TOPOLOGICAL SENSITIVITY ANALYSIS OF A MULTI-SCALE CONSTITUTIVE MODEL CONSIDERING A CRACKED MICROSTRUCTURE

A.A. NOVOTNY, J. SOKOŁOWSKI, AND E.A. DE SOUZA NETO

Abstract. This paper deals with the sensitivity analysis of the macroscopic elasticity tensor to topological microstructural changes of the underlying material. In particular, the microstucture is topologicaly perturbed by the nucleation of a small circular inclusion. The derivation of the proposed sensitivity relies on the concept of topological derivative, applied within a variational multi-scale constitutive framework where the macroscopic strain and stress at each point of the macroscopic continuum are defined as volume averages of their microscopic counterparts over a Representative Volume Element (RVE) of material associated with that point. We consider that the RVE can contain a number of voids, inclusions and/or cracks. It is assumed that non-penetration conditions are imposed at the crack faces which do not allow the opposite crack faces to penetrate each other. The derived sensitivity leads to a symmetric fourth order tensor field over the unperturbed RVE domain, which measures how the macroscopic elasticity parameters estimated within the multiscale framework changes when a small circular inclusion is introduced at the micro-scale level.

1. Introduction

Composite materials have become one of the most important classes of engineering materials. Their macroscopic mechanical behavior is of paramount importance in the design of load bearing components for a vast number of applications in civil, mechanical, aerospace, biomedical and nuclear industries. In a broad sense, one can argue that much of material science is about improving macroscopic material properties by means of topological and shape changes at a microstructural level. For example, changes in shape of graphite inclusions in a cast iron matrix may produce dramatic changes in the corresponding macroscopic properties of this material. In this context, the ability to accurately predict the macroscopic mechanical behavior from the corresponding microscopic properties as well as its sensitivity to changes in microstructure becomes essential in the analysis and potential purpose-design and optimisation of heterogeneous media. Such concepts have been successfully used, for instance, in [2, 15, 16] by means of a relaxation-based technique in the design of microstructural topologies that produce negative macroscopic Poisson's ratio. This type of approach relies on the use of a fictitious material density field and mimics, in a regularised sense, the introduction of localised topological microstructural changes (voids) wherever the artificial density is sufficiently close to zero (refer, for instance, to the fundamental papers [5, 27]).

In contrast to the regularised approach, in [11] was proposed a general *exact* analytical expression for the sensitivity of the two-dimensional macroscopic elasticity tensor to topological changes of the microstructure of the underlying material. The macroscopic linear elastic response is estimated by means of a well-established homogenisation-based multi-scale constitutive theory for elasticity problems [10, 18] where the macroscopic strain and stress tensors at each point of the macroscopic continuum are defined as the volume averages of their microscopic counterparts over a Representative Volume Element (RVE) of material associated with that point. In this paper we extend the results presented in [11]

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by considering that the RVE can contain, besides voids and inclusions, a number of cracks. It is assumed that non-penetration conditions are imposed at the crack faces which do not allow the opposite crack faces to penetrate each other. Since the problem is non-linear, we use the domain truncation technique together with the topological asymptotic expansion of the Steklov-Poincaré operator as proposed in [25] to derive the final closed formula for the topological derivative of the macroscopic specific energy. Then we consider two limit cases in which the cracks are either completely open or closed under compression. For these extrema situations, we obtain the topological derivative of the macroscopic homogenised elasticity tensor. The proposed sensitivity is a symmetric fourth order tensor field over the RVE that measures how the macroscopic elasticity constants estimated within the multi-scale framework changes when a small circular inclusion is introduced at the micro-scale. Its analytical formula is derived by making use of the concepts of *topological asymptotic expansion* and *topological derivative* [6, 24] within a variational formulation of the adopted multi-scale theory. The (relatively new) mathematical notions of topological asymptotic expansion and topological derivative allow the closed form exact calculation of the sensitivity of a given shape functional with respect to infinitesimal domain perturbations such as the insertion of voids, inclusions or source terms. Their use in the context of solid mechanics, topological optimisation of load bearing structures and inverse problems is reported in a number of recent publications [1, 3, 4, 9, 14, 17, 21, 22]. Concerning the theoretical development of the topological asymptotic analysis, the reader may refer to [20], for instance. In the present context, the variational setting for the multi-scale modelling methodology as described in [7] is found to be particularly well-suited for the application of the topological derivative formalism. The final format of the proposed analytical formula is strikingly simple and can be potentially used in applications such as the synthesis and optimal design of microstructures to meet a specified macroscopic behavior.

The work is organised as follows. We start by briefly describing the multi-scale constitutive framework adopted in the estimation of the macroscopic elasticity tensor associated to the cracked microstructure. The modelling approach is cast within the variational setting described in [7]. Before presenting the main contribution of the paper (the closed formula for the sensitivity of the macroscopic elasticity tensor to topological microstructural perturbations taking into account a cracked microcell) an overview of the topological derivative concept is given, followed by its application to the problem in question. This leads to the identification of the required sensitivity tensor field, that represents the topological derivative of the macroscopic homogenised elasticity tensor with respect to the nucleation of a small inclusion at the cracked microstructure.

2. Multi-scale constitutive modelling

This section describes a homogenisation-based variational multi-scale framework for classical elasticity problems which allows estimating the macroscopic elasticity tensor to be obtained from the given geometrical and elastic properties of a local Representative Volume Element (RVE) of material. This constitutive modelling approach follows closely the methodology of [10] . It is analogous to the multi-scale strategy presented, among others, by [19] and [18] – and whose variational structure is described in detail in [7].

The starting point of the multi-scale constitutive theory is the assumption that any point x of the macroscopic continuum (refer to Fig. 1) is associated to a local RVE whose domain Ω_μ has characteristic length L_μ, much smaller than the characteristic length L of the macro-continuum domain, Ω. The RVE domain consist of an elastic body containing a number of cracks γ_c. To simplify the formulation we shall consider here only cracks that do not intersect the boundary of the RVE.

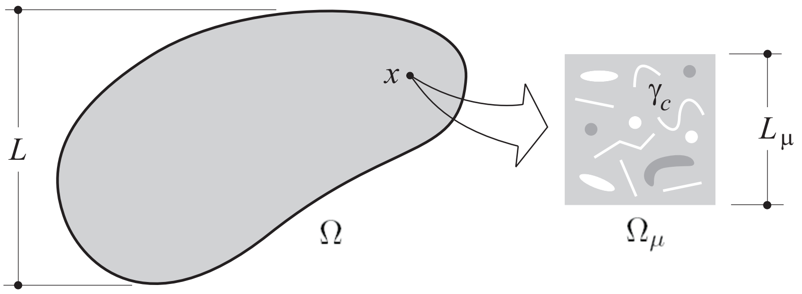

FIGURE 1. Macroscopic continuum with a locally attached microstructure.

