TOPOLOGICAL DERIVATIVE FOR AN ANISOTROPIC AND HETEROGENEOUS HEAT DIFFUSION PROBLEM

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Abstract. The topological derivative measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation. According to the literature, the topological derivative has been fully developed for a wide range of physical phenomenon modeled by partial differential equations, considering homogeneous and isotropic constitutive behavior. In fact, only a few works dealing with heterogeneous and anisotropic material behavior can be found in the literature, and, in general, the derived formulas are given in an abstract form. In this work, we derive the topological derivative in its closed form for the total potential energy associated to an anisotropic and heterogeneous heat diffusion problem, when a small circular inclusion of the same nature of the bulk phase is introduced at an arbitrary point of the domain. In addition, we provide a full mathematical justification for the derived formula and develop precise estimates for the remainders of the topological asymptotic expansion. Finally, the influence of the heterogeneity and anisotropy are shown through some numerical examples of heat conductor topology optimization.

1. Introduction

The topological derivative measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation, such as the insertion of holes, inclusions, source-terms or even cracks [8]. The topological derivative was rigorously introduced by [18]. Since then, this concept has proved to be extremely useful in the treatment of a wide range of problems, for instance, topology optimization [6, 16], inverse analysis [5, 13] and image processing [12, 14], and has became a subject of intensive research. See, for instance, applications of the topological derivative in the multi-scale constitutive modeling context [4, 9] and fracture mechanics sensitivity analysis [11]. Concerning the theoretical development of the topological asymptotic analysis, the reader may refer to the papers by [2] and [15], for instance.

In order to introduce these concepts, let us consider a bounded domain $\Omega \subset \mathbb{R}^2$, which is subject to a nonsmooth perturbation confined in a small ball $B_\varepsilon(\hat{x})$ of size ε and center at $\hat{x}$, as shown in Fig. 1. We introduce a characteristic function $x \mapsto \chi_\Omega(x)$, $x \in \mathbb{R}^2$, associated to the unperturbed domain, given by

$$\chi_\Omega(x) := \begin{cases} 1 & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \notin \Omega \end{cases} . \tag{1}$$

Then, we define a characteristic function associated to the topologically perturbed domain of the form $x \mapsto \chi_{\Omega_\varepsilon(\hat{x})}$, $x \in \mathbb{R}^2$. In the case of a perforation, for instance, $\chi_{\Omega_\varepsilon(\hat{x})} := \chi_\varepsilon(\hat{x}) = \chi_\Omega - \chi_{\overline{B_\varepsilon(\hat{x})}}$ and the perforated domain is obtained as $\Omega_\varepsilon(\hat{x}) = \Omega \setminus \overline{B_\varepsilon(\hat{x})}$. Then, we assume that a given shape functional $\psi(\chi_\varepsilon(\hat{x}))$, associated to the topologically perturbed domain, admits the following topological asymptotic expansion

$$\psi(\chi_\varepsilon(\hat{x})) = \psi(\chi_\Omega) + f(\varepsilon) D_T \psi(\hat{x}) + o(f(\varepsilon)) , \tag{2}$$

where $\psi(\chi_\Omega)$ is the shape functional associated to the original (unperturbed) domain and $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$, when $\varepsilon \to 0$. The function $\hat{x} \mapsto D_T\psi(\hat{x})$ is called the topological derivative of ψ at $\hat{x}$. Therefore, this derivative can be seen as a first order correction of $\psi(\chi_\Omega)$ to approximate $\psi(\chi_\varepsilon(\hat{x}))$. In fact, after rearranging (2) we have

$$\frac{\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi_\Omega)}{f(\varepsilon)} = D_T\psi(\hat{x}) + \frac{o(f(\varepsilon))}{f(\varepsilon)} . \tag{3}$$

Key words and phrases. topological derivative, topological asymptotic analysis, heterogeneous and anisotropic heat diffusion, heat conductor topology optimization.

The limit passage $\varepsilon \to 0$ in the above expression leads to

$$D_T\psi(\hat{x}) = \lim_{\varepsilon\to 0} \frac{\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi_\Omega)}{f(\varepsilon)} \ . \tag{4}$$

Since we are dealing with singular domain perturbations, the shape functionals $\psi(\chi_\varepsilon(\hat{x}))$ and $\psi(\chi_\Omega)$ are associated to topologically different domains. Therefore, the above limit is not trivial to calculate. In particular, we need to perform an asymptotic analysis of the shape functional $\psi(\chi_\varepsilon(\hat{x}))$ with respect to the small parameter ε.

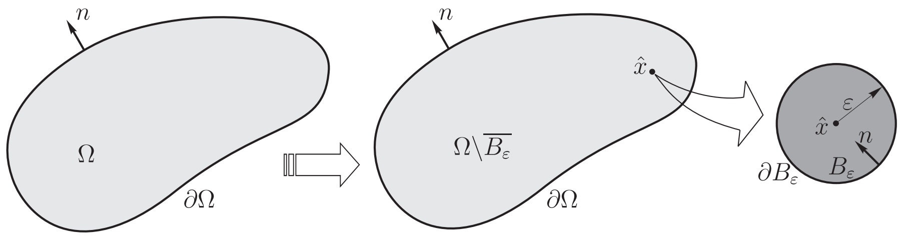


Figure 1. Topological derivative concept.

