{\partial\Omega} = 0$ and $V|_{\partial\mathcal{B}_\varepsilon} = -n$; then

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) &= -\int_{\partial\mathcal{B}_\varepsilon} [\![\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta_\varepsilon) - Q_\varepsilon(\theta_\varepsilon))\cdot\nabla u_\varepsilon^s)I - (\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta_\varepsilon)]\!] n\cdot n \\
&\quad - \int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon - u)^s - \int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla u^s. \qquad (3.17)
\end{aligned}
$$

By using the relation between the material and spatial derivatives of the temperature field, the above expression can be written as,

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) \;&=\; -\int_{\partial\mathcal{B}_\varepsilon} [\![\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta_\varepsilon)-Q_\varepsilon(\theta_\varepsilon))\cdot\nabla u^s_\varepsilon)I-(\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta_\varepsilon)]\!]n\cdot n\\
&\quad -\int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon-u)^s-\int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s\\
&\quad +\int_\Omega Q_\varepsilon(\nabla\theta_\varepsilon\cdot n)\cdot\nabla u^s\,. \qquad (3.18)
\end{aligned}
$$

On the other hand, the derivative of the state equation (2.24) with respect to parameter ε is given by

$$
\int_\Omega q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla\eta=-\int_\Omega[(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\eta)I-2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\eta]\cdot\nabla V\quad\forall\eta\in\mathcal{V}^T_\varepsilon\,. \qquad (3.19)
$$

Next, tacking $\eta=\varphi_\varepsilon$ in the above expression, we obtain

$$
\int_\Omega q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla\varphi_\varepsilon=-\int_\Omega[(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I-2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot\nabla V\,, \qquad (3.20)
$$

and tacking $\eta=\dot{\theta}_\varepsilon$ in the auxiliary problem (2.29), we obtain

$$
\int_\Omega q_\varepsilon(\varphi_\varepsilon)\cdot\nabla\dot{\theta}_\varepsilon=\int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s\,. \qquad (3.21)
$$

By using the definition of the heat flux operator (2.25) and comparing the two last expressions, the following identity holds

$$
\int_\Omega Q_\varepsilon(\dot{\theta}_\varepsilon)\cdot\nabla u^s=-\int_\Omega[(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I-2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot\nabla V. \qquad (3.22)
$$

From the above result, the derivative of the shape functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)$ can be written equivalently in the following form,

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) \;&=\; -\int_{\partial\mathcal{B}_\varepsilon} [\![\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta_\varepsilon)-Q_\varepsilon(\theta_\varepsilon))\cdot\nabla u^s_\varepsilon)I-(\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta_\varepsilon)]\!]n\cdot n\\
&\quad +\int_\Omega[(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I-2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot\nabla V\\
&\quad -\int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon-u)^s+\int_\Omega Q_\varepsilon(\nabla\theta_\varepsilon\cdot n)\cdot\nabla u^s\,. \qquad (3.23)
\end{aligned}
$$

By integrating by parts the second term in the above expression and using the definition of the Eshelby's tensor Σ_ε, we have

$$
\begin{aligned}
\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon) \;&=\; -\int_{\partial\mathcal{B}_\varepsilon}[\![\Sigma_\varepsilon]\!]n\cdot n-\int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon-u)^s\\
&\quad -\int_\Omega \operatorname{div}[(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I-2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot V\\
&\quad +\int_\Omega Q_\varepsilon(\nabla\theta_\varepsilon\cdot V)\cdot\nabla u^s\,. \qquad (3.24)
\end{aligned}
$$

Taking into account the state equation (2.24) and the auxiliary problem (2.29), we observe that the second term in the above expression satisfies the following identity

$$
\int_\Omega \operatorname{div}[(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I-2q_\varepsilon(\theta_\varepsilon)\otimes_s\nabla\varphi_\varepsilon]\cdot V=\int_\Omega Q_\varepsilon(\nabla\theta_\varepsilon\cdot V)\cdot\nabla u^s\,. \qquad (3.25)
$$

Then, the lats two terms in (3.24) vanish, leading to the result. □

Corollary 1. *By considering the relation between the material and spatial derivative of the temperature field, (3.2) can be written as:*

$$\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)=\int_\Omega \Sigma_\varepsilon\cdot\nabla V-\int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon-u)^s-\int_\Omega Q_\varepsilon(\nabla\theta_\varepsilon\cdot V)\cdot\nabla(u_\varepsilon-u)^s. \tag{3.26}$$

By integrating by part the first term of the above expression and using the restriction of the velocity field V on the boundaries $\partial\Omega$ and $\partial\mathcal{B}_\varepsilon$, we obtain

$$\begin{aligned}\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)= & -\int_{\partial\mathcal{B}_\varepsilon}[\![\Sigma_\varepsilon]\!]n\cdot n-\int_\Omega \mathrm{div}\Sigma_\varepsilon\cdot V-\int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon-u)^s\\ & -\int_\Omega Q_\varepsilon(\nabla\theta_\varepsilon\cdot V)\cdot\nabla(u_\varepsilon-u)^s\,.\end{aligned} \tag{3.27}$$

By comparing (3.14) with (3.27) and recalling that both identities are valid for all $V\in\Omega$, the follow result holds true

$$\int_\Omega(\mathrm{div}(\Sigma_\varepsilon)+\gamma_\varepsilon^M\gamma_\varepsilon^C(\mathbb{C}B\cdot\nabla(u_\varepsilon-u)^s)\nabla\theta_\varepsilon)\cdot V=0\quad\forall V\in\Omega, \tag{3.28}$$

Thus, the equation for the balance of the configurational forces [10] *can be written as:*

$$\mathrm{div}(\Sigma_\varepsilon)=-\gamma_\varepsilon^M\gamma_\varepsilon^C(\mathbb{C}B\cdot\nabla(u_\varepsilon-u)^s)\nabla\theta_\varepsilon\quad in\quad\Omega. \tag{3.29}$$

Note that the first term of the derivative $\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)$ in (3.14) is given by an integral concentrated on $\partial\mathcal{B}_\varepsilon$ depending on the solution to (2.21), (2.24) and (2.29). The second term, given by a integral over all domain Ω, will be treated carefully (see A).

To analytically solve the integrals expression of the derivative $\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta_\varepsilon)$ it is necessary to perform an asymptotic analysis of the solutions of the PDE's involved in these coupled problems. In order to simplify the analysis, let us use the linearity property of the shape functional with respect to the solution of the thermal problem (2.24) and split the analysis in two cases: *(i)* $\gamma^T=1$ and *(ii)* $\gamma^M=\gamma^C=1$.

3.1. **Case $\gamma^T=1$.** For this particular case, $\gamma^T=1$, we have that the temperature field is not perturbed by the presence of the inclusion $\mathcal{B}_\varepsilon$ in the mechanical problem. Then, the temperature for the unperturbed and perturbed problems coincides, i.e. $\theta_\varepsilon=\theta$. Thus, the derivative of the shape functional can be written as:

$$\begin{aligned}\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon,\theta) &= -\int_{\partial\mathcal{B}_\varepsilon}[\![\Sigma_\varepsilon]\!]n\cdot n\\ &= -\int_{\partial\mathcal{B}_\varepsilon}[\![\frac{1}{2}((S_\varepsilon(u_\varepsilon,\theta)-Q_\varepsilon(\theta))\cdot\nabla u_\varepsilon^s)I-(\nabla u_\varepsilon)^\top S_\varepsilon(u_\varepsilon,\theta)]\!]n\cdot n\,.\end{aligned} \tag{3.30}$$

