SHAPE SENSITIVITY ANALYSIS OF A QUASI-ELETROSTATIC PIEZOELECTRIC SYSTEM IN MULTILAYERED MEDIA

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ABSTRACT. The optimization of shape and topology of piezo-patches or layered piezo-electrical material attached to structural parts, like elastic bodies, plates and shells plays a major role in the design of smart structures, as piezo-mechanic-acoustic devices in loudspeakers or energy harvesters. While the design for time-harmonic motions is genuinely frequency-dependent, as has been reported in the literature in the context of density optimizaiton with the SIMP-method, time-varying piezoelectric material has not been investigated with respect to optimal design so far. Therefore, shape sensitivities for layered piezoelectric material and time-varying loads and charges are derived in this paper. In particular, we provide the shape-derivatives for nested piezo-layers associated to a class of shape functional. More general layers can be dealt with similarly.

1. INTRODUCTION

Piezo-electrical materials play an important role in sensor and actuator devices used in smart-materials. There are naturally occurring piezo-materials, typically revealing rather weak piezoelectric effects usable for sensor applications, and those synthetically manufactured, like ceramics, which exhibit high coupling effects and are therefore important in actuator devices. The spectrum of applications is becoming broader in recent years. In particular small loudspeakers, c-muts and piezoelectric harvesters contain layers of piezoelectric material. Most of the recent application of piezoelectric sensor- and actuator devices are on a small scale and require a minimum of such material while maximizing its effect. Because of such restrictions on weight and the cost of the material with respect to a particular performance of the desired device, the piezo-layers should be optimized with respect to their shape and their topology.

The mechanical properties of piezoelectric material are well understood. The literature is vast and therefore we refrain from attempting an appropriate account of publications. Let us mention instead surveys as [14, 15]. While well-posedness of the static equations has been handled in many publications, the full dynamic equations of piezoelectricity and also multi-component piezo-structures have been studied in e.g. [5, 16, 17, 23]. Again the list far from being complete.

When it comes to optimization in the context of piezoelectricity the literature is sparce. Topology optimization of piezo-patches has been considered by Silva and Kikuchi [20] and Kögel and Silva [12]. Coupling of piezoelectric patches and elastic material has been treated in the context of SIMP-optimization in Wein et al. [24, 25]. In particular in [26] a piezo-patch was considered as being glued to a 3-d elastic body. The piezo-patch was subject to a frequency input of a defined frequency ω and the maximal displacement of the elastic body was calculated. See Figure 1 for the set-up.

The question raised in [24, 25, 26] was concern as to whether the topology of the piezo-patch coupled to the elastic plate could be optimized with respect to a given cost-function, like maximal displacement at a given point, by using the material interpolation method SIMP [1, 20]. This question was considered under the assumption of time-harmonic motions, that is to say, based on a Helmholtz-like static model. The optimization was done for a given frequency or a given frequency band. A partial result can be seen in figures 2,3.

Sweeping over a frequency range, the overall behavior has been tested numerically. See Figure 4.

It is obvious that the optimal topology obtained this way is frequency dependent. Moreover, the SIMP-method as such provides optimal density distributions rather than shapes or true 0-1 designs. The question of optimal shapes of piezo-layers and their 0-1 topology designs was left open. Even though robust topology optimization with respect to frequency bands has been achieved using min-max-SIMP optimization, the question of optimal shape and topology for time-dependent problems, was also left open.

Key words and phrases. piezoelectricity, electromechanical interaction, shape sensitivity analysis.

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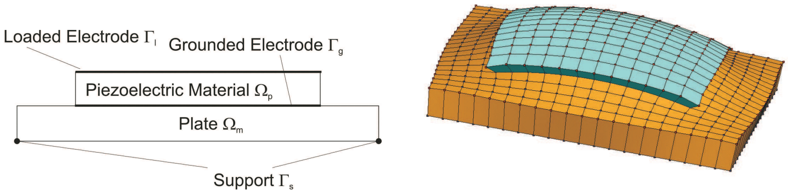
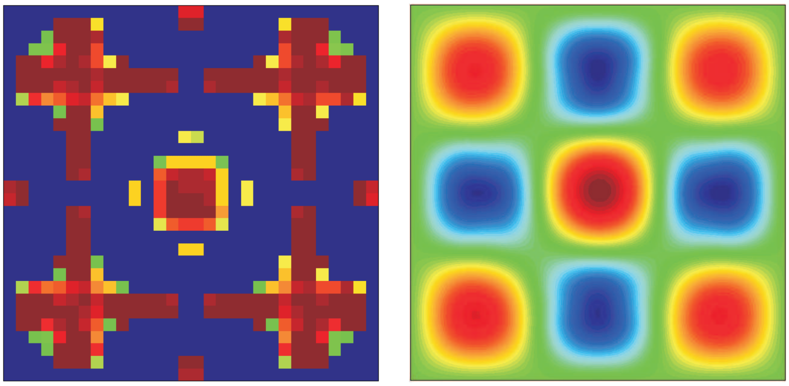


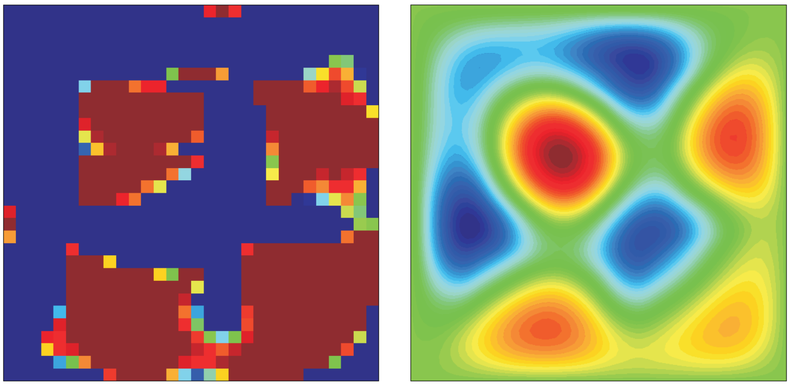
(a) an optimized topology (b) displacement

FIGURE 1. [26] set-up and corresponding displacement



(a) an optimized topology (b) displacement

FIGURE 2. [26] SIMP optimized topology and corresponding displacement



(a) an optimized topology (b) displacement

FIGURE 3. [26]SIMP optimized topology and corresponding displacement

We recall that shape derivatives obtained for solutions of boundary value problems lead to the shape gradients of the associated functionals. By the structure theorem for such a shape gradient of a shape differentiable functional, it follows that it is given by a distribution supported on the moving boundary. On the other hand, we require the shape gradient given by a function in order to apply the level-set strategy for numerical solution of the associated shape optimisation problem. This issue is also addressed in the paper, and the appropriate regularity of solutions to boundary value problems is discuss to guarantee the required regularity of shape gradients.

The more general boundary perturbations which are called singular domain perturbations cannot be directly analysed by the speed method. However, such boundary perturbations as well as the associated topological derivatives can be used in the numerical procedure in order to change the domain topology, by creating small voids, or adding small rigid inclusions in an elastic body. The analysis of singular domain perturbations is

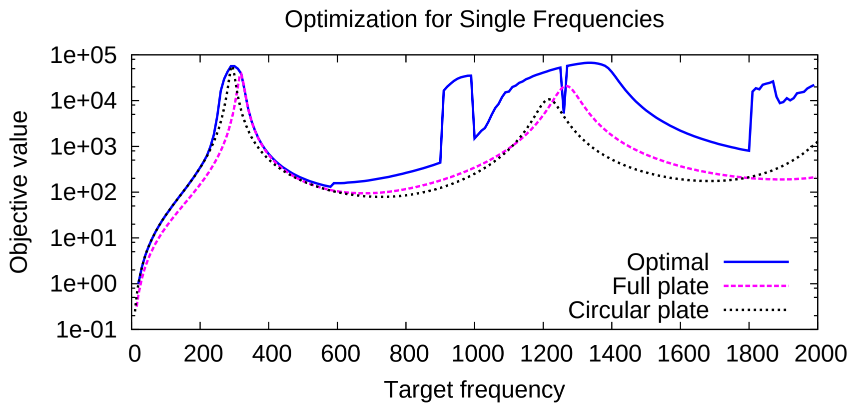


FIGURE 4. [26] a frequency sweep and gain

performed by means of matched and compound asymptotic expansions, with the estimates of asymptotic remainders in weighted spaces [2]. We point out that the topological gradients of shape functionals can be obtained in the form of singular limits of shape derivatives when the radius of a small void is going to zero. This observation indicates that the topological gradients are in fact of the same nature as the shape derivatives, even if the derivation procedure is more involved from the mathematical point of view. Another observation on this aspect of shape sensitivity analysis is that in the elliptic case the topological gradient are in fact continuous with respect to contrast, which turns out to be an easy way of the derivation. First, we consider the regular perturbations of the coefficient of the elliptic operator by adding a small inclusion, then perform the limit passage in the resulting topological derivative with respect to the contrast, i.e. the coefficient which transforms the elastic inclusion into a void or into a rigid inclusion. Unfortunately, this argument does not work for evolution problems, since the topological derivatives obtained for regular perturbations cannot furnish by such a limit passage the topological derivatives for singular perturbations. On the other hand, this passage is possible for time harmonic regime, if the frequency is fixed.

It is this set-up that we want to further investigate in this paper. In particular, once a topology optimization step is performed, either using topological gradients (see [2]) or material interpolation (e.g. by SIMP [26]) one may use shape optimization and the level-set method in order to promote the optimal shape using the speed-method. To this end one needs the shape-gradient for piezo-electric material. In this paper we go a step further, and ask for such gradients when the fully dynamic problem is considered. We consider the shape optimization problem of minimizing the shape functional $\mathcal{J}_\Omega(u,q)$ for the coupled fields: elastic displacement u and electric potential q. The model, for a given shape Ω, is given by a coupled electromechanical system. For the problem we establish:

- the existence of an optimal shape;
- the first order optimality conditions.