As mentioned in the previous section, the macroscopic strain tensor ε at a point x of the macroscopic continuum is the volume average of its microscopic counterpart ε_μ over the domain of the RVE:

$$\varepsilon \equiv \frac{1}{V_\mu} \int_{\Omega_\mu} \nabla^s u_\mu \ , \tag{2.1}$$

where V_μ is the total volume of the RVE and u_μ denoting the microscopic displacement field of the RVE. The use of Green's Theorem in definition (2.1) gives the following equivalent expression for ε

$$\varepsilon = \frac{1}{V_\mu} \int_{\partial\Omega_\mu} u_\mu \otimes_s n \ , \tag{2.2}$$

where n is the outward unit normal to the boundary $\partial\Omega_\mu$ and $\otimes_s$ denotes the symmetric tensor product.

Without loss of generality, it is possible split u_μ into a sum

$$u_\mu(y) = u + \overline{u}(y) + \widetilde{u}_\mu(y) \ , \tag{2.3}$$

of a constant (rigid) RVE displacement coinciding with the macro displacement $u = u(x)$, a field $\overline{u}(y) \equiv \varepsilon\, y$, linear in y and a fluctuation displacement field $\widetilde{u}_\mu(y)$ that, in general, varies with y. Following (2.3) the microscopic strain field (2.1) can be expressed as a sum

$$\nabla^s u_\mu = \varepsilon + \nabla^s \widetilde{u}_\mu \ , \tag{2.4}$$

of a homogeneous strain (uniform over the RVE) coinciding with the macroscopic strain, and a field $\nabla^s \widetilde{u}_\mu$ corresponding to a fluctuation of the microscopic strain about the homogenised (average) value.

2.1. **Admissible and virtual microscopic displacement fields.** Assumption (2.1) places a constraint on the admissible displacement fields of the RVE. That is, only fields u_μ that satisfy (2.1) can be said to be kinematically admissible. This condition can be formally expressed by requiring the set $\mathcal{V}_\mu$ of kinematically admissible displacements of the RVE to satisfy

$$\mathcal{V}_\mu \subset \mathcal{V}_\mu^* \equiv \left\{ v \in \left[H^1(\Omega_\mu)\right]^2 : \int_{\Omega_\mu} v = V_\mu u \,, \int_{\partial\Omega_\mu} v \otimes_s n = V_\mu\, \varepsilon \right\} , \tag{2.5}$$

where $\mathcal{V}_\mu^*$ is named the *minimally constrained set of kinematically admissible RVE displacement fields.*

In view of (2.3), constraint (2.5) can, without loss of generality, be made equivalent to requiring that the space $\widetilde{\mathcal{V}}_\mu$ of admissible displacement fluctuations of the RVE be a subspace of the *minimally constrained space of displacement fluctuations*, $\widetilde{\mathcal{V}}_\mu^*$:

$$\widetilde{\mathcal{V}}_\mu \subset \widetilde{\mathcal{V}}_\mu^* \equiv \left\{ v \in \left[H^1(\Omega_\mu)\right]^2 : \int_{\Omega_\mu} v = 0, \int_{\partial\Omega_\mu} v \otimes_s n = 0 \right\}. \tag{2.6}$$

Trivially, we have that the space of virtual displacement of the RVE, defined as

$$\mathcal{U}_\mu \equiv \left\{ \eta \in \left[H^1(\Omega_\mu)\right]^2 : \eta = v_1 - v_2;\ \forall v_1, v_2 \in \mathcal{V}_\mu \right\}, \tag{2.7}$$

coincides with the space of microscopic displacement fluctuations, i.e., $\mathcal{U}_\mu = \widetilde{\mathcal{V}}_\mu$. Since we are dealing with a cracked microstructure, we need to introduce the convex sets

$$\mathcal{K}_\mu = \{v \in \mathcal{V}_\mu : [\![v]\!]_{\gamma_c} \cdot n \geq 0 \text{ on } \gamma_c\} \quad \text{and} \quad \widetilde{\mathcal{K}}_\mu = \{v \in \widetilde{\mathcal{V}}_\mu : [\![v]\!]_{\gamma_c} \cdot n \geq 0 \text{ on } \gamma_c\}, \tag{2.8}$$

where $[\![v]\!]_{\gamma_c}$ denotes the jump of function v on γ_c, that is

$$[\![\cdot]\!]_{\gamma_c} := (\cdot)|_{\gamma_c^+} - (\cdot)|_{\gamma_c^-} \tag{2.9}$$

and $\pm$ fit positive and negative crack faces $\gamma_c^\pm$ with respect to n.

2.2. **Macroscopic stress.** Similarly to the macroscopic strain tensor (2.1), the macroscopic stress tensor, σ, is defined as the volume average of the microscopic stress field σ_μ, over the RVE:

$$\sigma \equiv \frac{1}{V_\mu} \int_{\Omega_\mu} \sigma_\mu(u_\mu). \tag{2.10}$$

In the present analysis, we shall assume the materials of the RVE matrix and inclusions to satisfy the classical linear elastic constitutive law:

$$\sigma_\mu(\xi) = \mathbb{C}_\mu \nabla^s \xi, \tag{2.11}$$

where $\mathbb{C}_\mu$ is the fourth order elasticity tensor, for the isotropic and homogeneous materials, defined as:

$$\mathbb{C}_\mu = \frac{E}{1-\nu^2}\left[(1-\nu)\,\mathbb{I} + \nu\,(\mathrm{I} \otimes \mathrm{I})\right], \tag{2.12}$$

with E and ν denoting, respectively, the Young's modulus and the Poisson's ratio of the domain Ω_μ. In addition, we use I and $\mathbb{I}$ to denote the second and fourth order identity tensors, respectively.

