SHAPE AND TOPOLOGY SENSITIVITY ANALYSIS FOR CRACKS IN ELASTIC BODIES ON BOUNDARIES OF RIGID INCLUSIONS

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Abstract. We consider an elastic body with a rigid inclusion and a crack located at the boundary of the inclusion. It is assumed that non-penetration conditions are imposed at the crack faces which do not allow the opposite crack faces to penetrate each other. We analyze the variational formulation of the problem and provide shape and topology sensitivity analysis of the solution in two and three spatial dimensions. The differentiability of the energy with respect to the crack length, for the crack located at the boundary of rigid inclusion, is established.

1. Introduction

The problem associated to cracks in elastic bodies on boundaries of rigid inclusions appears in a vast number of applications in civil, mechanical, aerospace, biomedical and nuclear industries. In particular, some classes of materials are composed by a bulk phase with inclusions inside. When the inclusions are much stiffer than the bulk material, we can treat them as rigid inclusions. In addition, it is quite common to have cracks between both phases. Thus, in this paper we deal with the mechanical modelling as well as the shape and topology sensitivity analysis associated to the limit case of rigid inclusions in elastic bodies with a crack at the interface.

The mechanical modelling is based on the assumption of non-penetration conditions at the crack faces between the elastic material and the rigid inclusion, which do not allow the opposite crack faces to penetrate each other, leading to a new class of variational inequalities. For the sensitivity analysis, we attempt to find the shape derivative of the elastic energy with respect to the perturbations of the crack tip. We also obtain the topological derivatives of the energy shape functional associated to the nucleation of a smooth imperfection in the bulk elastic material. These quantities are very important in design procedures and in numerical solution of some inverse problems. Both the analysis and the shape and topology optimization of this class of problems seem to be new and very useful from the mathematical and also the mechanical point of views.

The paper is organized as follows. The problem formulation associated to cracks in elastic bodies on boundaries of rigid inclusions is presented in Section 2. Some results concerning shape sensitivity analysis with respect to the perturbations of the crack tip are given with all details in Section 3. The topological derivatives associated to the energy shape functional are calculated in Section 4. We provide some closed formulas for the case of nucleation of spherical holes in three spatial dimensions and circular elastic inclusions in two spatial dimensions. In 2D case, two limit passages with respect to the so-called *contrast factor* are performed in such ways that the elastic inclusion become a void or a rigid inclusion.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and $\omega \subset \Omega$ be a subdomain with smooth boundary Ξ such that $\overline{\omega} \cap \Gamma = \emptyset$. We assume that Ξ consists of two parts γ and $\Xi \setminus \gamma$, $meas(\Xi \setminus \gamma) > 0$, where γ is a smooth 2D surface described as

$$x_i = x_i(y_1, y_2), \quad (y_1, y_2) \in D, \ i = 1, 2, 3,$$

with bounded domain $D \subset \mathbb{R}^2$ having a smooth boundary ∂D, and the rank of the matrix $\frac{\partial x}{\partial y}$ is equal to 2.

Denote by $\nu = (\nu_1, \nu_2, \nu_3)$ a unit outward normal vector to Ξ, see Fig. 1. The subdomain ω is assumed to correspond to a rigid inclusion, and the surface γ describes a crack located on Ξ.

Key words and phrases. Rigid inclusion, crack growth, unilateral condition, frictionless contact, shape derivative, topological derivative, asymptotic analysis.

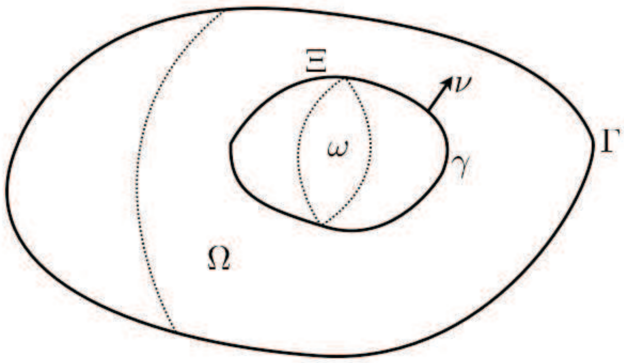

FIGURE 1. Domain Ω with rigid inclusion ω

Domain $\Omega \setminus \overline{\omega}$ corresponds to the elastic part of the body. For the further use we introduce the space of infinitesimal rigid displacements

$$R(\omega) = \{\rho = (\rho_1, \rho_2, \rho_3) \mid \rho(x) = Bx + C,\ x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}, \quad C = (c^1, c^2, c^3); \quad b_{ij}, c^i = const,\ i, j = 1, 2, 3.$$