According to the literature, the topological derivative has been fully developed for a wide range of physical phenomenon modeled by partial differential equations, considering homogeneous and isotropic constitutive behavior. In fact, only a few works dealing with heterogeneous and anisotropic material behavior can be found in the literature, and, in general, the derived formulas are given in an abstract form (see, for instance, [7]). In particular, the topological sensitivity associated to the nucleation of a hole in a domain characterized by an orthotropic and homogeneous heat diffusion problem was calculated by [18]. In order to simplify the analysis, the domain was perturbed by introducing an elliptical hole oriented in the directions of the orthotropy and with semi-axis proportional to the material properties coefficients in each orthogonal direction. More recently in [10], the previous result was extended by considering as topological perturbation a small circular inclusion of the same nature as the bulk material, instead of an elliptical hole. In this work, we derive the topological derivative in its closed form for the total potential energy associated to an anisotropic and heterogeneous heat diffusion problem, when a small circular inclusion of the same nature of the bulk phase is introduced at an arbitrary point of the domain. In addition, we provide a full mathematical justification for the derived formula and develop precise estimates for the remainders of the topological asymptotic expansion. Finally, the influence of the heterogeneity and anisotropy are shown through some numerical examples of heat conductor topology optimization. We note that this result can be applied in technological research areas such as topology design of piezoresistive membranes. In fact, under a deformation process, the constitutive properties of such membranes change according to the stress state. Hence, their material properties become highly anisotropic and heterogeneous.

This paper is organized as follows. Section 2 describes the model associated to an anisotropic and heterogeneous heat diffusion problem. In Section 3, we present the main result of the paper: a closed formula for the topological derivative. In Section 4 is presented a numerical experiment showing the influence of the conductivity tensor in the optimal design of heat conductor. The paper ends in Section 5 where concluding remarks are presented.

2. Formulation of the problem

As mentioned in the previous section, the topological asymptotic analysis of the total potential energy associated to an anisotropic and heterogeneous heat diffusion problem is calculated. Thus, the unperturbed shape functional is defined as:

$$\psi(\chi_\Omega) := \mathcal{J}_{\chi_\Omega}(\theta) = \frac{1}{2}\int_\Omega K\nabla\theta\cdot\nabla\theta + \int_{\Gamma_N} \overline{q}\theta \ , \tag{5}$$

where $K = K(x)$ is a symmetric second order conductivity tensor and θ is solution of the following variational problem: find the field $\theta \in \mathcal{U}$, such that

$$\int_\Omega K\nabla\theta \cdot \nabla\eta + \int_{\Gamma_N} \overline{q}\eta = 0 \qquad \forall \eta \in \mathcal{V} . \tag{6}$$

In the variational problem (6) the set $\mathcal{U}$ of admissible functions and the space $\mathcal{V}$ of admissible variations are given by

$$\mathcal{U} := \left\{\phi \in H^1(\Omega) : \phi|_{\Gamma_D} = \overline{\theta}\right\} \quad \text{and} \quad \mathcal{V} := \left\{\phi \in H^1(\Omega) : \phi|_{\Gamma_D} = 0\right\} . \tag{7}$$

In addition, $\partial\Omega = \overline{\Gamma_N \cup \Gamma_D}$ with $\Gamma_N \cap \Gamma_D = \emptyset$, where Γ_N and Γ_D are Neumann and Dirichlet boundaries, respectively. Thus, $\overline{\theta}$ is a Dirichlet data on Γ_D and $\overline{q}$ is a Neumann data on Γ_N, both assumed to be smooth enough, see Fig. 2.

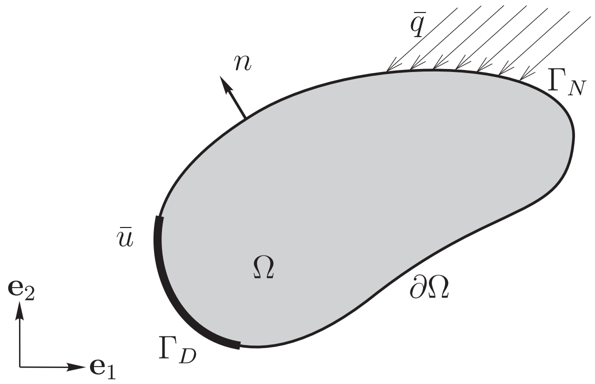


Figure 2. Formulation of the problem.

In our particular case, we consider a perturbation on the domain given by the nucleation of a small circular inclusion with conductivity γK, where the parameter $\gamma \in [0, \infty)$ represents the contrast in the material property. Taking into account the definition of the characteristic function associated to the perturbed domain $\chi_\varepsilon(\hat{x}) = \chi_\Omega - (1-\gamma)\chi_{B_\varepsilon(\hat{x})}$, the perturbed shape functional can be written as:

$$\psi(\chi_\varepsilon(\hat{x})) := \mathcal{J}_{\chi_\varepsilon}(\theta_\varepsilon) = \frac{1}{2}\int_\Omega \gamma_\varepsilon K\nabla\theta_\varepsilon \cdot \nabla\theta_\varepsilon + \int_{\Gamma_N} \overline{q}\theta_\varepsilon , \tag{8}$$

where parameter γ_ε is defined as

$$\gamma_\varepsilon := \begin{cases} 1 & \text{in} \quad \Omega \setminus \overline{B_\varepsilon} \\ \gamma & \text{in} \quad B_\varepsilon \end{cases} . \tag{9}$$