In order to obtain an explicit expression for the perturbed stress field, we consider the following ansatz for the displacement field u_ε:

$$u_\varepsilon(x)=u(x)+w_\varepsilon(x/\varepsilon)+\widetilde{u}_\varepsilon(x), \tag{3.31}$$

where $u(x)$ is the solution of the unperturbed problem in Ω, $w_\varepsilon(x/\varepsilon)$ the solution to an exterior perturbed problem in $\mathbb{R}^2$ and $\widetilde{u}_\varepsilon(x)$ the remainder, solution to a perturbed problem in Ω. The terms in the above expansion requires additional explanation. The function $w_\varepsilon(x/\varepsilon)$ decays to zero at the infinity, i.e., $w_\varepsilon\to 0$ at ∞, and compensates the discrepancy introduced by the lower order term of the Taylor series expansion of u around $\widehat{x}$. The remainder $\widetilde{u}_\varepsilon(x)$ is introduced to compensate the discrepancies left by w_ε on the exterior boundary $\partial\Omega$ as well as by the higher order term of the Taylor series expansion of u in the neighborhood of $\mathcal{B}_\varepsilon(\widehat{x})$. Then, the mechanical stress satisfies the identity

$$\sigma_\varepsilon(u_\varepsilon)=\gamma_\varepsilon^M\mathbb{C}\nabla u^s+\gamma_\varepsilon^M\mathbb{C}\nabla w_\varepsilon^s+\gamma_\varepsilon^M\mathbb{C}\nabla\widetilde{u}_\varepsilon^s\,. \tag{3.32}$$

Moreover, by introducing the term $-Q_\varepsilon(\theta)$ at both sides of the above expression, the stress field associated to the perturbed domain $S_\varepsilon(u_\varepsilon,\theta)$ admits the following asymptotic expansion

$$S_\varepsilon(u_\varepsilon,\theta)=\gamma_\varepsilon^M\sigma(u)+\sigma_\varepsilon(w_\varepsilon)+\sigma_\varepsilon(\widetilde{u}_\varepsilon)-\gamma_\varepsilon^M\gamma_\varepsilon^C Q(\theta)\,, \tag{3.33}$$

where $\sigma_\varepsilon(w_\varepsilon)$ is the solution of the exterior problem

$$\left\{\begin{array}{rcll} \operatorname{div}(\sigma_\varepsilon(w_\varepsilon)) &=& 0 & \text{in} \quad \mathbb{R}^2 \\ \sigma_\varepsilon(w_\varepsilon) &=& \gamma_\varepsilon^M \mathbb{C}\nabla w_\varepsilon^s & \\ \sigma_\varepsilon(w_\varepsilon) &\to& 0 & \text{at} \quad \infty \\ [\![\sigma_\varepsilon(w_\varepsilon)]\!]n &=& -s_u & \text{on} \quad \partial\mathcal{B}_\varepsilon \end{array}\right. , \tag{3.34}$$

and the residue $\widetilde{u}_\varepsilon$ satisfies the equation

$$\left\{\begin{array}{rcll} \operatorname{div}(\sigma_\varepsilon(\widetilde{u}_\varepsilon)) &=& 0 & \text{in} \quad \Omega\setminus\overline{\mathcal{B}_\varepsilon} \\ \operatorname{div}(\sigma_\varepsilon(\widetilde{u}_\varepsilon)) &=& (1-\gamma^C)\gamma^M\mathbb{C}B\nabla\theta & \text{in} \quad \mathcal{B}_\varepsilon \\ \sigma_\varepsilon(\widetilde{u}_\varepsilon) &=& \gamma_\varepsilon^M\mathbb{C}\nabla\widetilde{u}_\varepsilon^s & \\ \widetilde{u}_\varepsilon &=& -w_\varepsilon & \text{on} \quad \partial\Gamma_{D_u} \\ \sigma_\varepsilon(\widetilde{u}_\varepsilon)n &=& -\sigma_\varepsilon(w_\varepsilon)n & \text{on} \quad \partial\Gamma_{N_u} \\ [\![\sigma_\varepsilon(\widetilde{u}_\varepsilon)]\!]n &=& \varepsilon h_u & \text{on} \quad \partial\mathcal{B}_\varepsilon \end{array}\right. , \tag{3.35}$$

which has the following estimate $\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)} = o(\varepsilon)$ (see A). Moreover, the functions s_u and h_u in (3.34) and (3.35), respectively, are given by

$$s_u := (1-\gamma^M)\sigma(u(\widehat{x}))n - (1-\gamma^M\gamma^C)Q(\theta(\widehat{x}))n\,, \tag{3.36}$$

$$h_u := (1-\gamma^M)(\nabla\sigma(u(\zeta))n)n - (1-\gamma^M\gamma^C)(\nabla Q(\theta(\xi))n)n\,, \tag{3.37}$$

where the points ζ and ξ belong to the interval $(x,\widehat{x})$.

By considering a polar system of coordinates (r,ϕ) centered at point $\widehat{x}$ (center of the inclusion $\mathcal{B}_\varepsilon$) and aligned with the principal directions of the tensor $S(u,\theta)$ associated to the original domain Ω, the components of the tensor $\sigma_\varepsilon(w_\varepsilon)$ are given by (see, for instance, [13]:

- Exterior solution ($r \geq \varepsilon$)

$$\begin{aligned} \sigma_\varepsilon(w_\varepsilon)^{rr} = & -\frac{1-\gamma^M}{1+a\gamma^M}\frac{\varepsilon^2}{r^2}\left(\frac{\sigma_1+\sigma_2}{2}\right) \\ & -\frac{1-\gamma^M}{1+b\gamma^M}\frac{\varepsilon^2}{r^2}\left(4-3\frac{\varepsilon^2}{r^2}\right)\left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\phi \\ & +\frac{1-\gamma^M\gamma^C}{1+a\gamma^M}\frac{\varepsilon^2}{r^2}\left(\frac{Q_1+Q_2}{2}\right), \end{aligned} \tag{3.38}$$

$$\begin{aligned} \sigma_\varepsilon(w_\varepsilon)^{\phi\phi} = & \frac{1-\gamma^M}{1+a\gamma^M}\frac{\varepsilon^2}{r^2}\left(\frac{\sigma_1+\sigma_2}{2}\right) - 3\frac{1-\gamma^M}{1+b\gamma^M}\frac{\varepsilon^4}{r^4}\left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\phi \\ & -\frac{1-\gamma^M\gamma^C}{1+a\gamma^M}\frac{\varepsilon^2}{r^2}\left(\frac{Q_1+Q_2}{2}\right), \end{aligned} \tag{3.39}$$

$$\sigma_\varepsilon(w_\varepsilon)^{\phi r} = -\frac{1-\gamma^M}{1+b\gamma^M}\frac{\varepsilon^2}{r^2}\left(2-3\frac{\varepsilon^2}{r^2}\right)\left(\frac{\sigma_1-\sigma_2}{2}\right)\sin 2\phi\,. \tag{3.40}$$

- Interior solution ($0 < r < \varepsilon$)

$$\begin{aligned} \sigma_\varepsilon(w_\varepsilon)^{rr} = & \frac{a\gamma^M(1-\gamma^M)}{1+a\gamma^M}\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{b\gamma^M(1-\gamma^M)}{1+b\gamma^M}\left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\phi \\ & -\frac{a\gamma^M(1-\gamma^M\gamma^C)}{1+a\gamma^M}\left(\frac{Q_1+Q_2}{2}\right), \end{aligned} \tag{3.41}$$

$$\begin{aligned} \sigma_\varepsilon(w_\varepsilon)^{\phi\phi} = & \frac{a\gamma^M(1-\gamma^M)}{1+a\gamma^M}\left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{b\gamma^M(1-\gamma^M)}{1+b\gamma^M}\left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\phi \\ & -\frac{a\gamma^M(1-\gamma^M\gamma^C)}{1+a\gamma^M}\left(\frac{Q_1+Q_2}{2}\right), \end{aligned} \tag{3.42}$$

$$\sigma_\varepsilon(w_\varepsilon)^{\phi r} = -\frac{b\gamma^M(1-\gamma^M)}{1+b\gamma^M}\left(\frac{\sigma_1-\sigma_2}{2}\right)\sin 2\phi\,. \tag{3.43}$$

where $\sigma_{1,2}$ and $Q_{1,2}$ are, respectively, the principal stress associated to the tensor $\sigma(u)$ and $Q(\theta)$ of the unperturbed domain Ω, evaluated at the point $\widehat{x} \in \Omega$. In addition, the constants a and b in (3.38) to (3.43) depend only on Poisson's ratio ν of the matrix, and are given by

$$a = \frac{1+\nu}{1-\nu} \quad \text{and} \quad b = \frac{3-\nu}{1+\nu}. \tag{3.44}$$