Denote $\Omega_\gamma = \Omega \setminus \overline{\gamma}$. Problem formulation describing an equilibrium of the elastic body with the rigid inclusion ω and the crack γ is as follows. In the domain Ω_γ, we have to find functions $u = (u_1, u_2, u_3)$, $u = \rho_0$ on ω; $\rho_0 \in R(\omega)$; and in the domain $\Omega \setminus \overline{\omega}$ we have to find functions $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2, 3$, such that

$$-\mathrm{div}\sigma = F \quad \text{in} \quad \Omega \setminus \overline{\omega}, \tag{2.1}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in} \quad \Omega \setminus \overline{\omega}, \tag{2.2}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{2.3}$$

$$(u - \rho_0) \cdot \nu \geq 0 \quad \text{on} \quad \gamma^+, \tag{2.4}$$

$$\sigma_\tau = 0,\ \sigma_\nu \leq 0 \quad \text{on} \quad \gamma^+, \tag{2.5}$$

$$\sigma_\nu (u - \rho_0) \cdot \nu = 0 \quad \text{on} \quad \gamma^+, \tag{2.6}$$

$$-\int\limits_{\Xi} \sigma\nu \cdot \rho = \int\limits_{\omega} F \cdot \rho \quad \forall \rho \in R(\omega). \tag{2.7}$$

Here $F = (F_1, F_2, F_3) \in L^2(\Omega)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$

$$\sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2, \sigma_\tau^3), \quad \sigma\nu = \{\sigma_{ij}\nu_j\}_{i=1}^{i=3},$$

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ i, j = 1, 2, 3.$$

All functions with two lower indices are assumed to be symmetric in those indices. Summation convention over repeated indices is accepted throughout the paper. Elasticity tensor $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2, 3$, is given, and it satisfies usual symmetry and positive definiteness properties,

$$a_{ijkl} = a_{klij} = a_{jikl}, \quad a_{ijkl} \in L^\infty(\Omega),\ i, j, k, l = 1, 2, 3,$$

$$a_{ijkl}\xi_{kl}\xi_{ij} \geq c_0|\xi|^2, \quad \forall\ \xi_{ij} = \xi_{ji}, \quad c_0 = const.$$

In addition, we consider the isotropic case, namely

$$A = 2m\mathbb{I} + l\,(\mathrm{I} \otimes \mathrm{I})\ , \tag{2.8}$$

where I and $\mathbb{I}$ respectively are the second and fourth order identity tensors and, m and l are the Lamé coefficients, which can be defined in terms of the Young modulus E and the Poisson ratio υ as

$$m = \frac{E}{2(1+\upsilon)} \qquad \text{and} \qquad l = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)}\ . \tag{2.9}$$

Relations (2.1) are equilibrium equations, and (2.2) corresponds to the Hooke's law. Inequality (2.4) describes a mutual nonpenetration between crack faces $\gamma^{\pm}$. The first relation in (2.5) means a zero friction between the crack faces. For simplicity we assume clamping condition (2.3) on Γ.

Note that external forces F are applied to $\Omega \setminus \overline{\omega}$ as well as to ω, but there are no equilibrium equations in ω. Influence of these forces is taken into account through (2.7). If we have no crack γ on Ξ, relations (2.4)-(2.6) should be omitted. This specific problem formulation for the particular case $F = 0$ in ω can be found in [25]. The problem formulation with the crack and nonpenetration conditions seems to be new.