In addition, in (8) the function θ_ε is the solution of the following variational problem: find the field $\theta_\varepsilon \in \mathcal{U}_\varepsilon$, such that

$$\int_\Omega \gamma_\varepsilon K\nabla\theta_\varepsilon \cdot \nabla\eta + \int_{\Gamma_N} \overline{q}\eta = 0 \qquad \forall \eta \in \mathcal{V}_\varepsilon , \tag{10}$$

and the set $\mathcal{U}_\varepsilon$ and the space $\mathcal{V}_\varepsilon$ are defined as

$$\mathcal{U}_\varepsilon := \{\phi \in \mathcal{U} : [\![\phi]\!] = 0 \text{ on } \partial B_\varepsilon\} \quad \text{and} \quad \mathcal{V}_\varepsilon := \{\phi \in \mathcal{V} : [\![\phi]\!] = 0 \text{ on } \partial B_\varepsilon\} , \tag{11}$$

where we use $[\![(\cdot)]\!]$ to denotes the *jump* of function $(\cdot)$ across the boundary ∂B_ε. Note that the domain Ω is topologically perturbed by the introduction of an inclusion $B_\varepsilon(\hat{x})$ of the same nature as the bulk material, but with contrast γ. Finally, the Euler-Lagrange equation associated to the variational problem (10) reads: find field θ_ε, such that

$$\begin{cases} \operatorname{div}(\gamma_\varepsilon K\nabla\theta_\varepsilon) & = 0 \quad \text{in} \quad \Omega \\ \theta_\varepsilon & = \overline{\theta} \quad \text{on} \quad \Gamma_D \\ -K\nabla\theta_\varepsilon \cdot n & = \overline{q} \quad \text{on} \quad \Gamma_N \\ [\![\theta_\varepsilon]\!] & = 0 \quad \text{on} \quad \partial B_\varepsilon \\ [\![\gamma_\varepsilon K\nabla\theta_\varepsilon]\!] \cdot n & = 0 \quad \text{on} \quad \partial B_\varepsilon \end{cases} . \tag{12}$$

3. Topological derivative

In this Section we present the main result of this work: a closed formula for the topological derivative of the total potential energy associated to an anisotropic and heterogeneous heat diffusion problem. Tacking into account the problems defined over the original and perturbed domains, we can choose an admissible test function $\eta = \theta_\varepsilon - \theta$. Then the state equations, given by (6) and (10), can be respectively written as

$$\int_\Omega K\nabla\theta \cdot \nabla(\theta_\varepsilon - \theta) + \int_{\Gamma_N} \overline{q}(\theta_\varepsilon - \theta) = 0, \tag{13}$$

$$\int_\Omega \gamma_\varepsilon K\nabla\theta_\varepsilon \cdot \nabla(\theta_\varepsilon - \theta) + \int_{\Gamma_N} \overline{q}(\theta_\varepsilon - \theta) = 0. \tag{14}$$

After rearranging the above expressions, we obtain

$$\int_\Omega K\nabla\theta \cdot \nabla\theta = \int_\Omega K\nabla\theta_\varepsilon \cdot \nabla\theta + \int_{\Gamma_N} \overline{q}(\theta_\varepsilon - \theta), \tag{15}$$

$$\int_\Omega \gamma_\varepsilon K\nabla\theta_\varepsilon \cdot \nabla\theta_\varepsilon = \int_\Omega \gamma_\varepsilon K\nabla\theta_\varepsilon \cdot \nabla\theta - \int_{\Gamma_N} \overline{q}(\theta_\varepsilon - \theta). \tag{16}$$

Introducing the above expressions in the definition of the shape functionals $\psi(\chi_\Omega)$ and $\psi(\chi_\varepsilon(\hat{x}))$, we have that the total potential energy associated to both problems can be written alternatively as

$$\psi(\chi_\Omega) = \frac{1}{2}\int_\Omega K\nabla\theta_\varepsilon \cdot \nabla\theta + \frac{1}{2}\int_{\Gamma_N} \overline{q}(\theta_\varepsilon + \theta), \tag{17}$$

$$\psi(\chi_\varepsilon(\hat{x})) = \frac{1}{2}\int_\Omega \gamma_\varepsilon K\nabla\theta_\varepsilon \cdot \nabla\theta + \frac{1}{2}\int_{\Gamma_N} \overline{q}(\theta_\varepsilon + \theta). \tag{18}$$

With the above results in hand, the difference of the shape functionals associated to the unperturbed and perturbed problems reads

$$\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi_\Omega) = \frac{1}{2}\int_\Omega \gamma_\varepsilon K\nabla\theta_\varepsilon \cdot \nabla\theta - \frac{1}{2}\int_\Omega K\nabla\theta_\varepsilon \cdot \nabla\theta. \tag{19}$$

Next, by considering the definition of the contrast γ_ε in the previous results, we have that the difference of the total potential energy is given by an integral concentrated in the inclusion B_ε, namely

$$\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi_\Omega) = -\frac{1}{2}(1-\gamma)\int_{B_\varepsilon} K\nabla\theta_\varepsilon \cdot \nabla\theta. \tag{20}$$