Therefore, we need to perform the shape sensitivity analysis of solutions to the model. We also need to determine the shape gradients and their densities for the associated shape functionals. To this end the speed method is used (see [21]).

2. THE PROBLEM FORMULATION

Let us consider an open bounded domain Ω of $\mathbb{R}^3$ with smooth boundary $\partial\Omega = S$. We assume that Ω has the form $\Omega = \mathcal{B}_0 \diagdown \overline{\mathcal{B}_1}$, where $\mathcal{B}_0$ and $\mathcal{B}_1$ are open bounded domains with $\overline{\mathcal{B}_1} \subset \mathcal{B}_0$, with $\overline{(\cdot)}$ used to denote the closure of $(\cdot)$. In addition, $\partial\mathcal{B}_0 = S_0$ and $\partial\mathcal{B}_1 = S_1$, thus $S = S_0 \cup S_1$. Let $m > 1$ be a given integer. For each i with $1 \le i \le m$ let $\mathcal{D}_i$ be and open subset with smooth boundary Γ_i and such that $\overline{\mathcal{B}_1} \subset \mathcal{D}_i \subset \mathcal{B}_0$, $\overline{\mathcal{D}_i} \subset \mathcal{D}_{i+1}$. We set $\Omega_0 = \mathcal{D}_1 \diagdown \overline{\mathcal{B}_1}$, $\Omega_i = \mathcal{D}_{i+1} \diagdown \overline{\mathcal{D}_i}$ for $1 \le i \le m-1$ and $\Omega_m = \mathcal{B}_0 \diagdown \overline{\mathcal{D}_m}$. In summary, as shown in fig. 5, we have $\Omega = \cup_{i=0}^m \Omega_i$, such that $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$, with boundaries $\partial\Omega = S_0 \cup S_1$, $\partial\Omega_0 = S_1 \cup \Gamma_1$, $\partial\Omega_i = \Gamma_i \cup \Gamma_{i+1}$ for $i = 1, ..., m-1$, and $\partial\Omega_m = \Gamma_m \cup S_0$.

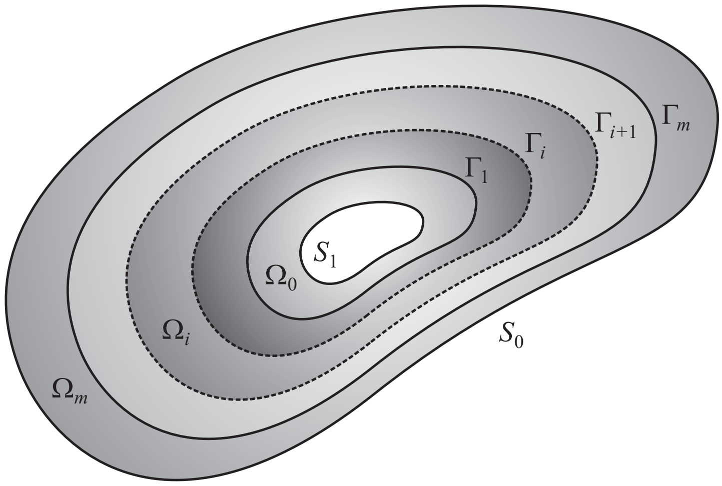


FIGURE 5. Domain Ω with boundary $\partial\Omega = S_0 \cup S_1$.

2.1. **The strong system.** Let $u : \Omega \times [0,T] \to \mathbb{R}^3$ be such that $u(x,t)$ is the displacement of the body Ω in $x \in \Omega$ at the time $t \in [0,T]$. Let moreover $q : \Omega \times [0,T]$ be such that $q(x,t)$ is the electric potential at $x \in \Omega$ and $t \in [0,T]$. We define $\nabla^s u := \frac{1}{2}(\nabla u + \nabla u^T)$. The electromechanical interaction phenomenon is modeled by the following coupled system [9]

$$\left\{ \begin{array}{rcl} u_{tt} - \mathrm{div}\sigma & = & F \\ -\mathrm{div}\varphi & = & G \end{array} \right. \quad \text{in} \quad \Omega \times (0,T) \ , \tag{2.1}$$

where σ is the mechanical stress tensor and φ the electrical displacement field, F is a given load and G a given field. The material law describing the piezoelectric effect in the linearized case of small mechanical deformations and electric fields reads as

$$\left\{ \begin{array}{rcl} \sigma(u,q) & = & C\varepsilon(u) - Pe(q) \ , \\ \varphi(u,q) & = & P^T\varepsilon(u) + De(q) \ , \end{array} \right. \tag{2.2}$$

where C is the elasticity fourth-order tensor, P the piezoelectric coupling third-order tensor and D the dielectric second-order tensor. As usual C, D satisfy the symmetry conditions $C_{ijkl} = C_{jikl} = C_{klij}$ and $D_{ij} = D_{ji}$, whereas P satisfies $P_{ijk} = P_{jik}$. Furthermore, there exist nonnegative constants c_0, d_0 such that

$$C_{ijkl}X_{ij}X_{kl} \geq c_0 X_{ij}^2, \quad D_{ij}y_iy_j \geq d_0 y_i^2,$$

where Einstein's summation convention is used. It is assumed for simplicity that all tensors are piecewise constant, i.e., constant in each layer. In addition, the mechanical strain tensor ε and the electric vector field e are given by

$$\varepsilon(u) = \nabla^s u \quad \text{and} \quad e(q) = -\nabla q \ . \tag{2.3}$$

We also associate with system (2.1) the following given initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \ , \tag{2.4}$$

and boundary conditions of the form

$$\left\{ \begin{array}{rcl} \sigma n & = & \Sigma \\ q & = & 0 \end{array} \right. \text{on } S_0 \times (0,T) \quad \text{and} \quad \left\{ \begin{array}{rcl} \varphi \cdot n & = & \Phi \\ u & = & 0 \end{array} \right. \text{on } S_1 \times (0,T) \ , \tag{2.5}$$

where n is the outward unit normal vector pointing toward the exterior of Ω. It is apparent that the inhomogeneous Dirichlet conditions can be shifted into the right hand sides of (2.1) and be incorporated to F and G. Thus, without loss of generality, we may consider homogenous Dirichlet boundary conditions in (2.5), namely $q = 0$ on $S_0 \times (0,T)$, in the sequel. Finally, we consider the following transmission conditions

$$\left\{ \begin{array}{rcl} [\![\sigma]\!]n & = & 0 \\ [\![u]\!] & = & 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} [\![\varphi]\!] \cdot n & = & 0 \\ [\![q]\!] & = & 0 \end{array} \right. , \tag{2.6}$$

where, for any $(x,t) \in \Gamma_i \times (0,T)$, $i = 1,2,...,m$, the symbol $[\![(\cdot)]\!]$ is used to denote the jump between quantities evaluated on the boundary Γ_i of each pair Ω_{i-1} and Ω_i, that is

$$[\![(\cdot)]\!] = (\cdot)^{(i)} - (\cdot)^{(i-1)} \ , \tag{2.7}$$

and $n = n^{(i)} = -n^{(i-1)}$ is the unit normal vector pointing toward the exterior of Ω_i.

Remark 1. *Notice that system* (2.1) *can be derived from the fully coupled dynamic equations governing the elasto-dynamic system and the Maxwell system with constitutive relations* (2.2)*:*

$$\begin{cases} u_{tt} - \mathrm{div}\sigma(u,E) = 0 \\ E_t = \mathrm{rot}H \\ \mu_0 H_t = -\mathrm{rot}E \end{cases} \tag{2.8}$$

Without loss of generality, we can assume the permeability $\mu_0 = 1$, $E(q)$ is regular enough such that the second equation can be differentiated with respect to time, the third then being inserted and finally the div *operator being applied. This deletes the term* $-\mathrm{rot}(\mathrm{rot}E)$. *After that one can integrate twice with respect time and obtain the second equation in* (2.1). *From this setting one also obtains various important simplifications such as transverse isotropic material, where the equations become much simpler. The full system* (2.8) *and its shape-sensitivity is subject to current research. We note that Nicaise* [18] *has treated well-posedness for a similar system using semi-group theory. See also* [10, 11].

Remark 2. *We may introduce time-harmonics upon introducing*

$$u(x,t) =: e^{-\omega t}\hat{u}(x), \quad q(x,t) =: e^{-i\omega t}\hat{q}(x).$$

Then the system (2.1) *reads as follows*

$$\begin{cases} \omega^2\hat{u} + \mathrm{div}\hat{\sigma} = 0 & \textit{in } \Omega \\ \mathrm{div}\hat{\varphi} = 0 & \textit{in } \Omega \end{cases} \tag{2.9}$$

where $\hat{\sigma} = \hat{\sigma}(\hat{u},\hat{q}) = C\varepsilon(\hat{u}) - Pe(\hat{q})$, $\hat{\varphi}$ accordingly. System (2.9) *can be considered as a Helmhotz-type system* [19]. *We associate to* (2.9) *the boundary and transmission conditions* (2.7),(2.6). *See e.g. Mercier and Nicaise* [17] *for well-posedness. The SIMP-based topology optimization in* [25, 24, 26] *has been based on such time-harmonic models. Moreover, for $\omega = 0$* (2.9) *reduces to a problem that has been discussed in Cardone et al.*[2] *also with respect to topological sensitivities.*