2.3. **The RVE equilibrium problem.** Let us consider that the RVE is in equilibrium if and only if the displacement field u_μ in Ω_μ satisfies the classical variational inequality [23]: Find $u_\mu \in \mathcal{K}_\mu$, such that

$$\int_{\Omega_\mu} \sigma_\mu(u_\mu) \cdot \nabla^s(\eta - u_\mu) \geq 0 \quad \forall \eta \in \mathcal{K}_\mu. \tag{2.13}$$

The linearity of (2.11) together with and the additive decomposition (2.4), allows the microscopic stress field to be split as

$$\sigma_\mu(u_\mu) = \sigma_\mu(\overline{u}) + \sigma_\mu(\widetilde{u}_\mu), \tag{2.14}$$

where $\sigma_\mu(\overline{u})$ is the stress field associated with the uniform strain induced by $\overline{u}$, i.e., $\sigma_\mu(\overline{u}) = \mathbb{C}_\mu \nabla^s \overline{u} = \mathbb{C}_\mu \varepsilon$ and $\sigma_\mu(\widetilde{u}_\mu)$ is the stress fluctuation field associated with $\widetilde{u}_\mu$, i.e., $\sigma_\mu(\widetilde{u}_\mu) = \mathbb{C}_\mu \nabla^s \widetilde{u}_\mu$. By introducing (2.14) into (2.13), we obtain that the *RVE equilibrium problem* consists of finding, for a given macroscopic strain ε, an admissible microscopic displacement fluctuation field $\widetilde{u}_\mu \in \widetilde{\mathcal{K}}_\mu$, such that

$$\int_{\Omega_\mu} \sigma_\mu(\widetilde{u}_\mu) \cdot \nabla^s(\eta - \widetilde{u}_\mu) \geq - \int_{\Omega_\mu} \overline{\sigma}_\mu \cdot \nabla^s(\eta - \widetilde{u}_\mu) \qquad \forall \eta \in \widetilde{\mathcal{K}}_\mu \;, \tag{2.15}$$

where $\overline{\sigma}_\mu = \sigma_\mu(\overline{u})$. By substituting $\eta = 0$ and $\eta = 2\widetilde{u}_\mu$ as test function in (2.15) and sum up the relation obtained, we prove the equality

$$\int_{\Omega_\mu} (\sigma_\mu(\widetilde{u}_\mu) + \sigma_\mu(\overline{u})) \cdot \nabla^s \widetilde{u}_\mu = 0 \;. \tag{2.16}$$

2.4. **Classes of multi-scale constitutive models.** To completely define a constitutive model of the present type, the choice of a space $\mathcal{U}_\mu \subset \widetilde{\mathcal{V}}_\mu^*$ of variations of admissible displacement must be made. We list below four classical possible choices:

(a) *Taylor model* or *Rule of Mixtures.* This model is obtained by simply defining

$$\mathcal{U}_\mu = \mathcal{U}_\mu^{\mathcal{T}} \equiv \{0\} \;. \tag{2.17}$$

In this case, the strain is homogeneous over the RVE, i.e., $\varepsilon_\mu = \varepsilon$.

(b) *Linear boundary displacement model.* For this class of models the choice is

$$\mathcal{U}_\mu = \mathcal{U}_\mu^{\mathcal{L}} \equiv \{\widetilde{u}_\mu \in \widetilde{\mathcal{V}}_\mu^* : \widetilde{u}_\mu\,(y) = 0\, \forall y \in \partial\Omega_\mu\} \;. \tag{2.18}$$

(c) *Periodic boundary fluctuations model.* The space of displacement fluctuations is defined as

$$\mathcal{U}_\mu = \mathcal{U}_\mu^{\mathcal{P}} \equiv \{\widetilde{u}_\mu \in \widetilde{\mathcal{V}}_\mu^* : \widetilde{u}_\mu(y^+) = \widetilde{u}_\mu(y^-) \;\forall \text{pair } (y^+, y^-) \in \partial\Omega_\mu\} \;. \tag{2.19}$$

(d) *Minimally constrained.* In this case, we chose,

$$\mathcal{U}_\mu = \mathcal{U}_\mu^{\mathcal{U}} \equiv \widetilde{\mathcal{V}}_\mu^* \;. \tag{2.20}$$

Remark 1. *Note that the spaces of displacement fluctuations (and virtual displacement) listed above satisfy*

$$\mathcal{U}_\mu^{\mathcal{T}} \subset \mathcal{U}_\mu^{\mathcal{L}} \subset \mathcal{U}_\mu^{\mathcal{P}} \subset \mathcal{U}_\mu^{\mathcal{U}} \;. \tag{2.21}$$

That is, the Taylor model gives the stiffest (most kinematically constrained) solution to the microscopic equilibrium problem, followed in order of decreasing stiffness, by the linear boundary displacement, the periodic displacement fluctuation and the uniform boundary traction model. The uniform traction model produces the most compliant (least kinematically constrained) solution.

2.5. **The homogenised elasticity tensor.** Let us introduce a convex function of class C^0 such that

$$W(\varepsilon) = \frac{1}{2}\sigma \cdot \varepsilon = \frac{1}{2V_\mu} \int_{\Omega_\mu} \sigma_\mu(u_\mu) \cdot \nabla^s u_\mu \;. \tag{2.22}$$

Then the macroscopic stress tensor σ is a subgradient of W, namely [23]

$$\delta W \geq \sigma \cdot \delta\varepsilon \;. \tag{2.23}$$

Thus we have

$$\sigma = \partial_\varepsilon W \;, \tag{2.24}$$

where $\partial_\varepsilon(\cdot)$ is used to denote the subgradient of $(\cdot)$ with respect to ε. Therefore, at the macroscopic level, the homogenised elasticity tensor $\mathbb{C}$ is obtained as

$$\mathbb{C} \equiv \partial_\varepsilon \sigma \;. \tag{2.25}$$

On the other hand,

$$\begin{aligned}
\partial_\varepsilon \sigma &= \partial_\varepsilon \left(\frac{1}{V_\mu} \int_{\Omega_\mu} \sigma_\mu(u_\mu) \right) \\
&= \partial_\varepsilon \left(\frac{1}{V_\mu} \int_{\Omega_\mu} \mathbb{C}_\mu \nabla^s u_\mu \right) \\
&= \frac{1}{V_\mu} \partial_\varepsilon \int_{\Omega_\mu} \mathbb{C}_\mu \left(\varepsilon + \nabla^s \widetilde{u}_\mu \right) \\
&= \frac{1}{V_\mu} \int_{\Omega_\mu} \mathbb{C}_\mu + \frac{1}{V_\mu} \int_{\Omega_\mu} \mathbb{C}_\mu \partial_\varepsilon (\nabla^s \widetilde{u}_\mu) \ . \qquad (2.26)
\end{aligned}$$

Thus we have,

$$\mathbb{C} = \mathbb{C}^{\mathcal{T}} + \widetilde{\mathbb{C}} \ , \qquad (2.27)$$

where

$$\mathbb{C}^{\mathcal{T}} = \frac{1}{V_\mu} \int_{\Omega_\mu} \mathbb{C}_\mu \quad \text{and} \quad \widetilde{\mathbb{C}} = \frac{1}{V_\mu} \int_{\Omega_\mu} \mathbb{C}_\mu \partial_\varepsilon (\nabla^s \widetilde{u}_\mu) \ . \qquad (2.28)$$