Let us assume that the conductivity tensor $K(x)$ is smooth enough such that it admits an expansion in Taylor series around the point $\hat{x}$ of the form $K(x) = K(\hat{x}) + \nabla K(\zeta)(x - \hat{x})$, where $\zeta \in (x, \hat{x})$. In order to analytically solve the above integral, we introduce the following ansatz for the solution associated to the perturbed problem θ_ε:

$$\theta_\varepsilon(x) = \theta(x) + w_\varepsilon(x) + \widetilde{\theta}_\varepsilon(x), \tag{21}$$

where the function w_ε is the solution of the following exterior problem

$$\left\{\begin{array}{rll} \operatorname{div}(\gamma_\varepsilon K(\hat{x})\nabla w_\varepsilon) & = 0 & \text{in} \quad \mathbb{R}^2 \\ w_\varepsilon & \to 0 & \text{at} \quad \infty \\ [\![w_\varepsilon]\!] & = 0 & \text{on} \quad \partial B_\varepsilon \\ [\![\gamma_\varepsilon K(\hat{x})\nabla w_\varepsilon]\!] \cdot n & = (1-\gamma)\, K(\hat{x})\nabla\theta(\hat{x}) \cdot n & \text{on} \quad \partial B_\varepsilon \end{array}\right. , \tag{22}$$

and the remainder $\widetilde{\theta}_\varepsilon$ must satisfy the following equations:

$$\left\{\begin{array}{rll} \operatorname{div}(\gamma_\varepsilon K\nabla\widetilde{\theta}_\varepsilon) & = \operatorname{div}(\gamma_\varepsilon \nabla K(\zeta)(x-\hat{x})\nabla w_\varepsilon) & \text{in} \quad \Omega \\ \widetilde{\theta}_\varepsilon & = -w_\varepsilon & \text{on} \quad \Gamma_D \\ K\nabla\widetilde{\theta}_\varepsilon \cdot n & = K\nabla w_\varepsilon \cdot n & \text{on} \quad \Gamma_N \\ [\![\widetilde{\theta}_\varepsilon]\!] & = 0 & \text{on} \quad \partial B_\varepsilon \\ [\![\gamma_\varepsilon K\nabla\widetilde{\theta}_\varepsilon]\!] \cdot n & = -\varepsilon [\![\gamma_\varepsilon (\nabla K(\zeta)\, n)(\nabla\theta(\hat{x}) + \nabla w_\varepsilon)]\!] \cdot n & \text{on} \quad \partial B_\varepsilon \end{array}\right. , \tag{23}$$

which yields the following estimate $\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^2$, with the constant C independent of ε (see A). Then, by introducing the first order Pólya-Szegö polarization tensor P [1, 17], the solution of the exterior problem (22) can be written as:

$$w_\varepsilon(x)|_{\mathbb{R}^2\setminus\overline{B_\varepsilon(\hat{x})}} = \frac{\varepsilon^2}{\|x-\hat{x}\|^2} P\nabla\theta(\hat{x})\cdot(x-\hat{x}) \,, \tag{24}$$

$$w_\varepsilon(x)|_{B_\varepsilon(\hat{x})} = P\nabla\theta(\hat{x})\cdot(x-\hat{x}) \,. \tag{25}$$

Taking into account (21), the difference between the shape functionals (20) reads

$$\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi_\Omega) = -\frac{1}{2}(1-\gamma)\int_{B_\varepsilon} K(\nabla\theta + \nabla w_\varepsilon)\cdot\nabla\theta + \mathcal{E}(\varepsilon), \tag{26}$$

where the term $\mathcal{E}(\varepsilon)$ is given by

$$\mathcal{E}(\varepsilon) = -\frac{1}{2}(1-\gamma)\int_{B_\varepsilon} K\nabla\widetilde{\theta}_\varepsilon\cdot\nabla\theta, \tag{27}$$

which has the following estimate $\mathcal{E}(\varepsilon) = o(\varepsilon^2)$ as shown in the A. Next, by using the interior elliptic regularity of the function θ in B_ε, the difference of the shape functionals (26) satisfies the following identity:

$$\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi_\Omega) = -\frac{1}{2}(1-\gamma)\int_{B_\varepsilon} K(\hat{x})\,(\nabla\theta(\hat{x}) + \nabla w_\varepsilon)\cdot\nabla\theta(\hat{x}) + o(\varepsilon^2), \tag{28}$$

where the expansion of the tensor $K(x)$ has been used again.

With the results (21), (24) and (25) in hand, the above expression can be analytically solved, leading to

$$\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi_\Omega) = \pi\varepsilon^2 R^\top T R\nabla\theta(\hat{x})\cdot\nabla\theta(\hat{x}) + o(\varepsilon^2), \tag{29}$$

where R is the rotation matrix that diagonalizes the conductivity tensor K at the point $\hat{x}$, the matrix T is given by

$$T = -\sqrt{\det(\widetilde{K}(\hat{x}))}\,\widetilde{K}(\hat{x})S(\hat{x}), \tag{30}$$

with $\widetilde{K}(\hat{x})$ the diagonalized conductivity tensor at the point $\hat{x}$, with eigenvalues k_1 and k_2, and the matrix $S(\hat{x})$ depending on the coefficients $\alpha = 1/\sqrt{k_1}$ and $\beta = 1/\sqrt{k_2}$, that is

$$S(\hat{x}) = \frac{1}{2}(1-\gamma)\alpha\beta\begin{pmatrix} \frac{\alpha+\beta}{\alpha+\gamma\beta} & 0 \\ 0 & \frac{\alpha+\beta}{\beta+\gamma\alpha} \end{pmatrix}. \tag{31}$$