First of all we provide a variational formulation of problem (2.1)-(2.7). To this end, let us consider the Sobolev space

$$H^{1,\omega}_{\Gamma}(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma)^3 \mid \varepsilon(v) = 0 \text{ on } \omega;\ v = 0 \text{ on } \Gamma\} \tag{2.10}$$

and define the set of admissible displacements

$$K_\omega = \{v \in H^{1,\omega}_{\Gamma}(\Omega_\gamma) \mid \varepsilon(v) = 0 \text{ on } \omega;\ (v^+ - v^-)\cdot \nu \geq 0 \text{ on } \gamma\}. \tag{2.11}$$

Let $(\cdot,\cdot)_{\Omega\setminus\overline{\omega}}$ be the inner product in $L^2(\Omega \setminus \overline{\omega})$. Consider the energy functional

$$\Pi(v) = \frac{1}{2}(\sigma(v), \varepsilon(v))_{\Omega\setminus\overline{\omega}} - (F, v)_{\Omega_\gamma}, \tag{2.12}$$

where the stress field $\sigma(v) = \sigma$ is defined in (2.2) for $u = v$, and introduce the following minimization problem

$$\inf_{v \in K_\omega} \Pi(v). \tag{2.13}$$

The convex cone K_ω is weakly closed in the space $H^{1,\omega}_{\Gamma}(\Omega_\gamma)$, and the functional Π is coercive and weakly lower semicontinuous on the same space. Hence, by the standard result in the calculus of variations problem (2.13) admits a solution satisfying the variational inequality

$$u \in K_\omega, \tag{2.14}$$

$$(\sigma(u), \varepsilon(\overline{u} - u))_{\Omega\setminus\overline{\omega}} \geq (F, \overline{u} - u)_{\Omega_\gamma} \quad \forall\, \overline{u} \in K_\omega. \tag{2.15}$$

Since bilinear form is coercive, the solution u of problem (2.14)-(2.15) is unique and Lipschitz continuous with respect to data.

Assuming that the solution of (2.14)-(2.15) is sufficiently smooth we can derive all relations (2.1)-(2.7), and conversely, any smooth solution of (2.1)-(2.7) satisfies (2.14)-(2.15). On the other hand, even if the solution to (2.14)-(2.15) is sufficiently smooth, it does not imply (2.5)-(2.6) in a pointwise sense, and this point requires further explanations. To this end, introduce the weighted Sobolev space

$$H^{1/2}_{00}(\gamma) = \{v \in H^{1/2}(\gamma) \mid \int\limits_\gamma \frac{v^2}{r} < +\infty\},$$

where the weight $1/r$ is the inverse of the distance function $r(x) = dist(x, \partial\gamma)$, and denote by $H^{-1/2}_{00}(\gamma)$ the dual space to $H^{1/2}_{00}(\gamma)$, with the duality pairing obtained by the extension of the $L^2(\gamma)$ scalar product to the pair $H^{-1/2}_{00}(\gamma)$ and $H^{1/2}_{00}(\gamma)$. Since σ, div$\sigma \in L^2(\Omega \setminus \overline{\omega})$ it follows that (cf. [21])

$$\sigma^i_\tau,\ \sigma_\nu \in H^{1/2}_{00}(\gamma),\ i = 1,2,3.$$

It can be shown that the first relation of (2.5) holds in the sense

$$\langle \sigma^i_\tau, \psi\rangle^{00}_{1/2,\gamma} = 0 \quad \forall\, \psi \in H^{1/2}_{00}(\gamma),\ i = 1,2,3, \tag{2.16}$$

and the second one holds as

$$\langle \sigma_\nu, \psi \rangle^{00}_{1/2,\gamma} \leq 0 \quad \forall\, \psi \in H^{1/2}_{00}(\gamma),\ \psi \geq 0, \tag{2.17}$$

where $\langle \cdot, \cdot \rangle^{00}_{1/2,\gamma}$ stands for the duality pairing between $H^{-1/2}_{00}(\gamma)$ and $H^{1/2}_{00}(\gamma)$. Condition (2.6) is fulfilled as follows

$$\langle \sigma_\nu, (u^+ - \rho_0) \cdot \nu \rangle^{00}_{1/2,\gamma} = 0,$$

and relation (2.7) holds in the sense

$$\langle \sigma\nu, \rho \rangle_{1/2,\Xi} = -\int\limits_\omega F \cdot \rho \quad \forall \rho \in R(\omega), \tag{2.18}$$

where $\langle \cdot, \cdot \rangle_{1/2,\Xi}$ stands for the duality pairing between $H^{-1/2}(\Xi)$ and $H^{1/2}(\Xi)$.