In particular, we are interested in the upper and lower bounds for the homogenised elasticity tensor $\mathbb{C}$, which are obtained as

$$\mathbb{C}^+ \equiv \partial_\varepsilon \sigma \big|_{\varepsilon=\varepsilon^+} \quad \text{and} \quad \mathbb{C}^- \equiv \partial_\varepsilon \sigma \big|_{\varepsilon=\varepsilon^-} \ , \qquad (2.29)$$

where ε^+ and ε^- represent respectively dilating and compressive spherical strain tensors. For these exceptional two cases, the convex set $\widetilde{\mathcal{K}}_\mu$ degenerates to Hilbert spaces and the variational inequality (2.15) leads to a variational problem given by: Find $\widetilde{u}_\mu^\pm \in \widetilde{\mathcal{S}}_\mu^\pm$, such that

$$\int_{\Omega_\mu} \sigma_\mu(\widetilde{u}_\mu^\pm) \cdot \nabla^s \eta = - \int_{\Omega_\mu} \overline{\sigma}_\mu \cdot \nabla^s \eta \qquad \forall \eta \in \widetilde{\mathcal{S}}_\mu^\pm \ , \qquad (2.30)$$

where $\widetilde{\mathcal{S}}_\mu^+$ and $\widetilde{\mathcal{S}}_\mu^-$ are defined as

$$\widetilde{\mathcal{S}}_\mu^+ = \mathcal{V}_\mu \quad \text{and} \quad \widetilde{\mathcal{S}}_\mu^- = \{ v \in \mathcal{V}_\mu : [\![v]\!]_{\gamma_c} \cdot n = 0 \text{ on } \gamma_c \} \ . \qquad (2.31)$$

It is important to observe that for ε^+ the cracks are opened, i.e.

$$[\![\widetilde{u}_\mu]\!]_{\gamma_c} \cdot n < 0 \quad \text{and} \quad [\![\sigma_\mu(u_\mu)]\!]_{\gamma_c} n \cdot n = 0 \qquad (2.32)$$

and for ε^- the cracks are under a compressive contact, that is

$$[\![\widetilde{u}_\mu]\!]_{\gamma_c} \cdot n = 0 \quad \text{and} \quad [\![\sigma_\mu(u_\mu)]\!]_{\gamma_c} n \cdot n < 0 \ . \qquad (2.33)$$

Now, we can write the macroscopic strain in Cartesian component form:

$$\varepsilon^\pm = (\varepsilon^\pm)_{ij} e_i \otimes e_j \ , \qquad (2.34)$$

where $\{e_i\}$ is an orthonormal basis of the two-dimensional Euclidean space and the scalars $(\varepsilon^\pm)_{ij}$ are the corresponding Cartesian components of the macroscopic strain $\varepsilon^\pm$. Therefore, after differentiating the state equation (2.30) with respect to $\varepsilon^\pm$ we obtain the following set of canonical variational problems: Find $\widetilde{u}_{\mu_{ij}}^\pm \in \widetilde{\mathcal{S}}_\mu^\pm$, such that

$$\int_{\Omega_\mu} \sigma_\mu(\widetilde{u}_{\mu_{ij}}^\pm) \cdot \nabla^s \eta = - \int_{\Omega_\mu} \mathbb{C}_\mu (e_i \otimes e_j) \cdot \nabla^s \eta \qquad \forall \eta \in \widetilde{\mathcal{S}}_\mu^\pm \ , \qquad (2.35)$$

where $\widetilde{u}_{\mu_{ij}}^\pm$ is the derivative of $\widetilde{u}_\mu^\pm$ with respect to each component $(\varepsilon^\pm)_{ij}$. Thus, since

$$\mathbb{C}_\mu = (\mathbb{C}_\mu)_{klpq} e_k \otimes e_l \otimes e_p \otimes e_q \quad \text{and} \quad \nabla^s \widetilde{u}_{\mu_{ij}}^\pm = (\nabla^s \widetilde{u}_{\mu_{ij}}^\pm)_{pq} e_p \otimes e_q \ , \qquad (2.36)$$

the tensor $\widetilde{\mathbb{C}}^{\pm}$ is obtained as

$$\widetilde{\mathbb{C}}^{\pm} = \left[\frac{1}{V_\mu} \int_{\Omega_\mu} (\mathbb{C}_\mu)_{ijpq} (\nabla^s \widetilde{u}^{\pm}_{\mu_{ij}})_{pq} \right] (e_i \otimes e_j \otimes e_k \otimes e_l) \ . \tag{2.37}$$

3. The topological sensitivity of the homogenised elasticity tensor

This section presents the main result of this paper. Here, we derive a closed formula for the sensitivity of the upper and lower bounds for the homogenised elasticity tensor $\mathbb{C}^{\pm}$ to the introduction of a circular inclusion centered at an arbitrary point of the RVE domain. To this end, let ψ be a functional that depends on a given domain and let it have sufficient regularity so that the following expansion is possible

$$\psi(\rho) = \psi(0) + f(\rho) D_T \psi + o(f(\rho)) \ , \tag{3.1}$$

where $\psi(0)$ is the functional evaluated for the original domain and $\psi(\rho)$ denotes the functional evaluated for a topologically perturbed domain. The parameter ρ defines the size of the topological perturbation, so that the original domain is retrieved when $\rho = 0$. In addition, $f(\rho)$ is a *regularising function* defined such that $f(\rho) \to 0$ with $\rho \to 0^+$ and $o(f(\rho))$ contains all terms of higher order in $f(\rho)$. The term $D_T\psi$ of (3.1) is defined as the *topological derivative* of ψ at the unperturbed (original) RVE domain.

The concept of topological derivative was rigorously introduced by [24]. Since then, the notion of topological derivative has proved extremely useful in the treatment of a wide range of problems in mechanics, optimisation, inverse analysis and image processing and has become a subject of intensive research, see for instance, [4, 6, 9, 20, 21].

3.1. **Application to the multi-scale elasticity model.** To begin the topological sensitivity analysis, it is appropriate to define the following functional

$$\psi(\rho) \equiv V_\mu \sigma_\rho \cdot \varepsilon \quad \Rightarrow \quad \psi(0) = V_\mu \sigma \cdot \varepsilon \ , \tag{3.2}$$

where σ_ρ denotes the macroscopic stress tensor associated with a RVE topologically perturbed by a small inclusion of radius ρ and center at $y_0 \in \Omega_\rho$ defined by $\mathcal{I}_\rho$ and σ is the macroscopic stress tensor associated to the unperturbed domain Ω_μ. More precisely, the perturbed domain is defined as $\Omega_{\mu_\rho} = (\Omega_\mu \diagdown \mathcal{H}_\rho) \cup \mathcal{I}_\rho$ (refer to Fig. 2).