Finally, using the definition for the topological derivative (4) and taking $f(\varepsilon) = |B_\varepsilon| = \pi\varepsilon^2$, the topological derivative for the problem under consideration is given explicitly by

$$D_T\psi(\hat{x}) = R^\top T R\nabla\theta(\hat{x})\cdot\nabla\theta(\hat{x}) \qquad \forall\hat{x}\in\Omega. \tag{32}$$

Note that in the topological derivative formula, the constitutive properties and the gradient of the field θ are evaluated at an arbitrary point $\hat{x}$ of the original domain. This means that in order to evaluate the topological derivative (32), we need to solve the unperturbed anisotropic and heterogeneous heat diffusion problem (6) and determine the bases (eigenvectors) in which the conductivity tensor is diagonal. In others words, the rotation matrix R and eigenvalues k_1 and k_2 should be determined for each point of the domain.

Remark 3.1. *From the final expression for the topological derivative associated to the anisotropic and heterogeneous heat diffusion problem (32), we can analyze the limits cases of the parameter γ, which are:*

- *ideal thermal insulator ($\gamma \to 0$):*

$$S(\hat{x}) = \frac{1}{2}\alpha\beta\begin{pmatrix} \frac{\alpha+\beta}{\alpha} & 0 \\ 0 & \frac{\alpha+\beta}{\beta} \end{pmatrix} \qquad \forall\hat{x}\in\Omega\,, \tag{33}$$

- *ideal thermal conductor ($\gamma \to \infty$):*

$$S(\hat{x}) = -\frac{1}{2}\alpha\beta \begin{pmatrix} \frac{\alpha+\beta}{\beta} & 0 \\ 0 & \frac{\alpha+\beta}{\alpha} \end{pmatrix} \qquad \forall \hat{x} \in \Omega\,. \tag{34}$$

Remark 3.2. *It is interesting to observe that for an homogeneous orthotropic or isotropic material behavior, we have $R = I$. In addition, in the second case we have $k_1 = k_2 = k$. Then, the final expressions for the topological derivative (32) degenerates to:*

- *orthotropic material behavior* [10]*:*

$$D_T\psi(\hat{x}) = -\sqrt{\det(\widetilde{K})}\widetilde{K}S\,\nabla\theta(\hat{x})\cdot\nabla\theta(\hat{x}) \qquad \forall \hat{x} \in \Omega\,, \tag{35}$$

- *isotropic material behavior* [2]*:*

$$D_T\psi(\hat{x}) = -k\frac{1-\gamma}{1+\gamma}\nabla\theta(\hat{x})\cdot\nabla\theta(\hat{x}) \qquad \forall \hat{x} \in \Omega\,. \tag{36}$$

4. Numerical example

To illustrate the applicability of expression (32) in the context of topology optimization, in this section we present an example considering different heterogeneous and anisotropic conductivity tensors $K(x)$. To this end we use the topology optimization algorithm proposed by [3]. In this example we consider a square domain $\Omega = (0, 1.0) \times (0, 1.0)$, subjected to Neumann data $\overline{q} = 1.0$ on Γ_{N_1} and Γ_{N_2} and homogeneous Dirichlet data on Γ_{D_1} and Γ_{D_2}. The remainder parts of the boundary remain isolated. The domain and boundary conditions for this example are shown in Fig. 3, where $a = 0.2$. The volume constraint is chosen to be 80% of the initial volume.

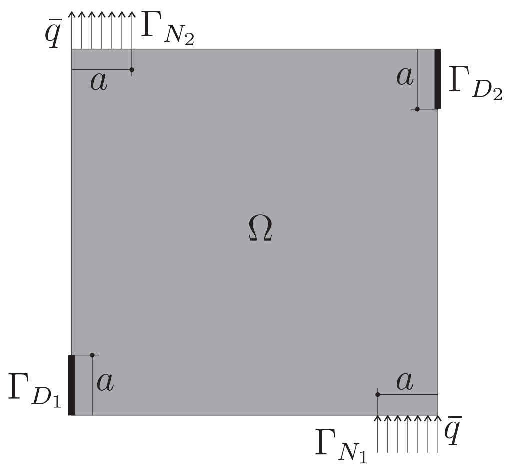


Figure 3. Domain of the numerical example.

In this example, we explore the influence of the heterogeneity and anisotropy of the conductivity tensor $K(x)$ in the numerical results. We also present the obtained results for the homogeneous case. The conductivity matrix $K(x)$ is constructed as:

$$K(x) = K_i p_j(x) \qquad \text{with } i \in [a, b, c] \text{ and } j \in [a, b, c, d]\,, \tag{37}$$