2.2. **The weak system.** The weak formulation of the piezoelectric problem reads as follow. Given the initial conditions (2.4), find for each $t \in (0,T)$ the displacement $u \in \mathcal{W}_M(\Omega)$ and the electric potential $q \in \mathcal{W}_E(\Omega)$, such that

$$\begin{cases} \langle u_{tt},\eta\rangle_\Omega + a_\Omega^{MM}(u,\eta) + a_\Omega^{EM}(q,\eta) &= \langle F,\eta\rangle_\Omega + \langle \Sigma,\eta\rangle_{S_0} \quad \forall \eta \in \mathcal{W}_M(\Omega) \\ a_\Omega^{EE}(q,\xi) - a_\Omega^{ME}(u,\xi) &= \langle G,\xi\rangle_\Omega + \langle \Phi,\xi\rangle_{S_1} \quad \forall \xi \in \mathcal{W}_E(\Omega) \end{cases} , \tag{2.10}$$

where

$$\langle u_{tt},\eta\rangle_\Omega = \int_\Omega u_{tt}\cdot\eta \ , \tag{2.11}$$

$$a_\Omega^{MM}(u,\eta) = \int_\Omega C\nabla^s u\cdot\nabla^s\eta \quad \text{and} \quad a_\Omega^{EM}(q,\eta) = \int_\Omega P\nabla q\cdot\nabla^s\eta \ , \tag{2.12}$$

$$a_\Omega^{EE}(q,\xi) = \int_\Omega D\nabla q\cdot\nabla\xi \quad \text{and} \quad a_\Omega^{ME}(u,\xi) = \int_\Omega P^T\nabla^s u\cdot\nabla\xi \ , \tag{2.13}$$

with $a_\Omega^{EM}(q,u) = a_\Omega^{ME}(u,q)$ and $\nabla := \partial/\partial x$. In addition, the spaces $\mathcal{W}_M(\Omega)$ and $\mathcal{W}_E(\Omega)$ are respectively defined as

$$\mathcal{W}_M(\Omega) = \{u \in [H^1(\Omega)]^3 : u|_{S_1} = 0,\ [\![u]\!]|_{\Gamma_i} = 0,\ i = 1,2,...,m\} \ , \tag{2.14}$$

$$\mathcal{W}_E(\Omega) = \{q \in H^1(\Omega) : q|_{S_0} = 0,\ [\![q]\!]|_{\Gamma_i} = 0,\ i = 1,2,...,m\} \ . \tag{2.15}$$

Theorem 3. *Let the initial data* $u(x,0) := f(x),\ u_t(x,0) := g(x)$ *satisfy*

$$f \in \mathcal{W}_M(\Omega), \quad g \in L^2(\Omega), \tag{2.16}$$

and consider distributed data

$$\begin{aligned} &F \in L^\infty(0,T;\mathcal{W}_M(\Omega)^*),\ G \in L^\infty(0,T;\mathcal{W}_E(\Omega)^*),\ G_t \in L^2(0,T;\mathcal{W}_E(\Omega)^*), \\ &\Sigma \in L^\infty(0,T;H^{\frac{1}{2}}(S_0)^*),\ \Phi \in L^\infty(0,T;H^{\frac{1}{2}}(S_1)^*),\ \Phi_t \in L^2(0,T;H^{\frac{1}{2}}(S_1)^*). \end{aligned} \tag{2.17}$$

Then there exists a unique weak solution (u,q) *to* (2.10) *such that*

$$u \in L^\infty(0,T;\mathcal{W}_M(\Omega)),\ u_t \in L^\infty(0,T;L^2(\Omega)),\ u_{tt} \in L^2(0,T;\mathcal{W}_M(\Omega)^*),\ q \in L^\infty(0,T;\mathcal{W}_E(\Omega)), \tag{2.18}$$

If in addition we assume that

$$f \in H^2(\Omega)^3,\ g \in \mathcal{W}_M(\Omega),\ \Phi(0) \in H^{\frac{1}{2}}(S_1)^*,\ \Sigma(0) \in H^{\frac{1}{2}}(S_0) \tag{2.19}$$

such that there is a $q_0 \in H^2(\Omega)$ *with* (f,q_0) *satisfying and the boundary conditions*

$$\left\{ \begin{array}{rcl} \sigma(f,q_0)n &=& \Sigma(0) \\ q_0 &=& 0 \end{array} \right. \text{ on } S_0 \times (0,T) \quad \text{and} \quad \left\{ \begin{array}{rcl} \varphi(f,q_0)\cdot n &=& \Phi(0) \\ f &=& 0 \end{array} \right. \text{ on } S_1 \times (0,T)\ , \tag{2.20}$$

together with the transmission conditions

$$\left\{ \begin{array}{rcl} [\![\sigma]\!] n &=& 0 \\ [\![u]\!] &=& 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} [\![\varphi]\!]\cdot n &=& 0 \\ [\![q]\!] &=& 0 \end{array} \right. ,\ \text{on } \Gamma_i \tag{2.21}$$

and

$$\begin{aligned} &F \in L^\infty(0,T;L^2(\Omega))\ F_t \in L^2(0,T;L^2(\Omega)),\ G \in L^\infty(0,T;\mathcal{W}_E(\Omega)^*),\ G_{tt} \in L^2(0,T;\mathcal{W}_E(\Omega)^*), \\ &\Sigma \in L^\infty(0,T;L^2(S_0))\ \Sigma_t \in L^2(0,T;L^2(S_0)),\ \Phi \in L^\infty(0,T;H^{\frac{1}{2}}(S_1)^*),\ \Phi_{tt} \in L^2(0,T;H^{\frac{1}{2}}(S_1)^*). \end{aligned} \tag{2.22}$$

Then the solution to (2.10)-(2.20)-(2.21) satisfies

$$\begin{aligned} &u \in L^\infty(0,T;H^2(\Omega)), \quad u_t \in L^\infty(0,T;\mathcal{W}_M(\Omega)), \quad u_{tt} \in L^\infty(0,T;L^2(\Omega)), \\ &q \in L^\infty(0,T;H^2(\Omega)), \quad q_t \in L^\infty(0,T;\mathcal{W}_E(\Omega))\ . \end{aligned} \tag{2.23}$$

Proof. The proof can be established by a Galerkin procedure. For weak solutions with different boundary conditions see the PhD thesis [13]. In [8] the authors consider a semigroup approach, based on a Shur-complement operator that reduces the piezoelectric system to an elliptic problem in u. As we treat different boundary conditions and also need weak *and* strong regularity of the solutions which is not revealed from [13, 8], for the sake of self-consistency, we provide the necessary a priori estimates. We consider sequences $\{\eta_j\},\ \{\xi_j\}$ in $\mathcal{W}_M(\Omega)$ and $\mathcal{W}_E(\Omega)$, respectively. Then we have the finite dimensional subspaces $\mathcal{W}_M(\Omega)^m =$ span $\{\eta,\dots,\eta_m\}$ and $\mathcal{W}_E(\Omega)^m =$ span $\{\xi_1,\dots\xi_m\}$, such that the union over all such spaces is mutually dense in $\mathcal{W}_M(\Omega)$, $\mathcal{W}_E(\Omega)$. Clearly, taking the test functions $\eta = u^m(t),\ \xi = q^m(t)$ we obtain for finite m that (2.10) has a unique solution $(u^m(t), q^m(t))$ with initial data $u^m(0) = u_0^m,\ u_t^m(0) = u_1^m$, where $u_0^m \to u(0) = f$ in $\mathcal{W}_M(\Omega)$ and $u_1^m \to u_t(0) = g$ in $L^2(\Omega)$, as $m \to \infty$. The finite dimensional system of ordinary differential equations takes the form

$$\begin{aligned} \langle u_{tt}^m(t),\eta\rangle &+ a_\Omega^{MM}(u^m(t),\eta) + a_\Omega^{EM}(q^m(t),\eta) + a_\Omega^{EE}(q^m(t),\xi) - a_\Omega^{MR}(u^m(t),\xi) \\ &= \langle F(t),\eta\rangle + \langle G(t),\xi\rangle + \langle \Sigma,\eta\rangle_{S_0} + \langle \Phi,\xi\rangle_{S_1},\ \forall(\eta,\xi) \in \mathcal{W}_M^m(\Omega)\times\mathcal{W}_E^m(\Omega) \end{aligned} \tag{2.24}$$

We then take test functions $(u_t^m(t),0)$ in (2.24) and, after differentiating with respect to time $(0,q^m(t))$, use the symmetry $a_\Omega^{ME}(u,q) - a_\Omega^{EM}(q,u) = 0$ and obtain after adding the results followed by integration with respect to time from 0 to t:

$$\begin{aligned}\langle u_t^m(t), u_t^m(t)\rangle \quad &+ \quad a_\Omega^{MM}(u^m(t), u^m(t)) + a_\Omega^{EE}(q^m(t), q^m(t)) \\ &= \quad \langle u_1^m, u_1^m\rangle + a_\Omega^{MM}(u_0^m, u_0^m) + a_\Omega^{EE}(q^m(0), q^m(0)) \\ &\quad +2\int_0^t (\langle F(s), u_t^m(s)\rangle + \langle G_t(s), q^m(s)\rangle)\, ds \\ &\quad +2\int_0^t (\langle \Sigma(s), u_t^m(s)\rangle_{S_0} + \langle \Phi_t(s), q^m(s)\rangle_{S_1})\, ds \end{aligned} \tag{2.25}$$

To obtain an initial condition for q^m we need to solve

$$a_\Omega^{EE}(q^m(0), \xi) = a_\Omega^{ME}(u_0^m, \xi) + \langle G(0), \xi\rangle + \langle \Phi(0), \xi\rangle_{S_1} \ \forall \xi \in \mathcal{W}_E^m(\Omega).$$

Since we know the regularity of u_0^m and $G(0)$, we can apply the Lax-Milgram Lemma to obtain a unique solution $q^m(0) \in \mathcal{W}_E(\Omega)$, such that

$$\|q^m(0)\|^2_{\mathcal{W}_E(\Omega)} \le C\{\|u_0^m\|^2_{\mathcal{W}_M(\Omega)} + \|G(0)\|^2_{\mathcal{W}_E(\Omega)^*} + \|\Phi(0)\|_{H^{\frac{1}{2}}(\Omega)^*}\}.$$