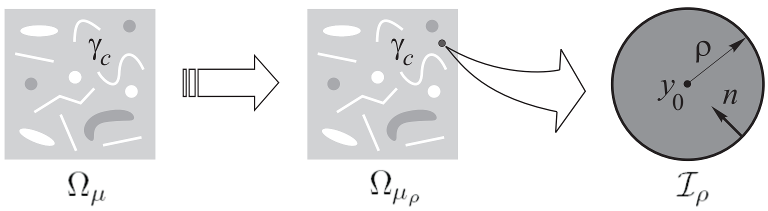

Figure 2. Microstructure perturbed with a inclusion $\mathcal{I}_\varepsilon$.

Thus, the asymptotic topological expansion of the functional (3.2) reads

$$\sigma_\rho \cdot \varepsilon = \sigma \cdot \varepsilon + \frac{1}{V_\mu} f(\rho) D_T \psi + o(f(\rho)) \ . \tag{3.3}$$

3.2. **Topological derivative calculation.** In order to obtain a closed form expression of the asymptotic expansion (3.3), we start here by deriving a closed formula for the associated topological derivative $D_T\psi$. To this end, we define the functional

$$\psi(\rho) \equiv \mathcal{J}_{\Omega_{\mu_\rho}}\left(u_{\mu_\rho}\right) = \int_{\Omega_{\mu_\rho}} \sigma_\mu^*(u_{\mu_\rho}) \cdot \nabla^s u_{\mu_\rho} \ , \tag{3.4}$$

where $\sigma_\mu^*(u_{\mu_\rho})$ is the microscopic stress field associated to perturbed domain Ω_{μ_ρ}. Analogously to the presented in the previous section, the stress tensor field σ_μ^* is defined as

$$\sigma_\mu^*(u_{\mu_\rho}) = \mathbb{C}_\mu^* \nabla^s u_{\mu_\rho} \ , \tag{3.5}$$

where the constitutive tensor $\mathbb{C}_\mu^*$, for $\gamma \in \mathbb{R}^+$, is given by

$$\mathbb{C}_\mu^* = \left\{ \begin{array}{ll} \mathbb{C}_\mu & \forall y \in \Omega_\mu \diagdown \overline{\mathcal{H}_\rho} \\ \gamma \mathbb{C}_\mu & \forall y \in \mathcal{I}_\rho \end{array} \right. . \tag{3.6}$$

Particularly, the microscopic displacement field $u_{\mu_\rho} \in \mathcal{K}_{\mu_\rho} \equiv \{v \in \mathcal{K}_\mu : [\![v]\!] = 0 \text{ on } \partial\mathcal{I}_\rho\}$ is the solution of the variational inequality for the perturbed domain Ω_{μ_ρ}: Find $u_{\mu_\rho} \in \mathcal{K}_{\mu_\rho}$, such that:

$$\int_{\Omega_{\mu_\rho}} \sigma_\mu^*(u_{\mu_\rho}) \cdot \nabla^s(\eta - u_{\mu_\rho}) \geq 0 \quad \forall \eta \in \mathcal{K}_{\mu_\rho} \ , \tag{3.7}$$

where $\mathcal{K}_{\mu_\rho}$ is the space of kinematically admissible displacement fluctuations of the perturbed RVE. In addition, u_{μ_ρ} is decomposed as

$$u_{\mu_\rho}(y) = u + \overline{u}(y) + \widetilde{u}_{\mu_\rho}(y) \ , \tag{3.8}$$

where $u = u(x)$, $\overline{u}(y) = \varepsilon y$ and the fluctuation displacement field $\widetilde{u}_{\mu_\rho} \in \widetilde{\mathcal{K}}_{\mu_\rho} \equiv \{v \in \widetilde{\mathcal{K}}_\mu : [\![v]\!] = 0 \text{ on } \partial\mathcal{I}_\rho\}$ is the solution of the following variational inequality: Find $\widetilde{u}_{\mu_\rho} \in \widetilde{\mathcal{K}}_{\mu_\rho}$, such that

$$\int_{\Omega_{\mu_\rho}} \sigma_\mu^*(\widetilde{u}_{\mu_\rho}) \cdot \nabla^s(\eta - \widetilde{u}_{\mu_\rho}) \geq -\int_{\Omega_{\mu_\rho}} \overline{\sigma}_\mu^* \cdot \nabla^s(\eta - \widetilde{u}_{\mu_\rho}) \qquad \forall \eta \in \widetilde{\mathcal{K}}_{\mu_\rho} \ , \tag{3.9}$$

where $\overline{\sigma}_\mu^* = \sigma_\mu^*(\overline{u})$ is the microscopic stress field, associated to Ω_{μ_ρ}, induced by $\overline{u}$, i.e., $\sigma_\mu^*(\overline{u}) = \mathbb{C}_\mu^*\varepsilon$ and $\sigma_\mu^*(\widetilde{u}_{\mu_\rho})$ is the stress fluctuation field associated with $\widetilde{u}_{\mu_\rho}$, i.e., $\sigma_\mu^*(\widetilde{u}_{\mu_\rho}) = \mathbb{C}_\mu^* \nabla^s \widetilde{u}_{\mu_\rho}$.

By substituting $\eta = 0$ and $\eta = 2\widetilde{u}_{\mu_\rho}$ as test function in (3.9) and sum up the relation obtained, we prove the equality

$$\int_{\Omega_{\mu_\rho}} \left(\sigma_\mu^*(\widetilde{u}_{\mu_\rho}) + \sigma_\mu^*(\overline{u})\right) \cdot \nabla^s \widetilde{u}_{\mu_\rho} = 0 \ . \tag{3.10}$$

Among the methods for calculation of the topological derivative available in the literature, we shall adopt the one proposed in [21], whereby the topological derivative is obtained as

$$D_T\psi = \lim_{\rho \to 0} \frac{1}{f'(\rho)} \frac{d}{d\rho} \mathcal{J}_{\Omega_{\mu_\rho}}\left(u_{\mu_\rho}\right) \ . \tag{3.11}$$

The derivative of the functional $\mathcal{J}_{\Omega_{\mu_\rho}}\left(u_{\mu_\rho}\right)$ with respect to the perturbation parameter ρ can be seen as the sensitivity of $\mathcal{J}_{\Omega_{\mu_\rho}}$, in the classical sense, to the change in shape produced by a uniform expansion of the inclusion.