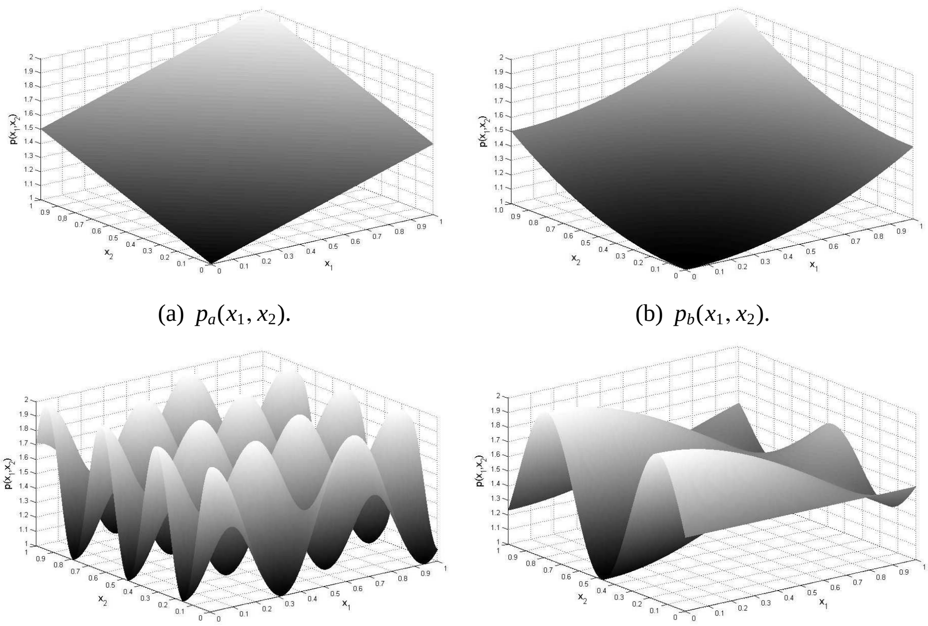
where K_i is a constant matrix and $p_i(x)$ are smooth functions that depend on the coordinate system $x = (x_1, x_2)$. In particular, four different functions $p_i(x)$ are introduced, see Fig. 4, which are:

$$\begin{aligned} p_a(x_1, x_2) &= \frac{1}{2}(x_1 + x_2) + 1; && (38) \\ p_b(x_1, x_2) &= \frac{1}{2}\left(x_1^2 + x_2^2\right) + 1; && (39) \\ p_c(x_1, x_2) &= \frac{1}{2}\cos(10x_1)\cos(20x_2) + \frac{3}{2}; && (40) \\ p_d(x_1, x_2) &= \frac{1}{2}\cos(2x_1)\sin(10x_2) + \frac{3}{2}. && (41) \end{aligned}$$

The constant matrixes K_i used in the numerical experiments are given by

$$K_a = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}; \quad K_b = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 2.0 \end{bmatrix}; \quad K_c = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 3.0 \end{bmatrix}, \tag{42}$$

and the parameter γ is fixed as 0.001.



(a) $p_a(x_1, x_2)$. (b) $p_b(x_1, x_2)$.

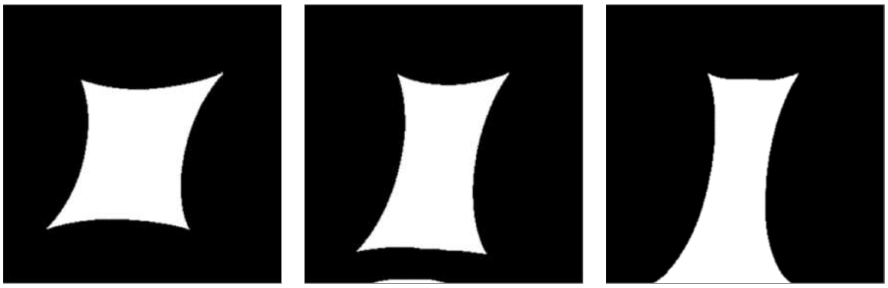
(c) $p_c(x_1, x_2)$. (d) $p_d(x_1, x_2)$.

FIGURE 4. Non-homogeneity profile $p_i(x)$.

In Figs. 5 to 9 we show the obtained topologies for the homogeneous and nonhomogeneous anisotropic material properties previously presented. According to Fig. 5, the results for the homogeneous case are qualitatively similar between them. This seems to indicate that the given anisotropy in the conductivity tensor does not affect the optimal topology, at least for this benchmark example. However, in the other cases the optimal topologies are strongly dependent on the heterogeneity profiles $p_i(x)$. In fact, in some of the obtained results, the topologies change drastically when the anisotropy of the tensor K becomes stronger, as can be seen in Figs. 6 – 9.

(a) Conductivity tensor K_a. (b) Conductivity tensor K_b. (c) Conductivity tensor K_c.

FIGURE 5. Optimized topologies for anisotropic homogeneous case.



(a) Conductivity tensor K_a. (b) Conductivity tensor K_b. (c) Conductivity tensor K_c.

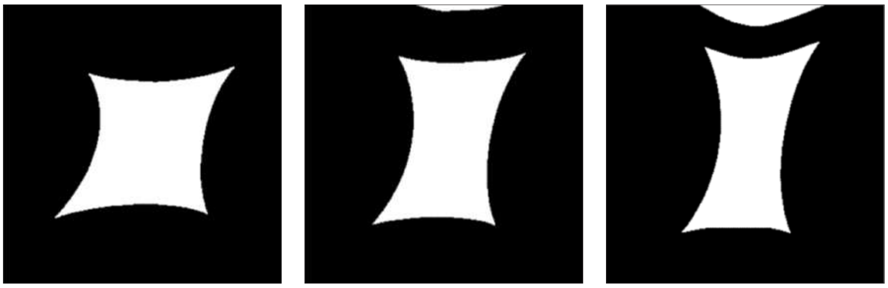
FIGURE 6. Optimized topologies for anisotropic nonhomogeneous case $p_a(x_1, x_2)$.

(a) Conductivity tensor K_a. (b) Conductivity tensor K_b. (c) Conductivity tensor K_c.