Now, using the coercivity of $a_\Omega^{MM}, a_\Omega^{EE}$ in $\mathcal{W}_M(\Omega)$ and $\mathcal{W}_E(\Omega)$, respectively, and applying the Gronwall-Lemma we obtain

$$\begin{aligned}&\|u_t^m(t)\|^2_{L^2(\Omega)} + \|u^m(t)\|^2_{\mathcal{W}_M(\Omega)} + \|q^m(t)\|^2_{\mathcal{W}_E(\Omega)} \\ &\quad \le C\Big\{\|u_1^m\|^2_{L^2(\Omega)} + \|u_0^m\|^2_{\mathcal{W}_M(\Omega)} + \|F\|^2_{L^2(0,T;L^2(\Omega))} + \|G_t\|^2_{L^2(0,T;\mathcal{W})_E(\Omega)^*} \\ &\qquad + \|G\|^2_{L^\infty(0,T;\mathcal{W}_E(\Omega)^*)} + \|\Sigma\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Omega)^*)} + \|\Phi_t\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Omega)^*)} + \|\Phi\|^2_{L^\infty(0,T;\mathcal{W}_E(\Omega))}\Big\}\end{aligned} \tag{2.26}$$

A standard argument give us the estimate

$$\|u_{tt}^m\|_{L^2(0,T;\mathcal{W}_M(\Omega)^*)} \le C. \tag{2.27}$$

Using the a priori energy estimates (2.26) and (2.27), one may then extract subsequences $\{u^m\}, \{u_t^m\}, \{u_{tt}^m\}$, which we relabel by original indices converging weak-($\star$) in $L^\infty(0,T;\mathcal{W}_M(\Omega))$, $L^\infty(0,T;\mathcal{W}_E(\Omega))$ and weak in $L^2(0,T;\mathcal{W}_M(\Omega)^*)$, respectively to elements $u*, u_t^*, u_{tt}^*$. Standard arguments reveal that these elements solve the weak system (2.10) and that the initial data are matched in the corresponding spaces as well.

As for the second part of the theorem, we first differentiate the weak system and take $(u_{tt}^m(t), 0)$ and $(0, q_t^m(t))$ as test functions. We obtain

$$\begin{aligned}&\frac{1}{2}\{\|u_{tt}\|^2_{L^2(\Omega)^3} + a_\Omega^{MM}(u_t^m(t), u_t^m(t)) + a_\Omega^{EE}(q_t^m(t), q_t^m(t))\} \\ &= \langle F_t(t), u_{tt}^m(t)\rangle + \langle G_{tt}(t), q_t^m(t)\rangle + \langle \Sigma_t(t), u_{tt}^m(t)\rangle_{S_0} + \langle \Phi_{tt}(t), q_t^m(t)\rangle_{S_1}\end{aligned} \tag{2.28}$$

Integration with respect to time give us

$$\begin{aligned}&\|u_{tt}^m(t)\|^2_{L^2(\Omega)} + \|u_t^m(t)\|^2_{\mathcal{W}_M(\Omega)} + \|q_t^m(t)\|^2_{\mathcal{W}_E(\Omega)} \\ &\le C\Big\{\|u_{tt}^m(0)\|^2_{L^2(\Omega)} + \|u_t^m(0)\|^2_{\mathcal{W}_M(\Omega)} + \|q_t^m(0)\|^2_{\mathcal{W}_E(\Omega)} \\ &+ \int_0^t \|F_t(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|G_{tt}(s)\|^2_{\mathcal{W}_E(\Omega)^*} ds + \int_0^t \|u_{tt}^m(s)\|^2_{L^2(\Omega)} ds \\ &+ \int_0^t \|q_t^m(s)\|^2_{\mathcal{W}_E(\Omega)^*} + \int_0^t \|\Sigma_t(s)\|^2_{L^2(S_0)} ds + \int_0^t \|\Phi_{tt}(s)\|^2_{H^{\frac{1}{2}}(\Omega)^*} ds\Big\}\end{aligned} \tag{2.29}$$

We need estimates on $\|u_{tt}^m(0)\|_{L^2(\Omega)^3}$ and $\|q_t^m(0)\|_{\mathcal{W}(\Omega)}$ in terms of our data. As now $u_t^m(0) \in \mathcal{W}_M(\Omega)$ and $G_t(0) \in \mathcal{W}_E(\Omega)^*$ we can uniquely solve the second equation of (2.10) to obtain

$$\|q_t^m(0)\|_{\mathcal{W}_E(\Omega)} \le C\{\|G_t(0)\|_{\mathcal{W}_E(\Omega)^*} + \|u_t^m(0)\|_{\mathcal{W}_M\Omega)} + \|\Phi_t(0)\|_{H^{\frac{1}{2}}(\Omega)^*}\}$$

Moreover, for given $u^m(0) \in H^2(\Omega)$ let $q^m(0) \in H^2(\Omega)$ be such that

$$\begin{cases} \text{div } D\nabla q &= G(0) + \text{div } P^T\nabla^s u^m(0), \quad \text{in } \Omega \\ D\nabla q \cdot n &= P^T\nabla^s u^m(0)\cdot n - \Phi(0) \quad \text{on } S_1 \\ q &= 0 \quad \text{on } S_0 \\ [\![\sigma]\!] n = 0, & [\![u]\!] = 0 \quad \text{on } S_i \\ [\![\varphi]\!]\cdot n = 0, & [\![q]\!] = 0 \quad \text{on } S_i \end{cases}$$

Then $\|q^m(0)\|_{H^2(\Omega)} \le \{\|G(0)\|_{L^2(\Omega)} + \|u^m(0)\|_{H^2(\Omega)^3}\}$. Evaluating the strong solution at $t=0$ and applying Gronwall's lemma we obtain the a priori estimate

$$\begin{aligned} &\|u^m_{tt}(t)\|^2_{H^2(\Omega)} + \|u^m_t(t)\|^2_{\mathcal{W}(\Omega)} + \|u^m_{tt}(t)\|^2_{L^2(\Omega)} + \|q^m(t)\|^2_{H^2(\Omega)} + \|q^m_t(t)\|^2_{\mathcal{W}(\Omega)} \\ &\quad \le C\{\|u^m(0)\|^2_{H^2(\Omega)} + \|u^m_t(0)\|^2_{\mathcal{W}(\Omega)} + \|F\|^2_{L^\infty(0,T;L^2(\Omega))} + \|F_t\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + \|G\|^2_{L^\infty(0,T;\mathcal{W}(\Omega)^*)} + \|G_{tt}\|^2_{L^2(0,T;\mathcal{W}(\Omega)^*)} + \|\Sigma\|^2_{L^\infty(0,T;L^2(S_0))} + \|\Sigma_t\|^2_{L^2(0,T;L^2(S_0))} \\ &\quad + \|\Phi\|^2_{L^\infty(0,T;H^{\frac12}(S_1)^*)} + \|\Phi_{tt}\|^2_{L^2(0,T;H^{\frac12}(S_1))^*}\} \end{aligned} \tag{2.30}$$

We then subtract weak-($\star$) convergent subsequences and pass to the limit in the equations. The fulfillment of the initial data is proved by a standard argument. Note that also non-homogenous boundary conditions for q (and u) can be easily handled. ☐

Remark 4. *The weak formulation of the piezoelectric problem reads:*

$$\begin{cases} \langle -\omega^2 u, \eta\rangle_\Omega + a^{MM}_\Omega(u,\eta) + a^{EM}_\Omega(q,\eta) &= \langle F,\eta\rangle_\Omega + \langle \Sigma,\eta\rangle_{S_0} \quad \forall \eta \in \mathcal{W}_M(\Omega) \\ a^{EE}_\Omega(q,\xi) - a^{ME}_\Omega(u,\xi) &= \langle G,\eta\rangle_\Omega + \langle \Phi,\eta\rangle_{S_1} \quad \forall \xi \in \mathcal{W}_E(\Omega) \end{cases},$$

This system with $\omega = 0$ has been investigated e.g. in [2]. *The corresponding differential operator in* (2.9) *has a compact resolvent. Hence it exhibits a pure point spectrum. Therefore, for ω^2 in the resolvent set of this operator the problem can be uniquely solved for u, q by Lax-Milgram lemma.*

2.3. **The shape functional.** We consider the shape functional of the form

$$\mathcal{J}_\Omega(u,q) = \int_0^T J_\Omega(u,q)\ . \tag{2.31}$$

Some particular examples of shape functionals are given explicitly in section 3.2. We assume that

- the sets Ω range in the set $\mathcal{O}$ of subsets of $\mathbb{R}^3$ satisfying the uniform cone property. Under the further assumption that $\Omega \to (u,q)_\Omega$ is continuous, then $\mathcal{G} = \{(\Omega,(u,q)_\Omega) | \Omega \in \mathcal{O}\}$ is compact;
- that $\mathcal{J}$ is lower semi-continuous in the sense that for $\Omega_n \in \mathcal{O}, (u_n,q_n) \in \mathcal{W}_M(\Omega_n)\times\mathcal{W}_E(\Omega_n)$

$$\left.\begin{array}{l} \Omega_n \to \Omega \text{ in } \mathcal{O},\ \Omega_n, \Omega \in \mathcal{O} \\ (u_n,q_n) \to (u,q) \in (u,q) \in \mathcal{W}_M(\Omega)\times\mathcal{W}_E(\Omega) \end{array}\right\} \Longrightarrow \lim\inf_{n\to\infty} \mathcal{J}_{\Omega_n}(u_n,q_n) \ge \mathcal{J}_\Omega(u,q)\ .$$