Finally, using the asymptotic expansions presented in (3.38) to (3.43), we have that the derivative $\dot{\mathcal{J}}_{\chi_\varepsilon}$ is given by the following expression:

$$\begin{aligned} \dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon, \theta) \quad = \quad & -\frac{\pi\varepsilon}{E}\left(\frac{1-\gamma^M}{1+b\gamma^M}\right)\left[4\sigma(u)\cdot\sigma(u) + \frac{\gamma^M(b-2a)-1}{1+a\gamma^M}(\mathrm{tr}\sigma(u))^2\right] \\ & -\frac{\pi\varepsilon}{2E}\left(\frac{1-\gamma^M\gamma^C}{1+a\gamma^M}\right)\left[(1-\gamma^M\gamma^C)(1+\nu)(\mathrm{tr}Q(\theta))^2 - 4\mathrm{tr}\sigma(u)\mathrm{tr}Q(\theta)\right] \\ & +o(\varepsilon), \end{aligned} \tag{3.45}$$

where $\mathrm{tr}(\cdot)$ denotes the trace operator of tensor $(\cdot)$.

3.2. **Case $\gamma^M = 1$ and $\gamma^C = 1$.** In this case the restriction $\gamma^M = 1$ and $\gamma^C = 1$ is introduced in expression (3.14), then the derivative of the shape functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon)$ is given by:

$$\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon) = \int_{\partial\mathcal{B}_\varepsilon} [\![(q_\varepsilon(\theta_\varepsilon)\cdot\nabla\varphi_\varepsilon)I - 2q_\varepsilon(\theta_\varepsilon)\otimes_s \nabla\varphi_\varepsilon]\!] n\cdot n + \mathcal{E}(\varepsilon), \tag{3.46}$$

where the term $\mathcal{E}(\varepsilon)$ is given by

$$\mathcal{E}(\varepsilon) = -\int_\Omega Q_\varepsilon(\theta'_\varepsilon)\cdot\nabla(u_\varepsilon - u)^s. \tag{3.47}$$

The temperature field θ_ε associated to the perturbed problem admits the following asymptotic expansion:

$$\theta_\varepsilon(x) = \theta(x) + v_\varepsilon(x/\varepsilon) + \widetilde{\theta}_\varepsilon(x), \tag{3.48}$$

where $\theta(x)$ is the solution of the unperturbed problem in Ω, $v_\varepsilon(x/\varepsilon)$ the solution to an exterior perturbed problem in $\mathbb{R}^2$ and $\widetilde{\theta}_\varepsilon(x)$ the remainder, solution to a perturbed problem in Ω. The function $v_\varepsilon(x/\varepsilon)$ is such that $v_\varepsilon \to 0$ at ∞ and it compensates the discrepancy left by the lower order term of the Taylor series expansion of θ in the neighborhood of $\widehat{x}$. The remainder $\widetilde{\theta}_\varepsilon(x)$ compensates the discrepancies introduced by v_ε on the exterior boundary $\partial\Omega$ and by the higher order term of the Taylor series expansion of θ around $\mathcal{B}_\varepsilon(\widehat{x})$. In particular, v_ε is the solution of the exterior problem

$$\left\{\begin{array}{rcll} \mathrm{div}(q_\varepsilon(v_\varepsilon)) & = & 0 & \text{in} \quad \mathbb{R}^2 \\ q_\varepsilon(v_\varepsilon) & = & -\gamma_\varepsilon^T K\nabla v_\varepsilon & \\ v_\varepsilon & \to & 0 & \text{at} \quad \infty \\ [\![v_\varepsilon]\!] & = & 0 & \text{on} \quad \partial\mathcal{B}_\varepsilon \\ [\![q_\varepsilon(v_\varepsilon)]\!]\cdot n & = & -(1-\gamma^T)\nabla\theta(\widehat{x})\cdot n & \text{on} \quad \partial\mathcal{B}_\varepsilon \end{array}\right. , \tag{3.49}$$

and the remainder $\widetilde{\theta}_\varepsilon$ must be satisfies the following equation:

$$\left\{\begin{array}{rcll} \mathrm{div}(q_\varepsilon(\widetilde{\theta}_\varepsilon)) & = & 0 & \text{in} \quad \Omega \\ q_\varepsilon(\widetilde{\theta}_\varepsilon) & = & -\gamma_\varepsilon^T K\nabla\widetilde{\theta}_\varepsilon & \\ \widetilde{\theta}_\varepsilon & = & -v_\varepsilon & \text{on} \quad \Gamma_{D_\theta} \\ q_\varepsilon(\widetilde{\theta}_\varepsilon)\cdot n & = & -q_\varepsilon(v_\varepsilon)\cdot n & \text{on} \quad \Gamma_{N_\theta} \\ [\![q_\varepsilon(\widetilde{\theta}_\varepsilon)]\!]\cdot n & = & \varepsilon(1-\gamma^T)(\nabla q(\theta(\zeta))n)\cdot n & \text{on} \quad \partial\mathcal{B}_\varepsilon \end{array}\right. , \tag{3.50}$$

which has the following estimate $\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} = o(\varepsilon)$ (see A). Moreover, the point ζ in (3.50) belongs to the interval $(x, \widehat{x})$. In addition, the solution v_ε to the exterior problem can be obtained by using a standard separation of variables technique, together with the Fourier series method. Then, the solution of the problem (3.49) is explicitly written in compact notation as:

- Exterior solution ($r \geq \varepsilon$)

$$v_\varepsilon(x/\varepsilon) = \frac{1-\gamma^T}{1+\gamma^T} \frac{\varepsilon^2}{\|x-\widehat{x}\|^2} \nabla\theta(\widehat{x}) \cdot (x-\widehat{x}). \tag{3.51}$$

- Interior solution ($0 < r < \varepsilon$)

$$v_\varepsilon(x/\varepsilon) = \frac{1-\gamma^T}{1+\gamma^T} \nabla\theta(\widehat{x}) \cdot (x-\widehat{x}). \tag{3.52}$$

Following the same steps as before, we assume that the field φ_ε, solution of the auxiliary problem (2.29), admits an asymptotic expansion of the form

$$\varphi_\varepsilon(x) = \varphi(x) + p_\varepsilon(x/\varepsilon) + \widetilde{\varphi}_\varepsilon(x), \tag{3.53}$$

where p_ε is the solution of the exterior problem

$$\begin{cases} \operatorname{div}(q_\varepsilon(p_\varepsilon)) &= 0 & \text{in} \quad \mathbb{R}^2 \\ q_\varepsilon(p_\varepsilon) &= -\gamma_\varepsilon^T K \nabla p_\varepsilon & \\ p_\varepsilon &\to 0 & \text{at} \quad \infty \\ [\![p_\varepsilon]\!] &= 0 & \text{on} \quad \partial\mathcal{B}_\varepsilon \\ [\![q_\varepsilon(p_\varepsilon)]\!] \cdot n &= -(1-\gamma^T)\nabla\varphi(\widehat{x}) \cdot n & \text{on} \quad \partial\mathcal{B}_\varepsilon \end{cases}, \tag{3.54}$$

and the remainder $\widetilde{\varphi}_\varepsilon$ must be satisfies the following equation:

$$\begin{cases} \operatorname{div}(q_\varepsilon(\widetilde{\varphi}_\varepsilon)) &= 0 & \text{in} \quad \Omega \setminus \overline{\mathcal{B}_\varepsilon} \\ \operatorname{div}(q_\varepsilon(\widetilde{\varphi}_\varepsilon)) &= -(1-\gamma^T)\mathbb{C}B \cdot \nabla u^s & \text{in} \quad \mathcal{B}_\varepsilon \\ q_\varepsilon(\widetilde{\varphi}_\varepsilon) &= -\gamma_\varepsilon^T K \nabla \widetilde{\varphi}_\varepsilon & \\ \widetilde{\varphi}_\varepsilon &= -p_\varepsilon & \text{on} \quad \Gamma_{D_\theta} \\ q_\varepsilon(\widetilde{\varphi}_\varepsilon) \cdot n &= -q_\varepsilon(p_\varepsilon) \cdot n & \text{on} \quad \Gamma_{N_\theta} \\ [\![q_\varepsilon(\widetilde{\varphi}_\varepsilon)]\!] \cdot n &= \varepsilon(1-\gamma^T)(\nabla q(\varphi(\xi))n) \cdot n & \text{on} \quad \partial\mathcal{B}_\varepsilon \end{cases}, \tag{3.55}$$

which has the following estimate $\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)} = o(\varepsilon)$ (see A). Moreover, the point ξ in (3.55) belongs to the interval $(x, \widehat{x})$. In addition, by applying separation of variables technique and the Fourier series method, the solution φ_ε to the exterior problem (3.54) can be explicitly written in compact notation as:

- Exterior solution ($r \geq \varepsilon$)

$$p_\varepsilon(x/\varepsilon) = \frac{1-\gamma^T}{1+\gamma^T} \frac{\varepsilon^2}{\|x-\widehat{x}\|^2} \nabla\varphi(\widehat{x}) \cdot (x-\widehat{x}). \tag{3.56}$$

- Interior solution ($0 < r < \varepsilon$)

$$p_\varepsilon(x/\varepsilon) = \frac{1-\gamma^T}{1+\gamma^T} \nabla\varphi(\widehat{x}) \cdot (x-\widehat{x}). \tag{3.57}$$

Finally, using the asymptotic expansions presented in (3.51), (3.52), (3.56) and (3.57), and recalling the estimate for (3.47) (see A); we have that the derivative of the functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon)$ is given by the following expression:

$$\dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon) = -4\pi\varepsilon \frac{1-\gamma^T}{1+\gamma^T} \nabla\theta \cdot \nabla\varphi + o(\varepsilon). \tag{3.58}$$

3.3. **Topological Derivative .** In order to calculate the topological derivative, we shall adopt the methodology developed in [15], whereby the topological derivative is obtained as

$$D_T(\widehat{x}) = \lim_{\varepsilon \to 0} \frac{1}{f'(\varepsilon)} \dot{\mathcal{J}}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon), \tag{3.59}$$

where the function $f(\varepsilon)$ is the size of the perturbation, i.e. $f(\varepsilon) = \pi\varepsilon^2 \Rightarrow f'(\varepsilon) = 2\pi\varepsilon$.

Due to the linearity property of the shape functional with respect to the thermal problem (2.24), it is possible to write the topological derivative of the functional $\mathcal{J}_{\chi_\varepsilon}(u_\varepsilon, \theta_\varepsilon)$ based on the results given in (3.45) and (3.58). Then, the final expression of the topological derivative becomes a scalar function defined over the unperturbed domain Ω, that is

$$\begin{aligned} D_T(\widehat{x}) \;=\; & -\frac{1}{2E}\left(\frac{1-\gamma^M}{1+b\gamma^M}\right)\left[4\sigma(u)\cdot\sigma(u)+\frac{\gamma^M(b-2a)-1}{1+a\gamma^M}(\mathrm{tr}\sigma(u))^2\right] \\ & -\frac{1}{4E}\left(\frac{1-\gamma^M\gamma^C}{1+a\gamma^M}\right)\left[(1-\gamma^M\gamma^C)(1+\nu)(\mathrm{tr}Q(\theta))^2-4\mathrm{tr}\sigma(u)\mathrm{tr}Q(\theta)\right] \\ & -2\frac{1-\gamma^T}{1+\gamma^T}\nabla\theta\cdot\nabla\varphi\,. \end{aligned} \tag{3.60}$$

Notice that the first term is classic in the topological asymptotic analysis for the elasticity problem. The linearity property mentioned previously appears explicitly in the last term of the above results, see term involving the contrast parameter γ^T. On the other hand, the non-linear dependence of the problem with the thermo-elastic constitutive properties appears, also explicitly, in the term with the contrast parameters $\gamma^M\gamma^C$. These two last terms represent the contribution of the thermal problem to the elastic stress problem.

3.4. **Numerical validation .** The analytical formula for the topological derivative presented in (3.60), can be validated by using the computational framework described in this section. To this end, we define (for a finite value of ε) the function $g_\varepsilon(\widehat{x})$ as:

$$g_\varepsilon(\widehat{x}) := \frac{\psi(\chi_\varepsilon(\widehat{x}))-\psi(\chi)}{f(\varepsilon)}. \tag{3.61}$$

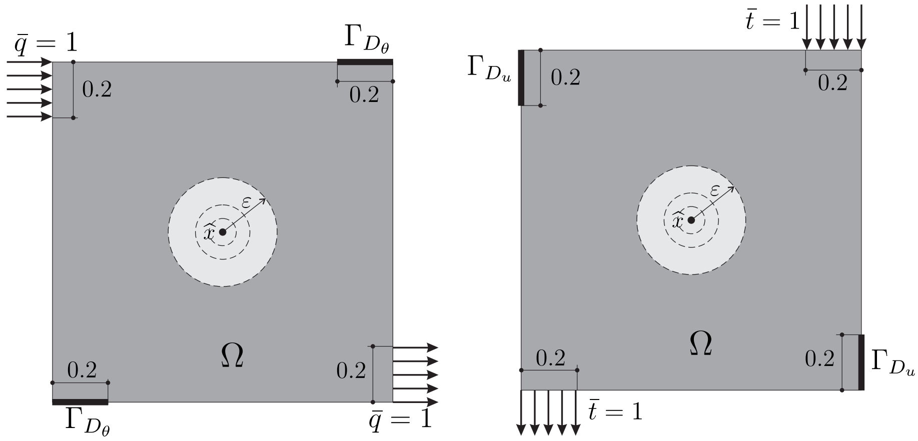
Clearly, the above definition have the following property,

$$\lim_{\varepsilon\to 0} g_\varepsilon(\widehat{x}) = D_T(\widehat{x})\,. \tag{3.62}$$

A numerical approximation of $D_T(\widehat{x})$ can be obtained by calculating the functions $\psi(\chi_\varepsilon(\widehat{x}))$ and $\psi(\chi)$, for a sequences of decreasing values of ε and then using (3.61) to compute the corresponding estimates $g_\varepsilon(\widehat{x})$ for $D_T(\widehat{x})$. The values of the function ψ are computed numerically by means of standard finite element procedure for the elasticity problem with thermal stresses. The domain considered in the verification is a unit square with material properties given by: Young's modulus $E = 1$, Poisson's ratio $\nu = 1/3$, thermal conductivity $k = 1$ and thermal expansion coefficient $\alpha = 1$. The perturbed domains are obtained by introducing circular inclusions of radii

$$\varepsilon \in \{0.160, 0.080, 0.040, 0.010, 0.005\}, \tag{3.63}$$

centered at $\widehat{x} = (0.5, 0.5)$, with the origin of the coordinate system positioned at the bottom left corner. The finite element mesh used to discretise the perturbed domain contains a total number of 962560 three-nodded elements and 481921 nodes. To solve the thermal problem, we set the temperature $\bar{\theta} = 0$ on the boundary denoted as Γ_{D_θ}. On the boundary Γ_{N_θ}, an external heat flux $\bar{q} = 1$ is prescribed, see fig. 2(a). In addition, the remainder part of the boundary remains insulated. For the mechanical problem, we prescribe the displacement on Γ_{D_u} to be $\bar{u} = 0$ and tractions $\bar{t} = 1$ on Γ_{N_u}, see fig. 2(b). Due to the definition of the auxiliary problems (2.16) and (2.29), the boundary conditions for these problems are the same as the thermal problem on Γ_{D_θ} and with homogeneous data on Γ_{N_θ}.