FIGURE 7. Optimized topologies for anisotropic nonhomogeneous case $p_b(x_1, x_2)$.

(a) Conductivity tensor K_a. (b) Conductivity tensor K_b. (c) Conductivity tensor K_c.

FIGURE 8. Optimized topologies for anisotropic nonhomogeneous case $p_c(x_1, x_2)$.



(a) Conductivity tensor K_a. (b) Conductivity tensor K_b. (c) Conductivity tensor K_c.

FIGURE 9. Optimized topologies for anisotropic nonhomogeneous case $p_d(x_1, x_2)$.

5. Final remarks

An analytical expression for the topological derivative of the total potential energy associated to an anisotropic and heterogeneous heat diffusion problem, when a circular inclusion of the same nature as the bulk material is introduced at an arbitrary point of the domain, has been derived. From the asymptotic analysis, it was proved that the heterogeneous behavior of the material properties does not contribute to the first order topological derivative. The final formula is a general simple analytical expression in terms of the solution of the state equation and the constitutive parameters evaluated at each point of the unperturbed domain. In fact, from the obtained result, the classical expression for the topological derivative for orthotropic and isotropic constitutive properties has been derived as particular cases. Finally, we remark that this information can be potentially used, as shown in the numerical example, in a number of applications of practical interest such as, for instance, inverse problem, image restoration, design and optimization of mechanical, thermal or electronic devices designed to achieve a specified behavior. In particular, the constitutive behavior of piezoresistive membranes, under a deformation process, becomes highly anisotropic and heterogeneous. Therefore, the obtained result can be directly applied in the topology design of such membranes.

6. Acknowledgments

This research was partly supported by CICAL program from the Ministry of Science of the Province of Cordoba and the National Agency for Scientific Promotion, Argentina, and by CNPq (Brazilian Research Council) and FAPERJ (Research Foundation of the State of Rio de Janeiro) under grants 470597/2010-0 and E-26/102.204/2009, respectively. All of this support is gratefully acknowledged.

Appendix A. Estimation of the remainders

In this section we proceed with the estimation of the remainders in the topological asymptotic expansion used in the derivation of the topological derivative expression (32) presented in Section 3. In particular, we study the asymptotic behavior of the remainder $\widetilde{\theta}_\varepsilon$ in (23) and the residue $\mathcal{E}(\varepsilon)$ defined in (27). Let us start introducing the following auxiliary functions:

Definition 1. *Let w_ε and θ be solutions to (22) and (6), respectively. Then, we introduce the following functions independent of the small parameter ε:*

$$g_1 = (1-\gamma)(\nabla K(\zeta) n)\nabla\theta(\hat{x})\cdot n \quad on \quad \partial B_\varepsilon,\ \forall\zeta\in(x,\hat{x}), \tag{43}$$

$$g_2 = -\varepsilon^{-2} K(\hat{x})\nabla w_\varepsilon(x)\cdot n \quad on \quad \Gamma_N, \tag{44}$$

$$g_3 = -\varepsilon^{-2} w_\varepsilon \quad on \quad \Gamma_D, \tag{45}$$

$$g_4 = \varepsilon^{-2}\gamma_\varepsilon \nabla K(\zeta)(x-\hat{x})\nabla w_\varepsilon \quad in \quad \Omega,\ \forall\zeta\in(x,\hat{x}), \tag{46}$$

$$g_5 = \varepsilon^{-2}\nabla K(\zeta)(x-\hat{x})\nabla w_\varepsilon\cdot n \quad on \quad \Gamma_D. \tag{47}$$

Lemma 1. *Let us consider the functions g_i, for $i = 1, ..., 5$, given in Definition 1 and let $\widetilde{\theta}_\varepsilon$ be a solution to (23) or equivalently a solution to the following variational problem: Find $\widetilde{\theta}_\varepsilon \in \widetilde{\mathcal{U}}_\varepsilon$, such that*

$$\int_\Omega \gamma_\varepsilon K \nabla \widetilde{\theta}_\varepsilon \cdot \nabla \eta = -\varepsilon^2 \int_\Omega g_4 \cdot \nabla \eta + \varepsilon^2 \int_{\Gamma_N} g_2 \eta + \varepsilon \int_{\partial B_\varepsilon} g_1 \eta \qquad \forall \eta \in \widetilde{\mathcal{V}}_\varepsilon \ , \tag{48}$$

where the set $\widetilde{\mathcal{U}}_\varepsilon$ and the space $\widetilde{\mathcal{V}}_\varepsilon$ are defined as

$$\widetilde{\mathcal{U}}_\varepsilon \ := \ \left\{ \phi \in H^1(\Omega) : [\![\phi]\!] = 0 \ on \ \partial B_\varepsilon, \phi = \varepsilon^2 g_3 \ on \ \Gamma_D \right\}, \tag{49}$$

$$\widetilde{\mathcal{V}}_\varepsilon \ := \ \left\{ \phi \in H^1(\Omega) : [\![\phi]\!] = 0 \ on \ \partial B_\varepsilon, \phi = 0 \ on \ \Gamma_D \right\}. \tag{50}$$

Then, for the tensor K smooth enough, we have the following estimate for the solution to (48):

$$\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^2, \tag{51}$$

with constant C independent of ε.