Theorem 5. *Assume that the admissible family of domains $\mathcal{U}_{ad} = \mathcal{O}$, the set $\mathcal{G}$ is compact and the function $\mathcal{J}_\Omega$ is lower semi-continuous. Then there exists a solution to the shape optimization problem.*

Proof. The proof is standard. See e.g., Sokolowski and Zolesio [21] as well as Delfour and Zolesio [3]. ☐

Remark 6. *We will prove, by an application of the speed method of the shape sensitivity analysis, that in fact the solutions (u,q) depend continuously on the domain $\Omega \in \mathcal{O}$.*

2.4. **The adjoint system.** In order to simplify the further calculation, let us introduce the adjoint states v and q, which solve the following variational system. Given the final conditions

$$v(x,T) = 0 \quad \text{and} \quad v_t(x,T) = 0\ , \tag{2.32}$$

find, for each $t \in (0,T)$, the adjoint displacement $v \in \mathcal{W}_M(\Omega)$ and the adjoint electrical potential $p \in \mathcal{W}_E(\Omega)$, such that,

$$\begin{cases} \langle v_{tt},\eta\rangle_\Omega + a^{MM}_\Omega(v,\eta) - a^{EM}_\Omega(p,\eta) &= -\langle D_u(J_\Omega(u,q)),\eta\rangle \quad \forall \eta \in \mathcal{W}_M(\Omega) \\ a^{EE}_\Omega(p,\xi) + a^{ME}_\Omega(v,\xi) &= -\langle D_q(J_\Omega(u,q)),\xi\rangle \quad \forall \xi \in \mathcal{W}_E(\Omega)\ . \end{cases} \tag{2.33}$$

The notation for right-hand sides could be misleading. Therefore, we explain here that the linear forms $D_u(J_\Omega(u,q))$ and $D_q(J_\Omega(u,q))$ are given in general by volume integrals and by the surface integrals, roughly speaking there exists functions $\mathfrak{F}_i$, $i = 1, \dots, 4$ such that

$$\langle D_u(J_\Omega(u,q)), \eta\rangle = \int_\Omega \mathfrak{F}_1\eta + \int_{\partial\Omega} \mathfrak{F}_2\eta \tag{2.34}$$

$$\langle D_q(J_\Omega(u,q)), \xi\rangle = \int_\Omega \mathfrak{F}_3\xi + \int_{\partial\Omega} \mathfrak{F}_4\xi \; . \tag{2.35}$$

In particular, $\mathfrak{F}_1 = F$, $\mathfrak{F}_2 = \Phi$, $\mathfrak{F}_3 = G$, $\mathfrak{F}_4 = \Sigma$. In order to assure the existence of solutions to the adjoint system, it is assumed that $\mathfrak{F}_i$ satisfy the assumptions of Theorem 3 for the respective regularity requirements.

From the above system, we can define the adjoint stress tensor σ_a and the adjoint electrical displacement φ_a as following

$$\left\{ \begin{array}{lcl} \sigma_a(v,p) & = & C\varepsilon(v) + Pe(p) \; , \\ \varphi_a(v,p) & = & -P^T\varepsilon(v) + De(p) \; . \end{array} \right. \tag{2.36}$$

Remark 7. *We can consider the weak adjoint system in the time-harmonic case:*

$$\left\{ \begin{array}{rcll} \langle -\omega^2 v, \eta\rangle_\Omega + a_\Omega^{MM}(v,\eta) - a_\Omega^{EM}(p,\eta) & = & -\langle D_u(J_\Omega(u,q)), \eta\rangle & \forall \eta \in \mathcal{W}_M(\Omega) \\ a_\Omega^{EE}(p,\xi) + a_\Omega^{ME}(v,\xi) & = & -\langle D_q(J_\Omega(u,q)), \xi\rangle & \forall \xi \in \mathcal{W}_E(\Omega). \end{array} \right.$$

Existence and uniqueness of weak (or more regular case) solutions can be done using the same arguments as in the original problem. See Remark 2.

3. SHAPE SENSITIVITY ANALYSIS

For sake of simplicity, in this section we consider that the Neumann data Σ on $S_0 \times (0,T)$ and Φ on $S_1 \times (0,T)$ in (2.5) are homogeneous. We also consider that the source terms F and G in (2.1) are identically zero. Thus, we focus our attention to the non-homogeneous initial conditions f and g in (2.4).

The perturbed domain, parameterized by $\tau \in \mathbb{R}^+$ small enough, is denoted as

$$\Omega_\tau = \{x_\tau \in \mathbb{R}^3 : x_\tau = x + \tau V, \; x \in \Omega, \; \tau \geq 0\} \; , \tag{3.1}$$

where V is a smooth vector field defined in Ω that represents the shape change velocity. Thus, the original domain is retrieved by setting $\tau = 0$, that is $\Omega_0 \equiv \Omega$. The shape functional defined in the perturbed domain reads

$$\mathcal{J}_{\Omega_\tau}(u_\tau, q_\tau) = \int_0^T J_{\Omega_\tau}(u_\tau, q_\tau) \; , \tag{3.2}$$

where the pair $u_\tau = u_\tau(x_\tau, t)$ and $q_\tau = q_\tau(x_\tau, t)$ are solutions of the following variational problem defined in the perturbed domain Ω_τ: given the initial conditions $u_\tau(x_\tau, 0) = f(x_\tau)$ and $u_{\tau_t}(x_\tau, 0) = g(x_\tau)$, find for each $t \in (0,T)$ the displacement $u_\tau \in \mathcal{W}_M(\Omega_\tau)$ and electrical potential $q_\tau \in \mathcal{W}_E(\Omega_\tau)$, such that

$$\left\{ \begin{array}{rcll} \langle u_{\tau_{tt}}, \eta\rangle_{\Omega_\tau} + a_{\Omega_\tau}^{MM}(u_\tau,\eta) + a_{\Omega_\tau}^{EM}(q_\tau,\eta) & = & 0 & \forall \eta \in \mathcal{W}_M(\Omega_\tau) \\ a_{\Omega_\tau}^{EE}(q_\tau,\xi) - a_{\Omega_\tau}^{ME}(u_\tau,\xi) & = & 0 & \forall \xi \in \mathcal{W}_E(\Omega_\tau), \end{array} \right. \tag{3.3}$$

where

$$\langle u_{\tau_{tt}}, \eta\rangle_{\Omega_\tau} = \int_{\Omega_\tau} u_{\tau_{tt}} \cdot \eta \; , \tag{3.4}$$

$$a_{\Omega_\tau}^{MM}(u_\tau,\eta) = \int_{\Omega_\tau} C\nabla^s u_\tau \cdot \nabla^s \eta \quad \text{and} \quad a_{\Omega_\tau}^{EM}(q_\tau,\eta) = \int_{\Omega_\tau} P\nabla q_\tau \cdot \nabla^s \eta \; , \tag{3.5}$$

$$a_{\Omega_\tau}^{EE}(q_\tau,\xi) = \int_{\Omega_\tau} D\nabla q_\tau \cdot \nabla \xi \quad \text{and} \quad a_{\Omega_\tau}^{ME}(u_\tau,\xi) = \int_{\Omega_\tau} P^T\nabla^s u_\tau \cdot \nabla \xi \; , \tag{3.6}$$

with $a^{EM}_{\Omega_\tau}(q_\tau, u_\tau) = a^{ME}_{\Omega_\tau}(u_\tau, q_\tau)$ and $\nabla := \partial/\partial x_\tau$. In addition, the spaces $\mathcal{W}_M(\Omega_\tau)$ and $\mathcal{W}_E(\Omega_\tau)$ are respectively defined as

$$\mathcal{W}_M(\Omega_\tau) = \{u_\tau \in [H^1(\Omega_\tau)]^3 : u_\tau|_{S_{1_\tau}} = 0,\ [\![u_\tau]\!]|_{\Gamma_{\tau_i}} = 0,\ i = 1, 2, ..., m\}\ , \tag{3.7}$$

$$\mathcal{W}_E(\Omega_\tau) = \{q_\tau \in H^1(\Omega_\tau) : q_\tau|_{S_{0_\tau}} = 0,\ [\![q_\tau]\!]|_{\Gamma_{\tau_i}} = 0,\ i = 1, 2, ..., m\}\ . \tag{3.8}$$

Theorem 8. *There exist shape derivatives u', u'_t and q' of solutions to system* (2.10)*, such that*

$$u' \in L^\infty(0, T; H^1(\Omega)), \quad u'_t \in L^\infty(0, T; L^2(\Omega)), \quad q' \in L^\infty(0, T; H^1(\Omega)), \tag{3.9}$$

given by weak solutions to the following system:

- *equations are given by (2.1)*
- *in general, the nonhomogeneous transmission conditions come out from (2.5) and (2.6)*

3.1. **Shape derivative calculation.** Our strategy can be described as follows. The first step is the proof of shape differentiability of solutions and of the shape functionals. So, at this stage the material derivatives are used.

When the shape differentiability is established, we are interested in the identification of the shape gradients as well as in the regularity of the obtained expressions for shape gradients. This step is crucial for numerical methods. The discretized shape gradient can be used e.g., for numerical solution of shape optimization problems. In the framework of the level-set strategy for solution of shape optimization problems we require in addition that the shape gradients are given by some functions. In general, however, the structure theorem for shape differentiable functionals leads only to the distributions supported on the boundary [21].

To obtain the expressions for the shape gradients, first by some manipulations including integration by parts we arrive at boundary integrals, cf. e.g., (3.50). Then using exclusively the velocity vector fields normal to the boundary we can identify the expressions for the shape gradients.