2.1. **Dual problem formulation.** We introduce the dual formulation of problem (2.14)-(2.15) in stresses. By this approach a solution of dual problem $\sigma = \{\sigma_{ij}\}$ in the domain $\Omega \setminus \overline{\omega}$ is defined, and moreover, we show that a solution of dual problem given by stresses σ_{ij} coincides with the solution $\sigma_{ij} = \sigma_{ij}(u)$ given by (2.14)-(2.15). Below we provide rigorous explanations of the procedure. First, we need the deformations in terms of stresses, thus we write Hooke's law (2.2) in the inverted form

$$A^{-1}\sigma = \varepsilon(u) \text{ in } \Omega \setminus \overline{\omega}. \tag{2.19}$$

Note that the tensor A^{-1} enjoys the properties similar to those of A, i.e., it is symmetric and positive definite. Consider the space of stresses

$$H = \{\sigma = \{\sigma_{ij}\} \mid \sigma_{ij} \in L^2(\Omega \setminus \overline{\omega}),\ i,j = 1,2,3\}$$

and the quadratic functional G defined on H,

$$G(\sigma) = \frac{1}{2}(A^{-1}\sigma, \sigma)_{\Omega\setminus\overline{\omega}}.$$

The set of admissible stresses is a cone in the space H with the elements which satisfy the sign condition for normal stresses on the crack as well as the global equilibrium condition over the inclusion, thus it is defined as follows

$$M = \{\sigma \in H \mid \text{equations (2.1) and conditions (2.5), (2.7) hold}\}.$$

The above cone is well defined since equations (2.1) in the definition of M are satisfied in the sense of distributions, and conditions (2.5), (2.7) hold in a weak sense defined in (2.16)-(2.18).

Let us recall that by weak convergence we mean the convergence in the scalar product only, a sequence v_n with $n \to \infty$ weakly converges in the Hilbert space $\mathcal{H}$ to an element $v \in \mathcal{H}$ provided that

$$\lim_{n\to\infty} (v_n, \varphi)_{\mathcal{H}} = (v, \varphi)_{\mathcal{H}} \quad \forall \varphi \in \mathcal{H}\ ,$$

which we use for the space of square integrable functions, e.g., $\mathcal{H} = L^2(\Omega \setminus \overline{\omega})$, as well as in the Sobolev spaces and its dual spaces. For more details on mathematical analysis in the same spirit, and in application to boundary value problems of linearized elasticity in nonsmooth domains we refer e.g. to [9].

Notice that M is a convex and weakly closed cone in the space H, i.e., the limit of a weakly convergent sequence in M belongs to M. Indeed, if $\sigma^n \in M$ and

$$\sigma^n \to \sigma \text{ weakly in } H,\ n \to \infty,$$

we have

$$-\text{div}\sigma^n = F \text{ in } \Omega \setminus \overline{\omega}.$$

Thus

$$-\text{div}\sigma = F \text{ in } \Omega \setminus \overline{\omega}. \tag{2.20}$$

Since

$$\sigma^{in}_\tau,\ \sigma^n_\nu \text{ are bounded in } H^{-1/2}_{00}(\gamma),\ i = 1,2,3,$$

we can assume that as $n \to \infty$

$$\sigma_\tau^{in},\ \sigma_\nu^n \to \sigma_\tau^i,\ \sigma_\nu \text{ weakly in } H_{00}^{-1/2}(\gamma),\ i = 1,2,3.$$

Consequently, the limit function σ satisfies (2.16)-(2.18) which, in view of (2.20), show that the cone M is weakly closed. We need the property since by direct method of the calculus of variations it follows that a weak limit of the minimizing sequence for our variational problem belongs to M, and therefore, there is a solution to the dual problem. On the other hand, the solution is given by a variational inequality, which admits a unique solution.

Let us consider now the dual problem in the form of the minimization problem for quadratic functional over a convex and weakly closed cone,

$$\inf_{\sigma \in M} G(\sigma). \tag{2.21}$$

Under our assumptions there exists a unique solution σ^0 of this problem which satisfies the following variational inequality

$$\sigma^0 \in M, \tag{2.22}$$

$$(A^{-1}\sigma^0, \overline{\sigma} - \sigma^0)_{\Omega \setminus \overline{\omega}} \geq 0 \quad \forall\, \overline{\sigma} \in M. \tag{2.23}$$

Now we denote by $\sigma = \sigma(u)$ the stress field found from (2.2) for the displacement field $u \in K_\omega$ of (2.14)-(2.15), and prove that we have the equality of stress fields $\sigma^0 = \sigma$ for dual and primal formulations of our problem.