(a) Thermal problem. (b) Mechanical problem

FIGURE 2. Domain and boundary conditions.

The study is conducted for two combinations of the contrast parameters γ^M, γ^C and γ^T. The analyzed cases are given by:

- Case A: $\gamma^M = \gamma^C = \gamma^T = 1/3$,
- Case B: $\gamma^M = \gamma^T = \gamma^C = 3$.

The normalized obtained results (g_ε/D_T) are plotted in fig. 3, where the analytical topological derivative and the numerical approximations for each value of ε are shown. It can be seen that the numerical topological derivatives converge to the corresponding analytical value for all cases. This confirms the validity of the proposed formula (3.60).

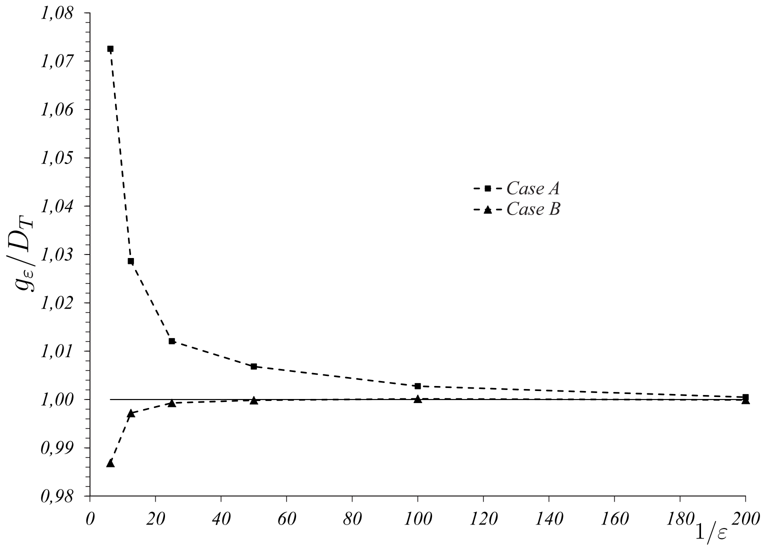


FIGURE 3. Results of numerical verification.

According to the numerical experiments, the obtained formula (3.60) remains valid only for small (infinitesimal) inclusions. The case associated to topological perturbations of finite size has been analyzed by [7, 3, 17, 12], for instance.

4. Final comments

The topological derivative in its closed form for the total potential mechanical energy associated to a thermo-mechanical semi-coupled system, when a circular inclusion is introduced at an arbitrary point of the domain, has been derived. In particular, the linear elasticity system (modeled by the Navier equation) coupled with the steady-state heat conduction problem (modeled by the Laplace equation) has been considered. The mechanical coupling term comes out from the thermal stresses induced by the temperature field. Since this term is non-local, a non-standard adjoint state has been introduced, which allowed to obtain a closed form for the topological derivative. In addition, a full mathematical justification for the derived formulas and precise estimates for the remainders of the topological asymptotic expansion have been provided. Finally, we remark that this information can be potentially used in a number of applications of practical interest such as multi-physic topology design of structures under mechanical and/or thermal loads.

Acknowledgments

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Appendix A. Estimation of the remainders

Lemma 1. *Let $\widetilde{u}_\varepsilon$ be solution to (3.35) or equivalently solution to the following variational problem: Find $\widetilde{u}_\varepsilon \in \widetilde{\mathcal{U}}_\varepsilon$, such that*

$$\int_\Omega \sigma_\varepsilon(\widetilde{u}_\varepsilon)\cdot\nabla\eta^s = \varepsilon^2\int_{\Gamma_{N_u}}\sigma(g_u)n\cdot\eta + \varepsilon\int_{\partial\mathcal{B}_\varepsilon} h_u\cdot\eta + \int_{\mathcal{B}_\varepsilon} b_u\cdot\eta \quad \forall\eta\in\widetilde{\mathcal{V}}_\varepsilon\ , \tag{A.1}$$

where the set $\widetilde{\mathcal{U}}_\varepsilon$ and the space $\widetilde{\mathcal{V}}_\varepsilon$ are defined as

$$\widetilde{\mathcal{U}}_\varepsilon := \{\phi\in H^1(\Omega;\mathbb{R}^2) : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \phi_{|\Gamma_{D_u}} = \varepsilon^2 g_u\}\ , \tag{A.2}$$

$$\widetilde{\mathcal{V}}_\varepsilon := \{\phi\in H^1(\Omega;\mathbb{R}^2) : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \phi_{|\Gamma_{D_u}} = 0\}\ , \tag{A.3}$$

and with functions g_u, h_u and b_u, independents of the small parameter ε, given by

$$g_u := -\varepsilon^{-2}w_\varepsilon\ , \tag{A.4}$$

$$h_u := (1-\gamma^M)(\nabla\sigma(u(\zeta))n)n - (1-\gamma^M\gamma^C)(\nabla Q(\theta(\xi))n)n)\ , \tag{A.5}$$

$$b_u := -(1-\gamma^C)\gamma^M\operatorname{div}(Q(\theta))\ , \tag{A.6}$$

where the points ζ and ξ belong to the interval $(x,\widehat{x})$. Then, we have the following estimate for the remainder $\widetilde{u}_\varepsilon$

$$\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)} \le C\varepsilon^{1+\delta}\ , \tag{A.7}$$

where C is a constant independent of the parameter ε and $\delta > 0$.

Proof. By taking $\eta = \widetilde{u}_\varepsilon - \phi_\varepsilon$ in (A.1), where ϕ_ε is the lifting of the Dirichlet boundary data $\varepsilon^2 g_u$ on Γ_{D_u}, we have

$$\int_\Omega \sigma_\varepsilon(\widetilde{u}_\varepsilon)\cdot\nabla\widetilde{u}^s_\varepsilon = \varepsilon^2\int_{\Gamma_{N_u}}\sigma(g_u)n\cdot\widetilde{u}_\varepsilon + \varepsilon^2\int_{\Gamma_{D_u}} g_u\cdot\sigma(\widetilde{u}_\varepsilon)n + \varepsilon\int_{\partial\mathcal{B}_\varepsilon} h_u\cdot\widetilde{u}_\varepsilon + \int_{\mathcal{B}_\varepsilon} b_u\cdot\widetilde{u}_\varepsilon\ . \tag{A.8}$$

From the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\int_\Omega \sigma_\varepsilon(\widetilde{u}_\varepsilon)\cdot\nabla\widetilde{u}^s_\varepsilon \le\ & \varepsilon^2\|\sigma(g_u)n\|_{H^{-1/2}(\Gamma_{N_u};\mathbb{R}^2)}\|\widetilde{u}_\varepsilon\|_{H^{1/2}(\Gamma_{N_u};\mathbb{R}^2)}\\ &+ \varepsilon^2\|g_u\|_{H^{1/2}(\Gamma_{D_u};\mathbb{R}^2)}\|\sigma(\widetilde{u}_\varepsilon)n\|_{H^{-1/2}(\Gamma_{D_u};\mathbb{R}^2)}\\ &+ \varepsilon\|h_u\|_{H^{-1/2}(\partial\mathcal{B}_\varepsilon;\mathbb{R}^2)}\|\widetilde{u}_\varepsilon\|_{H^{1/2}(\partial\mathcal{B}_\varepsilon;\mathbb{R}^2)}\\ &+ \|b_u\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)}\|\widetilde{u}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)}\ .\end{aligned} \tag{A.9}$$