Let us perform the shape sensitivity analysis of the functional $\mathcal{J}_{\Omega_\tau}(u_\tau, q_\tau)$. Thus, we need to calculate its derivative with respect to the parameter τ at $\tau = 0$, that is

$$\int_0^T \dot{J}_\Omega(u, q) = \dot{\mathcal{J}}_\Omega(u, q) := \frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(u_\tau, q_\tau)\bigg|_{\tau=0}\ . \tag{3.10}$$

In order to proceed, it is convenient to introduce an analogy to classical continuum mechanics [6] whereby the shape change velocity field V is identified with the classical velocity field of a deforming continuum and τ is identified as an artificial time parameter (refer to [22] for analogies of this type in the context of shape sensitivity analysis). Thus, the shape derivative of the functional $J_\Omega(u, q)$ is given by

$$\dot{J}_\Omega(u, q) = \langle D_\Omega(J_\Omega(u, q)), V\rangle + \langle D_u(J_\Omega(u, q)), \dot{u}\rangle + \langle D_q(J_\Omega(u, q)), \dot{q}\rangle\ . \tag{3.11}$$

Let us now calculate the derivative of the state system (3.3) with respect to the parameter τ at $\tau = 0$. Thus, by making use of the concept of material derivative of a spatial field [6, 7] and considering the Reynolds' Transport Theorem, we obtain

$$\langle u_{tt}, \eta\rangle\dot{}_\Omega = \int_\Omega \dot{u}_{tt}\cdot\eta + \int_\Omega (u_{tt}\cdot\eta)\mathrm{div}V\ , \tag{3.12}$$

$$\dot{a}^{MM}_\Omega(u, \eta) = a^{MM}_\Omega(\dot{u}, \eta) + \int_\Omega (C\nabla^s u\cdot\nabla^s\eta)\mathrm{div}V - \int_\Omega (\nabla u^T(C\nabla^s\eta) + \nabla\eta^T(C\nabla^s u)\cdot\nabla V\ , \tag{3.13}$$

$$\dot{a}^{EM}_\Omega(q, \eta) = a^{EM}_\Omega(\dot{q}, \eta) + \int_\Omega (P\nabla q\cdot\nabla^s\eta)\mathrm{div}V - \int_\Omega (\nabla q\otimes P^T\nabla^s\eta + \nabla\eta^T P\nabla q)\cdot\nabla V\ , \tag{3.14}$$

$$\dot{a}^{EE}_\Omega(q, \xi) = a^{EE}_\Omega(\dot{q}, \xi) + \int_\Omega (D\nabla q\cdot\nabla\xi)\mathrm{div}V - \int_\Omega (\nabla q\otimes D\nabla\xi + \nabla\xi\otimes D\nabla q)\cdot\nabla V\ , \tag{3.15}$$

$$\dot{a}^{ME}_\Omega(u, \xi) = a^{ME}_\Omega(\dot{u}, \xi) + \int_\Omega (P^T\nabla^s u\cdot\nabla\xi)\mathrm{div}V - \int_\Omega (\nabla u^T P\nabla\xi + \nabla\xi\otimes P^T\nabla^s u)\cdot\nabla V\ , \tag{3.16}$$

where we have used the fact that the admissible variations η and ξ do not depend on the parameter τ. Thus, the so-called material derivative of the state system, after some rearrangements, is given by the following identities

$$\begin{aligned}\langle \dot{u}_{tt}, \eta\rangle_\Omega + a_\Omega^{MM}(\dot{u},\eta) + a_\Omega^{EM}(\dot{q},\eta) &= \int_\Omega (\nabla u^T(C\nabla^s\eta) + \nabla\eta^T(C\nabla^s u) + \nabla q\otimes P^T\nabla^s\eta + \nabla\eta^T P\nabla q)\cdot\nabla V \\ &- \int_\Omega (u_{tt}\cdot\eta + C\nabla^s u\cdot\nabla^s\eta + P\nabla q\cdot\nabla^s\eta)\mathrm{div}V\ , \qquad (3.17)\\ a_\Omega^{EE}(\dot{q},\xi) - a_\Omega^{ME}(\dot{u},\xi) &= \int_\Omega (\nabla q\otimes D\nabla\xi + \nabla\xi\otimes D\nabla q - \nabla u^T P\nabla\xi - \nabla\xi\otimes P^T\nabla^s u)\cdot\nabla V \\ &- \int_\Omega (D\nabla q\cdot\nabla\xi - P^T\nabla^s u\cdot\nabla\xi)\mathrm{div}V\ , \qquad (3.18)\end{aligned}$$

supplemented with initial conditions (cf. (2.4))

$$\dot{u}(x,0) = (\nabla f(x))V \quad \text{and} \quad \dot{u}_t(x,0) = (\nabla g(x))V\ . \qquad (3.19)$$

Theorem 9. *Given initial conditions $(\nabla f)V \in H^1(\Omega)$ and $(\nabla g)V \in L^2(\Omega)$, there is a unique weak solution to system (3.17)-(3.18) and* (3.19)*, such that*

$$\dot{u}\in L^\infty(0,T;H^1(\Omega)),\quad \dot{u}_t\in L^\infty(0,T;L^2(\Omega)),\quad \dot{u}_{tt}\in L^\infty(0,T;H^{-1}(\Omega)),\quad \dot{q}\in L^\infty(0,T;H^1(\Omega)). \qquad (3.20)$$

If we assume the appropriate compatibility conditions for the initial and boundary conditions (cf. Theorem 3), then the weak solution becomes strong solution.

We return to the evaluation of shape gradients for the piezo system. To this end, by setting $\eta = v$ and $\xi = p$ in (3.17)-(3.18) we obtain

$$\begin{aligned}\langle v_{tt}, \dot{u}\rangle_\Omega + a_\Omega^{MM}(\dot{u},v) + a_\Omega^{EM}(\dot{q},v) &= \int_\Omega (\nabla u^T(C\nabla^s v) + \nabla v^T(C\nabla^s u) + \nabla q\otimes P^T\nabla^s v + \nabla v^T P\nabla q)\cdot\nabla V \\ &- \int_\Omega (u_{tt}\cdot v + C\nabla^s u\cdot\nabla^s v + P\nabla q\cdot\nabla^s v)\mathrm{div}V \\ &+ \langle v_{tt},\dot{u}\rangle_\Omega - \langle\dot{u}_{tt}, v\rangle_\Omega\ , \qquad (3.21)\\ a_\Omega^{EE}(\dot{q},p) - a_\Omega^{ME}(\dot{u},p) &= \int_\Omega (\nabla q\otimes D\nabla p + \nabla p\otimes D\nabla q - \nabla u^T P\nabla p - \nabla p\otimes P^T\nabla^s u)\cdot\nabla V \\ &- \int_\Omega (D\nabla q\cdot\nabla p - P^T\nabla^s u\cdot\nabla p)\mathrm{div}V\ , \qquad (3.22)\end{aligned}$$

where we have introduced the term $\pm\langle v_{tt},\dot{u}\rangle_\Omega$ in the left hand side of the first equality. Using integration by parts, we have

$$\begin{aligned}\int_0^T \langle v_{tt},\dot{u}\rangle_\Omega - \int_0^T \langle\dot{u}_{tt},v\rangle_\Omega &= \langle v_t,\dot{u}\rangle_\Omega|_0^T - \langle\dot{u}_t,v\rangle_\Omega|_0^T \\ &= \langle v_t(T),\dot{u}(T)\rangle_\Omega - \langle v_t(0),\dot{u}(0)\rangle_\Omega - \langle\dot{u}_t(T),v(T)\rangle_\Omega + \langle\dot{u}_t(0),v(0)\rangle_\Omega \\ &= \langle\dot{g},v(0)\rangle_\Omega - \langle v_t(0),\dot{f},\rangle_\Omega \\ &= \langle(\nabla g)V,v(0)\rangle_\Omega - \langle v_t(0),(\nabla f)V\rangle_\Omega\ , \qquad (3.23)\end{aligned}$$

which implies

$$\int_0^T (\langle v_{tt},\dot{u}\rangle_\Omega - \langle\dot{u}_{tt},v\rangle_\Omega) = \int_\Omega (\nabla g^T v(0) - \nabla f^T v_t(0))\cdot V\ . \qquad (3.24)$$

On the other hand,

$$\begin{aligned}\int_\Omega (u_{tt}\cdot v)\mathrm{div}V &= \int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V - \int_\Omega \nabla(u_{tt}\cdot v)\cdot V \\ &= \int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V - \int_\Omega (\nabla u_{tt}^T v + \nabla v^T u_{tt})\cdot V\ , \qquad (3.25)\end{aligned}$$

and from an integration by parts

$$\begin{aligned}\int_0^T\int_\Omega(\nabla u_{tt}^T v) &= \int_\Omega(\nabla u_t^T v)\Big|_0^T - \int_\Omega(\nabla u^T v_t)\Big|_0^T + \int_0^T\int_\Omega(\nabla u^T v_{tt}) \\ &= \int_0^T\int_\Omega(\nabla u^T v_{tt}) - \int_\Omega(\nabla g^T v(0) - \nabla f^T v_t(0)) \ . \end{aligned} \tag{3.26}$$

Therefore from (3.25) and (3.26) we have

$$\int_0^T\int_\Omega(u_{tt}\cdot v)\mathrm{div}V = \int_0^T\int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V - \int_0^T\int_\Omega(\nabla u^T v_{tt} + \nabla v^T u_{tt})\cdot V + \int_\Omega(\nabla g^T v(0) - \nabla f^T v_t(0))\cdot V \ . \tag{3.27}$$

Finally, we obtain

$$\int_0^T(\langle v_{tt},\dot u\rangle_\Omega - \langle \dot u_{tt}, v\rangle_\Omega) - \int_0^T\int_\Omega(u_{tt}\cdot v)\mathrm{div}V = \int_0^T\int_\Omega(\nabla u^T v_{tt} + \nabla v^T u_{tt})\cdot V - \int_0^T\int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V \ . \tag{3.28}$$