3.2.1. *Domain decomposition.* Since the problem is non-linear, let us introduce a domain decomposition given by $\Omega_{\mu_R} = \Omega_{\mu_\rho} \setminus \overline{B_R}$, where B_R is a ball of radius $R > \rho$ and center at $y_0 \in \Omega_\mu$, that is $B_R = \{y \in \mathbb{R}^2 : \|y - y_0\| < R\}$, $\overline{B_R}$ is the closure of B_R. Thus, we have the following linear elasticity system defined in B_R with an inclusion B_ρ inside: Find $\widetilde{w}_{\mu_\rho}$, such that

$$\left\{ \begin{array}{rcll} -\mathrm{div}\sigma^*_\mu(\widetilde{w}_{\mu_\rho}) & = & 0 & \text{in} \quad B_R \\ \sigma^*_\mu(\widetilde{w}_{\mu_\rho}) & = & \mathbb{C}^*_\mu \nabla^s \widetilde{w}_{\mu_\rho} & \text{in} \quad B_R \\ \widetilde{w}_{\mu_\rho} & = & \varphi & \text{on} \quad \partial B_R \\ [\![\widetilde{w}_{\mu_\rho}]\!] & = & 0 & \text{on} \quad \partial B_\rho \\ [\![\sigma^*_\mu(\widetilde{w}_{\mu_\rho})]\!]n & = & -(1-\gamma)(\mathbb{C}_\mu \varepsilon)n & \text{on} \quad \partial B_\rho \end{array} \right. . \tag{3.12}$$

We are interested in the Steklov-Poincaré operator on ∂B_R, that is

$$\mathcal{A}_\rho : \varphi \in H^{1/2}(\partial B_R) \mapsto \sigma^*_\mu(\widetilde{w}_{\mu_\rho})n \in H^{-1/2}(\partial B_R) . \tag{3.13}$$

Then we have $\sigma^*_\mu(\widetilde{u}_{\mu_R})n = \mathcal{A}_\rho(\widetilde{u}_{\mu_R})$ on ∂B_R, where $\widetilde{u}_{\mu_R}$ is solution of the variational inequality in Ω_{μ_R}, that is: Find $\widetilde{u}_{\mu_R} \in \widetilde{\mathcal{K}}_{\mu_\rho}$

$$\int_{\Omega_{\mu_R}} \sigma^*_\mu(\widetilde{u}_{\mu_R}) \cdot \nabla^s(\eta - \widetilde{u}_{\mu_R}) + \int_{\partial B_R} \mathcal{A}_\rho(\widetilde{u}_{\mu_R}) \cdot (\eta - \widetilde{u}_{\mu_R}) \geq - \int_{\Omega_{\mu_R}} \sigma^*_\mu(\overline{u}) \cdot \nabla^s(\eta - \widetilde{u}_{\mu_R}) \quad \forall \eta \in \widetilde{\mathcal{K}}_{\mu_\rho} . \tag{3.14}$$

Finally, in the disk B_R we have

$$\int_{B_R} \sigma^*_\mu(\widetilde{w}_{\mu_\rho}) \cdot \nabla^s \widetilde{w}_{\mu_\rho} = \int_{\partial B_R} \mathcal{A}_\rho(\widetilde{w}_{\mu_\rho}) \cdot \widetilde{w}_{\mu_\rho} - \int_{B_R} \sigma^*_\mu(\overline{u}) \cdot \nabla^s \widetilde{w}_{\mu_\rho} , \tag{3.15}$$

where $\widetilde{w}_{\mu_\rho}$ is the solution of the elasticity system in the disk B_R (3.12) or equivalently solution of the following variational problem: Find $\widetilde{w}_{\mu_\rho} \in \widetilde{\mathcal{W}}_{\mu_\rho}$, such that

$$\int_{B_R} \sigma^*_\mu(\widetilde{w}_{\mu_\rho}) \cdot \nabla^s \eta = - \int_{B_R} \sigma^*_\mu(\overline{u}) \cdot \nabla^s \eta \quad \forall \eta \in \widetilde{\mathcal{W}}^0_\mu , \tag{3.16}$$

with $\widetilde{\mathcal{W}}_{\mu_\rho}$ and $\widetilde{\mathcal{W}}^0_{\mu_\rho}$ such that

$$\widetilde{\mathcal{W}}_{\mu_\rho} = \{w \in H^1(B_R)^2 : [\![w]\!] = 0 \text{ on } \partial B_\rho, w = \varphi \text{ on } \partial B_R\} , \tag{3.17}$$

$$\widetilde{\mathcal{W}}^0_{\mu_\rho} = \{w \in H^1(B_R)^2 : [\![w]\!] = 0 \text{ on } \partial B_\rho, w = 0 \text{ on } \partial B_R\} . \tag{3.18}$$

Therefore, we can define the microscopic displacement field defined in the disk B_R as the sum

$$w_{\mu_\rho}(y) = u + \overline{u}(y) + \widetilde{w}_{\mu_\rho}(y) . \tag{3.19}$$

3.2.2. *Shape sensitivity analysis of the energy functional.* Let us introduced the energy-based shape functional defined in the disk B_R, that is

$$\mathcal{E}_\rho(w_{\mu_\rho}) := \int_{B_R} \sigma^*_\mu(w_{\mu_\rho}) \cdot \nabla^s w_{\mu_\rho} . \tag{3.20}$$

Thus, we need to calculate

$$\begin{aligned} \frac{d}{d\rho}\mathcal{E}_\rho(w_{\mu_\rho}) &= 2\int_{B_R} \sigma^*_\mu(w_{\mu_\rho}) \cdot \nabla^s \dot{\widetilde{w}}_{\mu_\rho} + \int_{B_R} \Sigma_\mu \cdot \nabla V \\ &= 2\int_{B_R} (\sigma^*_\mu(\widetilde{w}_{\mu_\rho}) - \sigma^*_\mu(\overline{u})) \cdot \nabla^s \dot{\widetilde{w}}_{\mu_\rho} + \int_{B_R} \Sigma_{\mu_\rho} \cdot \nabla V , \end{aligned} \tag{3.21}$$

which was obtained using the Reynold's transport theorem and the concept of material derivatives of spacial fields [12, 26]. Some of the terms in (3.21) require explanation. Vector

V represents the shape change velocity field defined on the disk B_R and such that $V = 0$ on ∂B_R and $V = -n$ on ∂B_ρ. Thus, $\dot{\widetilde{w}}_{\mu_\rho} \in \widetilde{\mathcal{W}}_\mu^0$ is the material (total) derivative with respect to ρ. Finally, the Eshelby energy-momentum tensor Σ_{μ_ρ} takes the form [8, 13]

$$\Sigma_{\mu_\rho} := \sigma_\mu^*(w_{\mu_\rho}) \cdot \nabla^s w_{\mu_\rho} \mathrm{I} - 2(\nabla \widetilde{w}_{\mu_\rho})^T \sigma_\mu^*(w_{\mu_\rho}) \ . \tag{3.22}$$