Proof. By taking $\eta = \widetilde{\theta}_\varepsilon - \varphi_\varepsilon$ in (48), where φ_ε is the lifting of the Dirichlet boundary data $\varepsilon^2 g_3$ on Γ_D, and after performing an integration by parts, we have

$$\begin{aligned} \int_\Omega \gamma_\varepsilon K \nabla \widetilde{\theta}_\varepsilon \cdot \nabla \widetilde{\theta}_\varepsilon \ &= \ -\varepsilon^2 \int_\Omega g_4 \cdot \nabla \widetilde{\theta}_\varepsilon + \varepsilon^2 \int_{\Gamma_N} g_2 \widetilde{\theta}_\varepsilon + \varepsilon \int_{\partial B_\varepsilon} g_1 \widetilde{\theta}_\varepsilon \\ &\quad + \varepsilon^2 \int_{\Gamma_D} (K \nabla \widetilde{\theta}_\varepsilon \cdot n) g_3 + \varepsilon^2 \int_{\Gamma_D} g_5 \widetilde{\theta}_\varepsilon, \end{aligned} \tag{52}$$

where we have considered the restriction of the function $\widetilde{\theta}_\varepsilon$ on the boundary Γ_D. From the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int_\Omega \gamma_\varepsilon K \nabla \widetilde{\theta}_\varepsilon \cdot \nabla \widetilde{\theta}_\varepsilon \ &\leq \ \varepsilon^2 \|g_4\|_{L^2(\Omega)} \|\nabla \widetilde{\theta}_\varepsilon\|_{L^2(\Omega)} + \varepsilon^2 \|g_2\|_{H^{-1/2}(\Gamma_N)} \|\widetilde{\theta}_\varepsilon\|_{H^{1/2}(\Gamma_N)} \\ &+ \ \varepsilon \|g_1\|_{H^{-1/2}(\partial B_\varepsilon)} \|\widetilde{\theta}_\varepsilon\|_{H^{1/2}(\partial B_\varepsilon)} + \varepsilon^2 \|g_3\|_{H^{1/2}(\Gamma_D)} \|K \nabla \widetilde{\theta}_\varepsilon \cdot n\|_{H^{-1/2}(\Gamma_D)} \\ &+ \ \varepsilon^2 \|g_5\|_{H^{-1/2}(\Gamma_D)} \|\widetilde{\theta}_\varepsilon\|_{H^{1/2}(\Gamma_D)}. \end{aligned} \tag{53}$$

Taking into account the trace theorem, we have

$$\begin{aligned} \int_\Omega \gamma_\varepsilon K \nabla \widetilde{\theta}_\varepsilon \cdot \nabla \widetilde{\theta}_\varepsilon \ &\leq \ \varepsilon^2 C_1 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_2 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon \|g_1\|_{L^2(B_\varepsilon)} \|\widetilde{\theta}_\varepsilon\|_{H^1(B_\varepsilon)} \\ &+ \ \varepsilon^2 C_3 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_4 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)}, \\ &\leq \ \varepsilon^2 C_1 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_2 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_5 \|\widetilde{\theta}_\varepsilon\|_{H^1(B_\varepsilon)} \\ &+ \ \varepsilon^2 C_3 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} + \varepsilon^2 C_4 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)}, \\ &\leq \ \varepsilon^2 C_6 \|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)}, \end{aligned} \tag{54}$$

where we have used the interior elliptic regularity of the function θ and the regularity of the tensor K. Next, from coercivity of the bilinear form on the left-hand side of (48), we have

$$\int_\Omega \gamma_\varepsilon K \nabla \widetilde{\theta}_\varepsilon \cdot \nabla \widetilde{\theta}_\varepsilon \geq c \|\widetilde{\theta}_\varepsilon\|^2_{H^1(\Omega)}. \tag{55}$$

Finally, from (54) and (55), we obtain

$$\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^2, \tag{56}$$

which leads to the result, with $C = C_6/c$ independent of ε. □

Lemma 2. *Let $\widetilde{\theta}_\varepsilon$ and θ be solutions to (23) and (6), respectively. Then, we have the following estimate for the remainder $\mathcal{E}(\varepsilon)$ in (27):*

$$\frac{1}{2}(1-\gamma) \int_{B_\varepsilon} K \nabla \widetilde{\theta}_\varepsilon(x) \cdot \nabla \theta = o(\varepsilon^2). \tag{57}$$

Proof. From the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \mathcal{E}(\varepsilon) &= -\frac{1}{2}(1-\gamma)\int_{B_\varepsilon} K\nabla\widetilde{\theta}_\varepsilon(x)\cdot\nabla\theta \\ &\leq C_1\|\nabla\theta\|_{L^2(B_\varepsilon)}\|\nabla\widetilde{\theta}_\varepsilon\|_{L^2(B_\varepsilon)} \\ &\leq \varepsilon C_2\|\nabla\widetilde{\theta}_\varepsilon\|_{L^2(B_\varepsilon)} \\ &\leq \varepsilon C_3\|\nabla\widetilde{\theta}_\varepsilon\|_{L^2(\Omega)} \\ &\leq \varepsilon C_4\|\widetilde{\theta}_\varepsilon\|_{H^1(\Omega)} \end{aligned} \tag{58}$$

where we have used the interior elliptic regularity of the function θ. Next, by taking into account Lemma 1, we have

$$\mathcal{E}(\varepsilon) \leq \varepsilon^3 C\ , \tag{59}$$

which leads to the result. □

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