Thus, (3.21)-(3.22) can be re-written as

$$\begin{aligned}\langle v_{tt},\dot u\rangle_\Omega + a_\Omega^{MM}(\dot u, v) + a_\Omega^{EM}(\dot q, v) &= \int_\Omega(\nabla u^T(C\nabla^s v) + \nabla v^T(C\nabla^s u) + \nabla q\otimes P^T\nabla^s v + \nabla v^T P\nabla q)\cdot\nabla V \\ &+ \int_\Omega(\nabla u^T v_{tt} + \nabla v^T u_{tt})\cdot V - \int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V \\ &- \int_\Omega(C\nabla^s u\cdot\nabla^s v + P\nabla q\cdot\nabla^s v)\mathrm{div}V \ , \end{aligned} \tag{3.29}$$

$$\begin{aligned} a_\Omega^{EE}(\dot q, p) - a_\Omega^{ME}(\dot u, p) &= \int_\Omega(\nabla q\otimes D\nabla p + \nabla p\otimes D\nabla q - \nabla u^T P\nabla p - \nabla p\otimes P^T\nabla^s u)\cdot\nabla V \\ &- \int_\Omega(D\nabla q\cdot\nabla p - P^T\nabla^s u\cdot\nabla p)\mathrm{div}V \ . \end{aligned} \tag{3.30}$$

In the same way, let us set $\eta = \dot u$ and $\xi = \dot q$ in the adjoint system (2.33), then

$$\left\{\begin{aligned}\langle v_{tt},\dot u\rangle_\Omega + a_\Omega^{MM}(v,\dot u) - a_\Omega^{EM}(p,\dot u) &= -\langle D_u(J_\Omega(u,q)),\dot u\rangle \\ a_\Omega^{EE}(p,\dot q) + a_\Omega^{ME}(v,\dot q) &= -\langle D_q(J_\Omega(u,q)),\dot q\rangle. \end{aligned}\right. \tag{3.31}$$

By comparison of (3.29)-(3.30) and (3.31), we observe that

$$\begin{aligned}\langle D_u(J_\Omega(u,q)),\dot u\rangle &= \int_\Omega(C\nabla^s u\cdot\nabla^s v + P\nabla q\cdot\nabla^s v)\mathrm{div}V + a_\Omega^{EM}(\dot q, v) + a_\Omega^{EM}(p,\dot u) \\ &- \int_\Omega(\nabla u^T(C\nabla^s v) + \nabla v^T(C\nabla^s u) + \nabla q\otimes P^T\nabla^s v + \nabla v^T P\nabla q)\cdot\nabla V \\ &- \int_\Omega(\nabla u^T v_{tt} + \nabla v^T u_{tt})\cdot V + \int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V \ , \end{aligned} \tag{3.32}$$

$$\begin{aligned}\langle D_q(J_\Omega(u,q)),\dot q\rangle &= \int_\Omega(D\nabla q\cdot\nabla p - P^T\nabla^s u\cdot\nabla p)\mathrm{div}V - a_\Omega^{ME}(\dot u, p) - a_\Omega^{ME}(v,\dot q) \\ &- \int_\Omega(\nabla q\otimes D\nabla p + \nabla p\otimes D\nabla q - \nabla u^T P\nabla p - \nabla p\otimes P^T\nabla^s u)\cdot\nabla V \ . \end{aligned} \tag{3.33}$$

where we have used the fact that the bilinear forms $a_\Omega^{MM}(\cdot,\cdot)$ and $a_\Omega^{EE}(\cdot,\cdot)$ are symmetric. In addition, since $a_\Omega^{EM}(p,\dot u) = a_\Omega^{ME}(\dot u, p)$ and $a_\Omega^{EM}(\dot q, v) = a_\Omega^{ME}(v,\dot q)$ we have

$$\langle D_u(J_\Omega(u,q)),\dot u\rangle + \langle D_q(J_\Omega(u,q)),\dot q\rangle = \int_\Omega S\cdot\nabla V + \int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V - \int_\Omega(\nabla u^T v_{tt} + \nabla v^T u_{tt})\cdot V \ , \tag{3.34}$$

where the Eshelby tensor S reads (see the fundamental paper [4])

$$S = (\sigma\cdot\nabla^s v - \varphi\cdot\nabla p)I - (\nabla u^T\sigma_a + \nabla v^T\sigma - \nabla q\otimes\varphi_a - \nabla p\otimes\varphi) \ , \tag{3.35}$$

with σ, φ and σ_a, φ_a given, respectively, by (2.2) and (2.36). In addition, we observe that

$$\int_\Omega S \cdot \nabla V = \int_{\partial\Omega} Sn \cdot V + \sum_{i=1}^m \int_{\Gamma_i} [\![S]\!] n \cdot V - \int_\Omega \mathrm{div} S \cdot V \ . \tag{3.36}$$

Let us calculate the divergence of tensor S, which leads

$$\mathrm{div} S = -(\nabla u^T \mathrm{div}\sigma_a - \nabla q \mathrm{div}\varphi_a) - (\nabla v^T \mathrm{div}\sigma - \nabla p \mathrm{div}\varphi) \ . \tag{3.37}$$

Taking into account that the pair σ, φ satisfies the system (2.1), then

$$\begin{aligned} \mathrm{div} S + (\nabla u^T v_{tt} + \nabla v^T u_{tt}) &= \nabla u^T (v_{tt} - \mathrm{div}\sigma_a) + \nabla q \mathrm{div}\varphi_a + \nabla v^T (u_{tt} - \mathrm{div}\sigma) + \nabla p \mathrm{div}\varphi \\ &= \nabla u^T (v_{tt} - \mathrm{div}\sigma_a) + \nabla q \mathrm{div}\varphi_a \ . \end{aligned} \tag{3.38}$$

Considering these last results together with (3.34) in (3.11), we obtain the shape derivative of the functional $\mathcal{J}_\Omega(u,q)$ independent of $\dot{u}$ and $\dot{q}$, namely

$$\begin{aligned} \dot{\mathcal{J}}_\Omega(u,q) &= \int_0^T \langle D_\Omega(J_\Omega(u,q)), V \rangle - \int_\Omega b \cdot V \\ &+ \int_0^T \int_{\partial\Omega} (u_{tt} \cdot v) n \cdot V + \int_0^T \int_{\partial\Omega} Sn \cdot V + \int_0^T \sum_{i=1}^m \int_{\Gamma_i} [\![S]\!] n \cdot V \ , \end{aligned} \tag{3.39}$$

where b is given by

$$b = \nabla u^T (v_{tt} - \mathrm{div}\sigma_a) + \nabla q \mathrm{div}\varphi_a \ . \tag{3.40}$$

3.2. **Examples of shape functional.** Let us present some examples of shape functional which should be useful for practical applications. In particular, the shape functional $J_\Omega(u,q)$ is defined as

$$J_\Omega(u,q) := \int_\Omega F_\Omega(u,q) + \int_{\partial\Omega} F_S(u,q) \ , \tag{3.41}$$

where, for the sake of simplicity we assume that $\partial_u F_S(u,q)|_{S_1} = 0$ and $\partial_q F_S(u,q)|_{S_0} = 0$. In this case, the adjoint system (2.33) becomes

$$\left\{ \begin{aligned} \langle v_{tt}, \eta \rangle_\Omega + a_\Omega^{MM}(v,\eta) - a_\Omega^{EM}(p,\eta) &= -\textstyle\int_\Omega \partial_u F_\Omega(u,q) \cdot \eta - \int_{S_0} \partial_u F_S(u,q) \cdot \eta \quad \forall \eta \in \mathcal{W}_M(\Omega) \\ a_\Omega^{EE}(p,\xi) + a_\Omega^{ME}(v,\xi) &= -\textstyle\int_\Omega \partial_q F_\Omega(u,q) \xi - \int_{S_1} \partial_q F_S(u,q) \xi \qquad \forall \xi \in \mathcal{W}_E(\Omega) \end{aligned} \right. \tag{3.42}$$

The strong system associated to (3.42) is given by

$$\left\{ \begin{aligned} v_{tt} - \mathrm{div}\sigma_a &= -\partial_u F_\Omega(u,q) \\ -\mathrm{div}\varphi_a &= -\partial_q F_\Omega(u,q) \end{aligned} \right. \quad \text{in} \quad \Omega \times (0,T) \ , \tag{3.43}$$

where the adjoint stress tensor σ_a and the adjoint electrical displacement φ_a are defined in (2.36). We associate with system (3.43) the final conditions (2.32). In addition, since $v \in \mathcal{W}_M(\Omega)$ and $p \in \mathcal{W}_E(\Omega)$, from (3.42) we have the boundary conditions

$$\left\{ \begin{aligned} \sigma_a n &= -\partial_u F_S(u,q) \\ p &= 0 \end{aligned} \right. \ \text{on } S_0 \times (0,T) \quad \text{and} \quad \left\{ \begin{aligned} \varphi_a \cdot n &= -\partial_q F_S(u,q) \\ v &= 0 \end{aligned} \right. \ \text{on } S_1 \times (0,T) \ , \tag{3.44}$$

and, for any $(x,t) \in \Gamma_i \times (0,T)$, $i = 1,2,...,m$, the transmission conditions of the form

$$\left\{ \begin{aligned} [\![\sigma_a]\!] n &= 0 \\ [\![v]\!] &= 0 \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} [\![\varphi_a]\!] \cdot n &= 0 \\ [\![p]\!] &= 0 \end{aligned} \right. \ . \tag{3.45}$$