The following relation holds

$$G(\sigma^0) = G(\sigma^0 - \sigma) + G(\sigma) + (A^{-1}(\sigma^0 - \sigma), \sigma)_{\Omega \setminus \overline{\omega}}. \tag{2.24}$$

Introduce the notation for the last term,

$$p = (A^{-1}(\sigma^0 - \sigma), \sigma)_{\Omega \setminus \overline{\omega}}$$

and let us show that $p \geq 0$. Indeed, by the Green formula we derive

$$\begin{aligned} p &= \int\limits_{\Omega \setminus \overline{\omega}} (\sigma_{ij}^0 - \sigma_{ij})\varepsilon_{ij}(u) \\ &= -\int\limits_{\Omega \setminus \overline{\omega}} \operatorname{div}(\sigma^0 - \sigma) \cdot u - \int\limits_{\Xi^+} (\sigma^0 - \sigma)\nu \cdot u. \end{aligned}$$

Since the stresses σ^0, σ satisfy (2.1), this relation implies

$$p = -\int\limits_{\Xi^+} (\sigma^0 - \sigma)\nu \cdot u. \tag{2.25}$$

In its own turn, (2.25) can be rewritten as

$$p = \int\limits_{\Xi^+} (\sigma^0 - \sigma)\nu \cdot u - \int\limits_{\gamma^+} (\sigma^0 - \sigma)\nu \cdot (u - \rho_0), \tag{2.26}$$

where the rigid body motion ρ_0 is given by the restriction of the displacement field u to the rigid inclusion ω. Stresses σ, σ^0 satisfy also (2.7), thus, by accounting (2.5), formula (2.26) takes the form

$$p = -\int\limits_{\gamma^+} (\sigma_\nu^0 - \sigma_\nu)(u - \rho_0) \cdot \nu.$$

Moreover, σ, u satisfy (2.6), hence

$$p = -\int\limits_{\gamma^+} \sigma_\nu^0 (u - \rho_0) \cdot \nu. \tag{2.27}$$

In view of (2.4) $\sigma^0 \in M$, whence, from (2.27) it follows that $p \geq 0$. So we have $G(\sigma^0 - \sigma) \geq 0$, $p \geq 0$, and the relation (2.24) implies

$$G(\sigma^0) \geq G(\sigma).$$

By the uniqueness of the solution σ^0 to (2.21), we obtain $\sigma^0 = \sigma$ which completes the proof.

2.2. **Passage from elastic inclusion to rigid inclusion.** In fact, problem (2.1)-(2.7) can be considered as a limit problem for a family of elasticity problems with the crack γ formulated in the domain Ω_γ. This means that we can construct a family of problems depending on a positive parameter λ such that for any fixed $\lambda > 0$ the problem describes the equilibrium state of an elastic body occupying the domain Ω_γ with the crack γ. We expect that a rigid inclusion ω is obtained for $\lambda \to 0$, i.e., for such a limit any point $x \in \omega$ has a displacement $\rho_0(x)$, $\rho_0 \in R(\omega)$. In what follows we provide a rigorous proof of the above statement. Introduce the tensor $A^\lambda = \{a^\lambda_{ijkl}\}$, $i, j, k, l = 1, 2, 3$,

$$a^\lambda_{ijkl} = \begin{cases} a_{ijkl} & \text{in} \quad \Omega \setminus \overline{\omega} \\ \lambda^{-1} a_{ijkl} & \text{in} \quad \omega, \end{cases}$$

and consider the following problem.
In the domain Ω_γ, we have to find functions $u^\lambda = (u^\lambda_1, u^\lambda_2, u^\lambda_3)$, $\sigma^\lambda = \{\sigma^\lambda_{ij}\}$, $i, j = 1, 2, 3$, such that

$$-\mathrm{div}\sigma^\lambda = F \quad \text{in} \quad \Omega_\gamma, \tag{2.28}$$

$$\sigma^\lambda - A^\lambda \varepsilon(u^\lambda) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{2.29}$$

$$u^\lambda = 0 \quad \text{on} \quad \Gamma, \tag{2.30}$$

$$[u^\lambda] \cdot \nu \geq 0, \ [\sigma^\lambda_\nu] = 0, \ \sigma^\lambda_\nu [u] \cdot \nu = 0 \quad \text{on} \quad \gamma, \tag{2.31}$$

$$\sigma^\lambda_\nu \leq 0, \ \sigma^\lambda_\tau = 0 \quad \text{on} \quad \gamma^\pm. \tag{2.32}$$

Here we use notations of the previous section, and $[v] = v^+ - v^-$ is a jump of v on γ, where $\pm$ fit positive and negative crack faces $\gamma^\pm$ with respect to the normal vector ν.