Since $\dot{\widetilde{w}}_{\mu_\rho} \in \widetilde{\mathcal{W}}_{\mu_\rho}^0$ and considering that Σ_{μ_ρ} is a free-divergence tensor field ($\mathrm{div}\Sigma_{\mu_\rho} = 0$), the shape derivative of the energy functional becomes

$$\frac{d}{d\rho}\mathcal{E}_\rho(w_{\mu_\rho}) = -\int_{\partial B_\delta} [\![\Sigma_{\mu_\rho}]\!] n \cdot n \ , \tag{3.23}$$

where we have used the fact that $\widetilde{w}_{\mu_\rho}$ is solution to (3.16), namely

$$\int_{B_R} (\sigma_\mu^*(\widetilde{w}_{\mu_\rho}) - \sigma_\mu^*(\overline{u})) \cdot \nabla^s \dot{\widetilde{w}}_{\mu_\rho} = 0 \quad \forall \dot{\widetilde{w}}_{\mu_\rho} \in \widetilde{\mathcal{W}}_{\mu_\rho}^0 \ . \tag{3.24}$$

3.2.3. *Calculation of the limit $\rho \to 0$.* By using formula (3.23) together with (3.11), the topological derivative can be obtained from the following result

$$D_T\psi = -\lim_{\rho \to 0} \frac{1}{f'(\rho)} \int_{\partial \mathcal{I}_\delta} [\![\Sigma_{\mu_\rho}]\!] n \cdot n \ . \tag{3.25}$$

Remark 2. *For the simplest class of multi-scale models given by the rule of mixtures (or Taylor) model, the tensor Σ_{μ_ρ} is given by*

$$\Sigma_{\mu_\rho} = (\sigma_\mu^* \cdot \varepsilon)\mathrm{I} \ . \tag{3.26}$$

Then, substituting (3.26) into definition (3.25) of the topological derivative and identifying function $f(\rho)$ as the size of the perturbation $\mathcal{I}_\rho$, i.e., $f(\rho) = \pi\rho^2$, we find that, for the rule of mixtures model, the topological derivative is given by

$$D_T^{\mathcal{T}}\psi = -(1-\gamma)\sigma_\mu \cdot \varepsilon \ . \tag{3.27}$$

In order to derive an explicit expression for the integrand on the right hand side of (3.25), we use the existence of the asymptotic expansions for $\widetilde{w}_{\mu_\rho}$, solution of the elasticity system (3.12) defined in the disk $B_R \subset \mathbb{R}^2$, in the neighborhood of B_ρ, namely

$$\widetilde{w}_{\mu_\rho}(y) = \widetilde{w}_\mu(y) + \widetilde{w}_\mu^\infty(y) + o(\rho) \ . \tag{3.28}$$

In addition, $\widetilde{w}_\mu^\infty$ is proportional to ρ, $\|\widetilde{w}_\mu^\infty\|_{\mathbb{R}^2} = O(\rho)$, on the surface ∂B_ρ of the ball. The expansion of $\sigma_\mu(\widetilde{w}_{\mu_\rho})$ corresponding to (3.12) has the form

$$\sigma_\mu(\widetilde{w}_{\mu_\rho}) = \sigma_\mu^\infty(\widetilde{w}_{\mu_0}(y_0), y) + O(\rho) \ . \tag{3.29}$$

where σ_μ^∞ is the stress distribution around a circular inclusion in an infinity medium and $\widetilde{w}_\mu$ is solution of the elasticity system (3.12) defined in the disk $B_R \subset \mathbb{R}^2$ for $\rho = 0$. Thus, σ_μ^∞ can be calculated explicitly, which is given in a polar coordinate system (r, θ) by:

- for $r \geq \rho$

$$
\begin{aligned}
(\sigma_\mu^\infty)^{rr} &= \widetilde{\alpha}\left(1-\frac{1-\gamma}{1+\gamma a}\frac{\rho^2}{r^2}\right)+\widetilde{\beta}\left(1-4\frac{1-\gamma}{1+\gamma b}\frac{\rho^2}{r^2}+3\frac{1-\gamma}{1+\gamma b}\frac{\rho^4}{r^4}\right)\cos 2\theta \\
&- \overline{\alpha}\frac{1-\gamma}{1+\gamma a}\frac{\rho^2}{r^2}-\overline{\beta}\left(4\frac{1-\gamma}{1+\gamma b}\frac{\rho^2}{r^2}-3\frac{1-\gamma}{1+\gamma b}\frac{\rho^4}{r^4}\right)\cos 2(\theta+\phi)\ , \qquad (3.30) \\
(\sigma_\mu^\infty)^{\theta\theta} &= \widetilde{\alpha}\left(1+\frac{1-\gamma}{1+\gamma a}\frac{\rho^2}{r^2}\right)-\widetilde{\beta}\left(1+3\frac{1-\gamma}{1+\gamma b}\frac{\rho^4}{r^4}\right)\cos 2\theta \\
&+ \overline{\alpha}\frac{1-\gamma}{1+\gamma a}\frac{\rho^2}{r^2}-3\overline{\beta}\frac{1-\gamma}{1+\gamma b}\frac{\rho^4}{r^4}\cos 2(\theta+\phi)\ , \qquad (3.31) \\
(\sigma_\mu^\infty)^{r\theta} &= -\widetilde{\beta}\left(1+2\frac{1-\gamma}{1+\gamma b}\frac{\rho^2}{r^2}-3\frac{1-\gamma}{1+\gamma b}\frac{\rho^4}{r^4}\right)\sin 2\theta \\
&- \overline{\beta}\left(2\frac{1-\gamma}{1+\gamma b}\frac{\rho^2}{r^2}-3\frac{1-\gamma}{1+\gamma b}\frac{\rho^4}{r^4}\right)\sin 2(\theta+\phi)\ ; \qquad (3.32)
\end{aligned}
$$

- for $0 < r < \rho$

$$
\begin{aligned}
(\sigma_\mu^\infty)^{rr} &= 2\frac{\gamma a}{1+\gamma a}\frac{\widetilde{\alpha}}{1-\nu}+4\frac{\gamma b}{1+\gamma b}\frac{\widetilde{\beta}}{3-\nu}\cos 2\theta \\
&+ \gamma a\frac{1-\gamma}{1+\gamma a}\overline{\alpha}+\gamma b\frac{1-\gamma}{1+\gamma b}\overline{\beta}\cos 2(\theta+\phi)\ , \qquad (3.33) \\
(\sigma_\mu^\infty)^{\theta\theta} &= 2\frac{\gamma a}{1+\gamma a}\frac{\widetilde{\alpha}}{1-\nu}-4\frac{\gamma b}{1+\gamma b}\frac{\widetilde{\beta}}{3-\nu}\cos 2\theta \\
&+ \gamma a\frac{1-\gamma}{1+\gamma a}\overline{\alpha}-\gamma b\frac{1-\gamma}{1+\gamma b}\overline{\beta}\cos 2(\theta+\phi)\ , \qquad (3.34) \\
(\sigma_\mu^\infty)^{r\theta} &= -4\frac{\gamma b}{1+\gamma b}\frac{\widetilde{\beta}}{3-\nu}\sin 2\theta-\gamma b\frac{1-\gamma}{1+\gamma b}\overline{\beta}\sin 2(\theta+\phi)\ . \qquad (3.35)
\end{aligned}
$$