3.2.1. *Domain integral.* We set $F_\Omega(u,q)$ in (3.41) as following

$$F_\Omega(u,q) := \frac{1}{2}\left(\alpha(u-u^*_\Omega)^2 + \beta(q-q^*_\Omega)^2\right) \ , \tag{3.46}$$

where u^*_Ω and q^*_Ω are given functions defined in Ω such that $u^*_\Omega|_{S_1} = 0$ and $q^*_\Omega|_{S_0} = 0$, and $\alpha = 1-\beta$ with $\beta \in [0,1]$. Thus, since the pair σ_a, φ_a satisfies the adjoint system (3.43), then vector b defined through (3.40) can be written as

$$\begin{aligned} b &= -\nabla u^T \partial_u F_\Omega(u,q) - \nabla q \partial_q F_\Omega(u,q) \\ &= -\alpha \nabla u^T (u-u^*_\Omega) - \beta \nabla q (q-q^*_\Omega) \ . \end{aligned} \tag{3.47}$$

For this case, the derivative of the shape functional $J_\Omega(u,q)$ with respect to the domain reads

$$\begin{aligned} \langle D_\Omega(J_\Omega(u,q)), V\rangle &= \frac{1}{2}\left(\alpha \int_\Omega (u-u^*_\Omega)^2 \mathrm{div} V + \beta \int_\Omega (q-q^*_\Omega)^2 \mathrm{div} V\right) \\ &= \alpha\frac{1}{2}\left(\int_\Omega \mathrm{div}[(u-u^*_\Omega)^2 V] - \int_\Omega \nabla[(u-u^*_\Omega)^2]\cdot V\right) \\ &+ \beta\frac{1}{2}\left(\int_\Omega \mathrm{div}[(q-q^*_\Omega)^2 V] - \int_\Omega \nabla[(q-q^*_\Omega)^2]\cdot V\right) \\ &= \alpha\frac{1}{2}\int_{\partial\Omega}(u-u^*_\Omega)^2 n\cdot V - \alpha\int_\Omega \nabla u^T(u-u^*_\Omega)\cdot V \\ &+ \beta\frac{1}{2}\int_{\partial\Omega}(q-q^*_\Omega)^2 n\cdot V - \beta\int_\Omega (q-q^*_\Omega)\nabla q\cdot V \ . \end{aligned} \tag{3.48}$$

From the above results we observe that

$$\langle D_\Omega(J_\Omega(u,q)), V\rangle - \int_\Omega b\cdot V = \frac{1}{2}\left(\alpha\int_{\partial\Omega}(u-u^*_\Omega)^2 n\cdot V + \beta\int_{\partial\Omega}(q-q^*_\Omega)^2 n\cdot V\right) \ . \tag{3.49}$$

By considering this last result in (3.39) we obtain

$$\begin{aligned} \dot{\mathcal{J}}_\Omega(u,q) &= \frac{1}{2}\alpha\int_0^T\int_{S_0}(u-u^*_\Omega)^2 n\cdot V + \frac{1}{2}\beta\int_0^T\int_{S_1}(q-q^*_\Omega)^2 n\cdot V \\ &+ \int_0^T\int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V + \int_0^T\int_{\partial\Omega} Sn\cdot V + \int_0^T\sum_{i=1}^m\int_{\Gamma_i}[\![S]\!]n\cdot V \ . \end{aligned} \tag{3.50}$$

The above form of shape derivative of the distributed functional can serve us to identify the shape gradient. Since the shape functional in question is differentiable in the sense of the shape sensitivity analysis in [21], we can apply the structure theorem to this end. In particular, from the boundary and transmission conditions, namely, (2.5)-(3.44) and (2.6)-(3.45), respectively, it is straightforward to verify that the above equation holds the structure theorem. Therefore, it is sufficient to take into consideration the speed vector fields normal to the boundaries and the interfaces. This observation influences only two boundary integrals with the Eshelby tensor, and the result is the following.

Lemma 10. *The density $\mathfrak{g}$ of the boundary shape gradient of the distributed shape functional is given by the following expression*

$$\begin{aligned} \langle \mathfrak{g}, V\cdot n\rangle &= \frac{1}{2}\alpha\int_0^T\int_{S_0}(u-u^*_\Omega)^2 V\cdot n + \frac{1}{2}\beta\int_0^T\int_{S_1}(q-q^*_\Omega)^2 V\cdot n \\ &+ \int_0^T\int_{\partial\Omega}(u_{tt}\cdot v)V\cdot n + \int_0^T\int_{\partial\Omega}(Sn\cdot n)V\cdot n + \int_0^T\sum_{i=1}^m\int_{\Gamma_i}([\![S]\!]n\cdot n)V\cdot n \ . \end{aligned} \tag{3.51}$$

As it is indicated before, in order to apply the level-set strategy of shape optimization, it is required that the density $\mathfrak{g}$ of the boundary shape gradient is given by functions supported on the boundaries and on the interfaces.

Remark 11. *For the distributed functional in the time harmonic case the boundary shape gradient is determined in the following form*

$$\begin{aligned}\langle \mathfrak{g}, V\cdot n\rangle &= \frac{1}{2}\alpha\int_{S_0}(u-u^*_\Omega)^2 V\cdot n+\frac{1}{2}\beta\int_{S_1}(q-q^*_\Omega)^2 V\cdot n\\ &- \int_{\partial\Omega}(\omega^2 u\cdot v)V\cdot n+\int_{\partial\Omega}(Sn\cdot n)V\cdot n+\sum_{i=1}^m\int_{\Gamma_i}([\![S]\!]n\cdot n)V\cdot n\,. \end{aligned} \tag{3.52}$$

3.2.2. *Boundary integral.* Now, we set $F_S(u,q)$ in (3.41) as following

$$F_S(u,q) := \frac{1}{2}\left(\alpha(u-u^*_S)^2+\beta(q-q^*_S)^2\right)\ , \tag{3.53}$$

where u^*_S and q^*_S are given functions defined on S such that $u^*_S|_{S_1}=0$ and $q^*_S|_{S_0}=0$, and $\alpha=1-\beta$ with $\beta\in[0,1]$. Thus, since the pair σ_a,φ_a satisfies the adjoint system (3.43), then vector b defined through (3.40) vanishes, that is, $b=0$. For this case, the derivative of the shape functional $J_\Omega(u,q)$ with respect to the domain reads

$$\langle D_\Omega(J_\Omega(u,q)),V\rangle = \frac{1}{2}\left(\alpha\int_{S_0}(u-u^*_S)^2\mathrm{div}_{\partial\Omega}V+\beta\int_{S_1}(q-q^*_S)^2\mathrm{div}_{\partial\Omega}V\right)\ , \tag{3.54}$$

where $\mathrm{div}_{\partial\Omega}V=(I-n\otimes n)\cdot\nabla V$ is the superficial divergence of the velocity field. By considering these last results in (3.39) and recalling (2.31) we obtain

$$\begin{aligned}\dot{\mathcal{J}}_\Omega(u,q) &= \alpha\frac{1}{2}\int_0^T\int_{S_0}(u-u^*_S)^2\mathrm{div}_{\partial\Omega}V+\beta\frac{1}{2}\int_0^T\int_{S_1}(q-q^*_S)^2\mathrm{div}_{\partial\Omega}V\\ &+ \int_0^T\int_{\partial\Omega}(u_{tt}\cdot v)n\cdot V+\int_0^T\int_{\partial\Omega}Sn\cdot V+\int_0^T\sum_{i=1}^m\int_{\Gamma_i}[\![S]\!]n\cdot V\,. \end{aligned} \tag{3.55}$$

Let us point out that in the above expression the integration by parts on the boundaries S_0 and S_1 in two integrals is necessary (cf. Lemma 2.14 in [21]) to obtain the expression for the shape gradient. In addition, by taking into account the boundary and transmission conditions respectively given by (2.5)-(3.44) and (2.6)-(3.45), it is straightforward to verify again that the above equation holds the structure theorem, leading to the result below.

Lemma 12. *The shape gradient for the boundary functional is given in the following form*

$$\begin{aligned}\langle \mathfrak{g}, V\cdot n\rangle &= \alpha\frac{1}{2}\int_0^T\int_{S_0}(u-u^*_S)^2\varkappa V\cdot n+\beta\frac{1}{2}\int_0^T\int_{S_1}(q-q^*_S)^2\varkappa V\cdot n\\ &+ \int_0^T\int_{\partial\Omega}(u_{tt}\cdot v)V\cdot n+\int_0^T\int_{\partial\Omega}(Sn\cdot n)V\cdot n+\int_0^T\sum_{i=1}^m\int_{\Gamma_i}([\![S]\!]n\cdot n)V\cdot n\ , \end{aligned} \tag{3.56}$$

where $\varkappa$ stands for the mean curvature [21] *on the boundaries S_0 and S_1.*

Remark 13. *For the distributed functional in the time harmonic case the boundary shape gradient is determined in the following form*

$$\begin{aligned}\langle \mathfrak{g}, V\cdot n\rangle &= \frac{1}{2}\alpha\int_{S_0}(u-u^*_S)^2\varkappa V\cdot n+\frac{1}{2}\beta\int_{S_1}(q-q^*_S)^2\varkappa V\cdot n\\ &- \int_{\partial\Omega}(\omega^2 u\cdot v)V\cdot n+\int_{\partial\Omega}(Sn\cdot n)V\cdot n+\sum_{i=1}^m\int_{\Gamma_i}([\![S]\!]n\cdot n)V\cdot n\,. \end{aligned} \tag{3.57}$$

4. CONCLUSION AND OUTLOOK

We have derived shape sensitivities for time-varying solutions of the piezoelectric system. The results also apply almost directly to time-harmonic solutions. The corresponding numerical simulations are under way. Given the shape sensitivities and topological sensitivities for piezoelectric material, the full alternating scheme for sensitivity-based topology optimization can be applied, where one performs a topological gradient descent followed by a level-set based shape gradient descent. The numerical implementation is beyond this paper and will be subject to a forthcoming publication.

The acoustic- and piezo-electric and elastodynamic-coupling is also subject of current research. See [26] for a first treatment.

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