Taking into account the trace theorem, we have

$$\begin{aligned}\int_\Omega \sigma_\varepsilon(\widetilde{u}_\varepsilon)\cdot\nabla\widetilde{u}^s_\varepsilon &\leq (\varepsilon^2 C_1+\varepsilon\|h_u\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)})\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}\\ &+ \|b_u\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)}\|\widetilde{u}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)}\\ &\leq \varepsilon^2 C_2\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}+\varepsilon C_3\|\widetilde{u}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)}\ ,\end{aligned} \tag{A.10}$$

where we have used the interior elliptic regularity of functions u and θ, solution to problems (2.8) and (2.11), respectively. For the estimation of the last term in the right-hand side of the above expression we will use the Hölder inequality together with the Sobolev embedding theorem. In fact, we can find an estimate for the remainder $\widetilde{u}_\varepsilon$ of the form $\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}\leq C\varepsilon^{1+\delta}$, with $\delta>0$ small. In particular, for $1/p+1/q=1$, we have

$$\begin{aligned}\|\widetilde{u}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)} &\leq \left[\left(\int_{\mathcal{B}_\varepsilon}\|\widetilde{u}_\varepsilon\|^{2p}\right)^{\frac{1}{p}}\left(\int_{\mathcal{B}_\varepsilon}1^{2q}\right)^{\frac{1}{q}}\right]^{\frac{1}{2}}\\ &= \pi^{1/2q}\,\varepsilon^{1/q}\|\widetilde{u}_\varepsilon\|_{L^{2p}(\mathcal{B}_\varepsilon;\mathbb{R}^2)}\\ &= \pi^{1/2q}\,\varepsilon^{1/q}\|\widetilde{u}_\varepsilon\|_{L^{2q/(q-1)}(\mathcal{B}_\varepsilon;\mathbb{R}^2)}\\ &\leq \varepsilon^\delta C\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}\ ,\end{aligned} \tag{A.11}$$

where $\delta=1/q$, with $q>1$, and the constant C independent of the small parameter ε. Next, by introducing the above result in (A.10) we have,

$$\int_\Omega \sigma_\varepsilon(\widetilde{u}_\varepsilon)\cdot\nabla\widetilde{u}^s_\varepsilon \leq \varepsilon^{1+\delta}C_4\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}\ . \tag{A.12}$$

Finally, from the coercivity of the bilinear form on the left-hand side of (A.1), namely,

$$c\|\widetilde{u}_\varepsilon\|^2_{H^1(\Omega;\mathbb{R}^2)} \leq \int_\Omega \sigma_\varepsilon(\widetilde{u}_\varepsilon)\cdot\nabla\widetilde{u}^s_\varepsilon\ , \tag{A.13}$$

we obtain

$$\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}\leq C_5\varepsilon^{1+\delta}\ , \tag{A.14}$$

which leads to the result, with $C_5=C_4/c$. □

Lemma 2. *Let $\widetilde{\theta}_\varepsilon$ be solution to (3.50) or equivalently solution to the following variational problem: Find $\widetilde{\theta}_\varepsilon\in\widetilde{\mathcal{U}}_\varepsilon$ such that*

$$-\int_\Omega q_\varepsilon(\widetilde{\theta}_\varepsilon)\cdot\nabla\eta=\varepsilon^2\int_{\Gamma_{N_\theta}} q(g_\theta)\cdot n\,\eta+\varepsilon\int_{\partial\mathcal{B}_\varepsilon} h_\theta\,\eta \quad \forall\eta\in\widetilde{\mathcal{V}}_\varepsilon\ , \tag{A.15}$$

where the set $\widetilde{\mathcal{U}}_\varepsilon$ and the space $\widetilde{\mathcal{V}}_\varepsilon$ are defined as

$$\widetilde{\mathcal{U}}_\varepsilon := \{\phi\in H^1(\Omega) : [\![\phi]\!]=0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \phi_{|_{\Gamma_{D_\theta}}}=-\varepsilon^2 g_\theta\}\ , \tag{A.16}$$

$$\widetilde{\mathcal{V}}_\varepsilon := \{\phi\in H^1(\Omega) : [\![\phi]\!]=0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \phi_{|_{\Gamma_{D_\theta}}}=0\}\ , \tag{A.17}$$

and with functions g_θ and h_θ, independents of the small parameter ε, given by

$$g_\theta := \varepsilon^{-2}v_\varepsilon\ , \tag{A.18}$$

$$h_\theta := -(1-\gamma^T)(\nabla q(\theta(\xi))n)\cdot n\ , \tag{A.19}$$

where the point ξ belongs to the interval $(x,\widehat{x})$. Then, we have the following estimate for the remainder $\widetilde{\theta}_\varepsilon$

$$\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)}\leq C\varepsilon^2\ , \tag{A.20}$$

where C is a constant independent of the parameter ε.

Proof. By taking $\eta = \widetilde{\theta}_\varepsilon - \phi_\varepsilon$ in (A.15), where ϕ_ε is the lifting of the Dirichlet boundary data $\varepsilon^2 g_\theta$ on Γ_{D_θ}, we have

$$-\int_\Omega q_\varepsilon(\widetilde{\theta}_\varepsilon)\cdot\nabla\widetilde{\theta}_\varepsilon = \varepsilon^2\int_{\Gamma_{N_\theta}} q(g_\theta)\cdot n\,\widetilde{\theta}_\varepsilon + \varepsilon^2\int_{\Gamma_{D_\theta}} g_\theta\, q(\widetilde{\theta}_\varepsilon)\cdot n + \varepsilon\int_{\partial\mathcal{B}_\varepsilon} h_\theta\,\widetilde{\theta}_\varepsilon\ . \tag{A.21}$$

From the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} -\int_\Omega q_\varepsilon(\widetilde{\theta}_\varepsilon)\cdot\nabla\widetilde{\theta}_\varepsilon &\le \varepsilon^2\|q(g_\theta)\cdot n\|_{H^{-1/2}(\Gamma_{N_\theta})}\|\widetilde{\theta}_\varepsilon\|_{H^{1/2}(\Gamma_{N_\theta})} \\ &+ \varepsilon^2\|g_\theta\|_{H^{1/2}(\Gamma_{D_\theta})}\|q(\widetilde{\theta}_\varepsilon)\cdot n\|_{H^{-1/2}(\Gamma_{D_\theta})} \\ &+ \varepsilon\|h_\theta\|_{H^{-1/2}(\partial\mathcal{B}_\varepsilon)}\|\widetilde{\theta}_\varepsilon\|_{H^{1/2}(\partial\mathcal{B}_\varepsilon)}\ . \end{aligned} \tag{A.22}$$

Taking into account the trace theorem, we have

$$\begin{aligned} -\int_\Omega q_\varepsilon(\widetilde{\theta}_\varepsilon)\cdot\nabla\widetilde{\theta}_\varepsilon &\le \varepsilon^2 C_1\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_2\|\nabla\widetilde{\theta}_\varepsilon\|_{L^2(\Omega)} \\ &+ \varepsilon\|h_\theta\|_{L^2(\mathcal{B}_\varepsilon)}\|\widetilde{\theta}_\varepsilon\|_{H^1(\mathcal{B}_\varepsilon)} \\ &\le \varepsilon^2 C_1\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_3\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_4\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} \\ &\le \varepsilon^2 C_5\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)}\ , \end{aligned} \tag{A.23}$$

where we have used the interior elliptic regularity of function θ, solution to problem (2.11). Finally, from the coercivity of the bilinear form on the left-hand side of (A.15), namely,

$$c\|\widetilde{\theta}_\varepsilon\|^2_{H^1(\Omega)} \le -\int_\Omega q_\varepsilon(\widetilde{\theta}_\varepsilon)\cdot\nabla\widetilde{\theta}_\varepsilon\ , \tag{A.24}$$

we obtain

$$\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^2\ , \tag{A.25}$$

which leads to the result, with $C = C_5/c$. □

Lemma 3. *Let $\widetilde{\theta}_\varepsilon$ be solution to (3.50). Then, its derivative with respect to ε has the following estimate*

$$\|\widetilde{\theta}'_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon\,, \tag{A.26}$$

where C is a constant independent of the parameter ε.