In the above formulas, ϕ denotes the angle between the eigenvector of tensors $\sigma_\mu(w_\mu(y_0))$ and $\sigma_\mu(\overline{u})$. The coefficients $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\overline{\alpha}$, $\overline{\beta}$ are given respectively by

$$
\widetilde{\alpha}=\frac{1}{2}(\widetilde{\sigma}_1+\widetilde{\sigma}_2),\ \widetilde{\beta}=\frac{1}{2}(\widetilde{\sigma}_1-\widetilde{\sigma}_2) \quad \text{and} \quad \overline{\alpha}=\frac{1}{2}(\overline{\sigma}_1+\overline{\sigma}_2),\ \overline{\beta}=\frac{1}{2}(\overline{\sigma}_1-\overline{\sigma}_2)\ , \qquad (3.36)
$$

where $\widetilde{\sigma}_{1,2}$ and $\overline{\sigma}_{1,2}$ are the eigenvalues of tensors $\sigma_\mu(w_\mu(y_0))$ and $\sigma_\mu(\overline{u})$, respectively. In addition, constants a and b are given by

$$
a=\frac{1+\nu}{1-\nu} \quad \text{and} \quad b=\frac{3-\nu}{1+\nu}\ . \qquad (3.37)
$$

Finally, considering formulas (3.30)-(3.34) together with (3.19) in (3.25), we can calculate the integral on $\partial\mathcal{I}_\rho$ explicitly, which allows to identify function $f(\rho)=\pi\rho^2$. Then, after calculating the limit $\rho\to 0$, we obtain the following result:

Theorem 3. *The energy shape functional admits for $\rho \to 0$ the following topological asymptotic expansion*

$$
\sigma_\rho\cdot\varepsilon=\sigma\cdot\varepsilon+v(\rho)\mathbb{H}_\gamma\sigma_\mu(u_\mu)\cdot\sigma_\mu(u_\mu)+o\left(v\left(\rho\right)\right)\ , \qquad (3.38)
$$

where u_μ is solution of the variational inequality (2.13) in $\Omega_\mu \subset \mathbb{R}^2$, $v(\rho)$ is the volume fraction of the inclusion, namely,

$$
v(\rho)=\frac{\pi\rho^2}{V_\mu} \qquad (3.39)
$$

and $\mathbb{H}_\gamma$ is a forth-order tensor defined as

$$\mathbb{H}_\gamma = -\frac{1}{E}\left(\frac{1-\gamma}{1+a\gamma}\right)\left[4\mathbb{I} + \frac{\gamma(a-2b)-1}{1+b\gamma}\mathrm{I}\otimes\mathrm{I}\right] . \tag{3.40}$$

Corollary 4. *Let us consider the contrast $\gamma \to 0$. Thus, the elastic inclusion degenerates to a circular cavity with homogeneous Neumann boundary condition and the tensor $\mathbb{H}_0$ becomes*

$$\mathbb{H}_0 = -\frac{1}{E}\left[4\mathbb{I} - \mathrm{I}\otimes\mathrm{I}\right] . \tag{3.41}$$

Corollary 5. *Let us consider the contrast $\gamma \to \infty$. Thus, the elastic inclusion degenerates to rigid one and the tensor $\mathbb{H}_\infty$ takes the form*

$$\mathbb{H}_\infty = \frac{1}{aE}\left[4\mathbb{I} + \frac{a-2b}{b}\mathrm{I}\otimes\mathrm{I}\right] . \tag{3.42}$$

3.3. **The sensitivity of the macroscopic elasticity tensor.** Let us consider again the particular case associated to the dilating ε^+ and compressive ε^- spherical strain tensors. Thus, by differentiating twice the topological asymptotic expansion (3.38) with respect to the macroscopic strain tensor $\varepsilon^\pm$ we obtain

$$\mathbb{C}^\pm_\rho = \mathbb{C}^\pm + v(\rho)\mathbb{D}^\pm_{T\mu} + o\left(v\left(\rho\right)\right) , \tag{3.43}$$

where $\mathbb{C}^\pm_\rho$ are the upper and lower bounds for the homogenised elasticity tensor of the topologically perturbed RVE, $\mathbb{C}^\pm$ are the upper and lower bounds for the homogenised elasticity tensor of the unperturbed RVE and $v(\rho) = \pi\rho^2/V_\mu$ is the volume fraction of the perturbation. In addition, $\mathbb{D}^\pm_{T\mu}$ is the fourth order symmetric tensor field over Ω_μ defined by (with $i, j, k, l = 1, 2$)

$$\mathbb{D}^\pm_{T\mu} = (\mathbb{H}_\gamma\sigma^\pm_{\mu_{ij}} \cdot \sigma^\pm_{\mu_{kl}})e_i \otimes e_j \otimes e_k \otimes e_l , \tag{3.44}$$

where $\sigma^\pm_{\mu_{ij}}$ is the canonical stress field defined analogously to (2.14), that is

$$\sigma^\pm_{\mu_{ij}} = \mathbb{C}^\pm_\mu(e_i \otimes e_j) + \sigma_\mu(\widetilde{u}^\pm_{\mu_{ij}}) , \tag{3.45}$$

with $\widetilde{u}^\pm_{\mu_{ij}}$ solution to the set of canonical variational problems given by (2.35).

Remark 6. *The topological sensitivity tensor (3.44) provides a first order accurate measure of how the macroscopic elasticity tensor varies when a topological perturbation is added to the RVE. Each Cartesian component $(\mathbb{D}^\pm_{T\mu})_{ijkl}$ represents the derivative of the component $ijkl$ of the macroscopic elasticity tensor with respect to the volume fraction $v(\rho)$ of a circular inclusion of radius ρ inserted at an arbitrary point y of the RVE. The remarkable simplicity of the closed form sensitivity given by (3.44) is to be noted. Once the vector fields $\widetilde{u}^\pm_{\mu_{ij}}$ have been obtained as solutions of (2.35) for the* original *RVE domain, the sensitivity tensor can be trivially assembled.*

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