Proof. The convergence follows by the property of the asymptotic expansions of solutions which can be differentiated term by term under the appropriate decrease order rule for the remainders of the expansions, namely $O'(\varepsilon^m) = O(\varepsilon^{m-1})$. See the work by [14], for instance. Therefore, the result follows by Lemma 2 together with the rule $O'(\varepsilon^2) = O(\varepsilon)$. □

Lemma 4. *Let $\widetilde{\varphi}_\varepsilon$ be solution to (3.55) or equivalently solution to the following variational problem: Find $\widetilde{\varphi}_\varepsilon \in \widetilde{\mathcal{U}}_\varepsilon$ such that*

$$-\int_\Omega q_\varepsilon(\widetilde{\varphi}_\varepsilon)\cdot\nabla\eta = \varepsilon^2\int_{\Gamma_{N_\theta}} q(g_\varphi)\cdot n\,\eta + \varepsilon\int_{\partial\mathcal{B}_\varepsilon} h_\varphi\,\eta + \int_{\mathcal{B}_\varepsilon} b_\varphi\,\eta \quad \forall\eta\in\widetilde{\mathcal{V}}_\varepsilon\ , \tag{A.27}$$

where the set $\widetilde{\mathcal{U}}_\varepsilon$ and the space $\widetilde{\mathcal{V}}_\varepsilon$ are defined as

$$\widetilde{\mathcal{U}}_\varepsilon := \{\phi\in H^1(\Omega) : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \phi_{|\Gamma_{D_\theta}} = -\varepsilon^2 g_\varphi\}\ , \tag{A.28}$$

$$\widetilde{\mathcal{V}}_\varepsilon := \{\phi\in H^1(\Omega) : [\![\phi]\!] = 0 \text{ on } \partial\mathcal{B}_\varepsilon,\ \phi_{|\Gamma_{D_\theta}} = 0\}\ , \tag{A.29}$$

and with functions g_φ, h_φ and b_φ, independent of the small parameter ε, given by

$$g_\varphi := \varepsilon^{-2}p_\varepsilon\ , \tag{A.30}$$

$$h_\varphi := -(1-\gamma^T)(\nabla q(\theta(\xi))n)\cdot n\ , \tag{A.31}$$

$$b_\varphi := -(1-\gamma^T)\alpha\operatorname{tr}\sigma(u)\ . \tag{A.32}$$

where the point ξ belongs to the interval $(x, \widehat{x})$. Then, we have the following estimate for the remainder $\widetilde{\varphi}_\varepsilon$

$$\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{1+\delta}\,, \tag{A.33}$$

where the constant C is independent of the parameter ε and $\delta > 0$.

Proof. By taking $\eta = \widetilde{\varphi}_\varepsilon - \phi_\varepsilon$ in (A.15), where ϕ_ε is the lifting of the Dirichlet boundary data $\varepsilon^2 g_\varphi$ on Γ_{D_θ}, we have

$$\begin{aligned} -\int_\Omega q_\varepsilon(\widetilde{\varphi}_\varepsilon)\cdot\nabla\widetilde{\varphi}_\varepsilon &= \varepsilon^2\int_{\Gamma_{N_\theta}} q(g_\varphi)\cdot n\,\widetilde{\varphi}_\varepsilon + \varepsilon^2\int_{\Gamma_{D_\theta}} g_\varphi\, q(\widetilde{\varphi}_\varepsilon)\cdot n \\ &+ \varepsilon\int_{\partial\mathcal{B}_\varepsilon} h_\varphi\,\widetilde{\varphi}_\varepsilon + \int_{\mathcal{B}_\varepsilon} b_\varphi\,\widetilde{\varphi}_\varepsilon\,. \end{aligned} \tag{A.34}$$

From the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} -\int_\Omega q_\varepsilon(\widetilde{\varphi}_\varepsilon)\cdot\nabla\widetilde{\varphi}_\varepsilon &\leq \varepsilon^2\|q(g_\varphi)\cdot n\|_{H^{-1/2}(\Gamma_{N_\theta})}\|\widetilde{\varphi}_\varepsilon\|_{H^{1/2}(\Gamma_{N_\theta})} \\ &+ \varepsilon^2\|g_\varphi\|_{H^{1/2}(\Gamma_{D_\theta})}\|q(\widetilde{\varphi}_\varepsilon)\cdot n\|_{H^{-1/2}(\Gamma_{D_\theta})} \\ &+ \varepsilon\|h_\varphi\|_{H^{-1/2}(\partial\mathcal{B}_\varepsilon)}\|\widetilde{\varphi}_\varepsilon\|_{H^{1/2}(\partial\mathcal{B}_\varepsilon)} \\ &+ \|b_\varphi\|_{L^2(\mathcal{B}_\varepsilon)}\|\widetilde{\varphi}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon)}\,. \end{aligned} \tag{A.35}$$

Taking into account the trace theorem, we have

$$\begin{aligned} -\int_\Omega q_\varepsilon(\widetilde{\varphi}_\varepsilon)\cdot\nabla\widetilde{\varphi}_\varepsilon &\leq \varepsilon^2 C_1\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_2\|\nabla\widetilde{\varphi}_\varepsilon\|_{L^2(\Omega)} \\ &+ \varepsilon\|h_\varphi\|_{L^2(\mathcal{B}_\varepsilon)}\|\widetilde{\varphi}_\varepsilon\|_{H^1(\mathcal{B}_\varepsilon)} + \|b_\varphi\|_{L^2(\mathcal{B}_\varepsilon)}\|\widetilde{\varphi}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon)} \\ &\leq \varepsilon^2 C_1\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_3\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)} \\ &+ \varepsilon^2 C_4\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)} + \varepsilon C_5\|\widetilde{\varphi}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon)}\,, \end{aligned} \tag{A.36}$$

where we have used the interior elliptic regularity of functions θ and u. By using the Hölder inequality together with the Sobolev embedding theorem, the last term in the right-hand side of the above expression is given by

$$\|\widetilde{\varphi}_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon)} \leq \varepsilon^\delta C\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)}\,, \tag{A.37}$$

where $\delta = 1/q$, with $q > 1$, and the constant C independent of the small parameter ε. Next, by introducing the above result in (A.36) we have,

$$-\int_\Omega q_\varepsilon(\widetilde{\varphi}_\varepsilon)\cdot\nabla\widetilde{\varphi}_\varepsilon \leq \varepsilon^{1+\delta} C_6\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)}\,. \tag{A.38}$$

Finally, from the coercivity of the bilinear form on the left-hand side of (A.15), namely,

$$c\|\widetilde{\varphi}_\varepsilon\|^2_{H^1(\Omega)} \leq -\int_\Omega q_\varepsilon(\widetilde{\varphi}_\varepsilon)\cdot\nabla\widetilde{\varphi}_\varepsilon\,, \tag{A.39}$$

we obtain

$$\|\widetilde{\varphi}_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{1+\delta}\,, \tag{A.40}$$

which leads to the result, with $C = C_6/c$ □

Lemma 5. *Let θ_ε, u_ε and u solution of the problems (2.24), (2.21) and (2.8). Then, we have the following estimate for the remainder $\mathcal{E}(\varepsilon)$ in (3.47):*

$$\mathcal{E}(\varepsilon) \leq C\varepsilon^2\,, \tag{A.41}$$

where C is a constant independent of the parameter ε.

Proof. By tacking into account the definition of the mechanical and thermal stress operators σ_ε and Q_ε, respectively, the remainder term $\mathcal{E}(\varepsilon)$ in (3.47) can be alternatively written as:

$$\mathcal{E}(\varepsilon) = -\int_\Omega \gamma_\varepsilon^C B\theta'_\varepsilon \cdot \sigma_\varepsilon(u_\varepsilon - u). \tag{A.42}$$

Next, by considering the definition of the contrast γ_ε^C and the ansatz (3.31) and (3.48), the above expression is given by

$$\begin{aligned}\mathcal{E}(\varepsilon) &= -\int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} Bv'_\varepsilon \cdot \sigma_\varepsilon(w_\varepsilon) - \int_{\mathcal{B}_\varepsilon} \gamma^C Bv'_\varepsilon \cdot \sigma_\varepsilon(w_\varepsilon) - \int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} B\widetilde{\theta}'_\varepsilon \cdot \sigma_\varepsilon(w_\varepsilon) \\ &\quad - \int_{\mathcal{B}_\varepsilon} \gamma^C B\widetilde{\theta}'_\varepsilon \cdot \sigma_\varepsilon(w_\varepsilon) - \int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} Bv'_\varepsilon \cdot \sigma_\varepsilon(\widetilde{u}_\varepsilon) - \int_{\mathcal{B}_\varepsilon} \gamma^C Bv'_\varepsilon \cdot \sigma_\varepsilon(\widetilde{u}_\varepsilon) \\ &\quad - \int_\Omega B\widetilde{\theta}'_\varepsilon \cdot \sigma_\varepsilon(\widetilde{u}_\varepsilon)\,.\end{aligned} \tag{A.43}$$

Since $v'_\varepsilon = 0$ in $\mathcal{B}_\varepsilon$ and $\sigma_\varepsilon(\phi) = \sigma(\phi)$ in $\Omega \setminus \overline{\mathcal{B}_\varepsilon}$, the remainder $\mathcal{E}(\varepsilon)$ is given by

$$\begin{aligned}\mathcal{E}(\varepsilon) &= -\int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} Bv'_\varepsilon \cdot \sigma(w_\varepsilon) - \int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} B\widetilde{\theta}'_\varepsilon \cdot \sigma(w_\varepsilon) - \int_{\mathcal{B}_\varepsilon} \gamma^C B\widetilde{\theta}'_\varepsilon \cdot \sigma_\varepsilon(w_\varepsilon) \\ &\quad - \int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} Bv'_\varepsilon \cdot \sigma(\widetilde{u}_\varepsilon) - \int_\Omega B\widetilde{\theta}'_\varepsilon \cdot \sigma_\varepsilon(\widetilde{u}_\varepsilon)\,.\end{aligned} \tag{A.44}$$

From the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}\mathcal{E}(\varepsilon) &\leq C_1\|v'_\varepsilon\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon})}\|\sigma(w_\varepsilon)\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon};\mathbb{R}^2)} + C_1\|\widetilde{\theta}'_\varepsilon\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon})}\|\sigma(w_\varepsilon)\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon};\mathbb{R}^2)} \\ &+ C_2\|\widetilde{\theta}'_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon)}\|\sigma_\varepsilon(w_\varepsilon)\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)} + C_1\|v'_\varepsilon\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon})}\|\sigma(\widetilde{u}_\varepsilon)\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon};\mathbb{R}^2)} \\ &+ C_3\|\widetilde{\theta}'_\varepsilon\|_{L^2(\Omega)}\|\sigma_\varepsilon(\widetilde{u}_\varepsilon)\|_{L^2(\Omega;\mathbb{R}^2)}\,.\end{aligned} \tag{A.45}$$

By considering the asymptotic expansion of v_ε presented in (3.51) its derivative with respect to ε can be written as $v'_\varepsilon = \varepsilon g_v(x)$ in $\Omega \setminus \overline{\mathcal{B}_\varepsilon}$, with the function g_v independent of the parameter ε. After introducing a change of variables of the form $y = \varepsilon^{-1}x$, we have

$$\begin{aligned}\|v'_\varepsilon\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon})} &= \left(\int_{\Omega\setminus\overline{\mathcal{B}_\varepsilon}} |\varepsilon g_v(x)|^2\right)^{\frac{1}{2}} \\ &= \varepsilon\left(\int_{\mathbb{R}^2\setminus\overline{\mathcal{B}_1}} |g_v(y)|^2\right)^{\frac{1}{2}} \\ &\leq \varepsilon C_4\,,\end{aligned} \tag{A.46}$$

where we have used the fact that function $g_v(y)$ is regular at infinity, i.e., $g_v(y) \to 0$ when $\|y\| \to \infty$. In the same way, by considering (3.38) to (3.40) and the same change of variables, we obtain

$$\begin{aligned}\|\sigma(w_\varepsilon)\|_{L^2(\Omega\setminus\mathcal{B}_\varepsilon;\mathbb{R}^2)} &= \varepsilon\left(\int_{\mathbb{R}^2\setminus\overline{\mathcal{B}_1}} \|\sigma(h_\sigma(y))\|^2\right)^{\frac{1}{2}} \\ &\leq \varepsilon C_5\,,\end{aligned} \tag{A.47}$$

being the function $h_\sigma(x)$ independent of the parameter ε and regular at infinity, such that, $\sigma(h_\sigma(y)) \to 0$ when $\|y\| \to \infty$. By taking into account the fact that function $\sigma_\varepsilon(w_\varepsilon)$ is independent of the parameter ε in $\mathcal{B}_\varepsilon$, we can write $\sigma_\varepsilon(w_\varepsilon) = \sigma(b_\sigma(x))$ in $\mathcal{B}_\varepsilon$. Then, we have

$$\begin{aligned}\|\sigma_\varepsilon(w_\varepsilon)\|_{L^2(\mathcal{B}_\varepsilon;\mathbb{R}^2)} &= \left(\int_{\mathcal{B}_\varepsilon} \|\sigma(b_\sigma(x))\|^2\right)^{\frac{1}{2}} \\ &\leq \varepsilon C_6\,.\end{aligned} \tag{A.48}$$

Next, by introducing the above results in (A.45), we have

$$\begin{aligned}\mathcal{E}(\varepsilon) &\leq \varepsilon^2 C_7 + \varepsilon C_8 \|\widetilde{\theta}'_\varepsilon\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon})} + \varepsilon C_9 \|\widetilde{\theta}'_\varepsilon\|_{L^2(\mathcal{B}_\varepsilon)} \\ &+ \varepsilon C_{10} \|\sigma(\widetilde{u}_\varepsilon)\|_{L^2(\Omega\setminus\overline{\mathcal{B}_\varepsilon};\mathbb{R}^2)} + C_3 \|\widetilde{\theta}'_\varepsilon\|_{L^2(\Omega)} \|\sigma_\varepsilon(\widetilde{u}_\varepsilon)\|_{L^2(\Omega;\mathbb{R}^2)}\,, \\ &\leq \varepsilon^2 C_7 + \varepsilon C_{11} \|\widetilde{\theta}'_\varepsilon\|_{H^1(\Omega)} + \varepsilon C_{12} \|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)} \\ &+ C_{13} \|\widetilde{\theta}'_\varepsilon\|_{H^1(\Omega)} \|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}\,,\end{aligned} \tag{A.49}$$

Finally, by considering the Lemmas 1, 2 and 3, we obtain

$$\mathcal{E}(\varepsilon) \leq C\varepsilon^2\,, \tag{A.50}$$

which leads to the results, with C independent of the parameter ε. □

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(S.M. Giusti) Departamento de Ingeniería Civil, Facultad Regional Córdoba, Universidad Tecnológica Nacional (UTN/FRC - CONICET) Maestro M. López esq. Cruz Roja Argentina, X5016ZAA - Córdoba, Argentina.
E-mail address: sgiusti@civil.frc.utn.edu.ar

(A.A. Novotny) Laboratório Nacional de Computação Científica LNCC/MCT, Coordenação de Matemática Aplicada e Computacional, Av. Getúlio Vargas 333, Quitandinha, 25651-075, Petrópolis, RJ, Brazil.
E-mail address: novotny@lncc.br

(J.E. Muñoz Rivera) Laboratório Nacional de Computação Científica LNCC/MCT, Coordenação de Matemática Aplicada e Computacional, Av. Getúlio Vargas 333, Quitandinha, 25651-075, Petrópolis, RJ, Brazil.
E-mail address: rivera@lncc.br

(J.E. Esparta Rodriguez) Laboratório Nacional de Computação Científica LNCC/MCT, Coordenação de Matemática Aplicada e Computacional, Av. Getúlio Vargas 333, Quitandinha, 25651-075, Petrópolis, RJ, Brazil.
E-mail address: esparta@lncc.br