For any fixed $\lambda > 0$ problem (2.28)-(2.32) is well known (see [21, 22, 19]). Such a problem admits a variational formulation. Indeed, introduce the set of admissible displacements

$$K = \{v \in H^1_\Gamma(\Omega_\gamma)^3 \mid [v] \cdot \nu \geq 0 \text{ on } \gamma\},$$

where

$$H^1_\Gamma(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma\}.$$

There exists a unique solution u^λ of the minimization problem

$$\inf_{v \in K} \{\frac{1}{2}(\sigma^\lambda(v), \varepsilon(v))_{\Omega_\gamma} - (F, v)_{\Omega_\gamma}\} \tag{2.33}$$

with the stress field $\sigma^\lambda(v)$ determined by (2.29) for $u^\lambda = v$. Solution u^λ of the minimization problem satisfies the variational inequality

$$u^\lambda \in K, \tag{2.34}$$

$$(\sigma^\lambda(u^\lambda), \varepsilon(\overline{u} - u^\lambda))_{\Omega_\gamma} \geq (F, \overline{u} - u^\lambda)_{\Omega_\gamma} \ \forall \, \overline{u} \in K. \tag{2.35}$$

By the convexity of the quadratic functional in (2.33) with respect to v, it follows that problems (2.33) and (2.34)-(2.35) are equivalent. Moreover, all relations (2.28)-(2.32) can be obtained from (2.34)-(2.35), and conversely, relations (2.28)-(2.32) imply (2.34)-(2.35).

Below we justify the limit passage with $\lambda \to 0$ in (2.34)-(2.35). Substitute $\overline{u} = 0$, $\overline{u} = 2u^\lambda$ as test functions in (2.35), and sum up the obtained relations. It implies the equality

$$(\sigma^\lambda(u^\lambda), \varepsilon(u^\lambda))_{\Omega_\gamma} = (F, u^\lambda)_{\Omega_\gamma}. \tag{2.36}$$

Assuming that $\lambda \in (0, \lambda_0)$, from (2.36) we obtain

$$\|u^\lambda\|_{H^1_\Gamma(\Omega_\gamma)^3} \leq c_1, \tag{2.37}$$

$$\frac{1}{\lambda} \int\limits_\omega a_{ijkl}\varepsilon_{kl}(u^\lambda)\varepsilon_{ij}(u^\lambda) \leq c_2 \tag{2.38}$$

with constants c_1, c_2 being uniform with respect to $\lambda \in (0, \lambda_0)$. Choosing a subsequence, if necessary, it can be assumed as $\lambda \to 0$

$$u^\lambda \to u \text{ weakly in } H^1_\Gamma(\Omega_\gamma)^3.$$

Then by (2.38)

$$\varepsilon_{ij}(u) = 0 \text{ in } \omega,\ \ i, j = 1, 2, 3.$$

This means that a function ρ_0 exists such that

$$u = \rho_0 \text{ in } \omega;\ \ \rho_0 \in R(\omega).$$

Since u^λ converge weakly in $H^1_\Gamma(\Omega_\gamma)^3$, the limit function u satisfies the inequality

$$(u^+ - \rho_0) \cdot \nu \geq 0 \text{ on } \gamma.$$

In particular, $u \in K_\omega$.

Let us take any fixed element $\overline{u} \in K_\omega$. Then, there exists $\rho \in R(\omega)$ such that $\overline{u} = \rho$ in ω, and $\overline{u}$ can be taken as a test function in (2.35). In such a case, inequality (2.35) implies

$$(\sigma^\lambda(u^\lambda), \varepsilon(\overline{u} - u^\lambda))_{\Omega_\gamma} \geq (F, \overline{u} - u^\lambda)_{\Omega_\gamma}. \tag{2.39}$$

By accounting $A^\lambda = A$ in $\Omega \setminus \overline{\omega}$, we can pass to the limit in (2.39) as $\lambda \to 0$ which implies

$$u \in K_\omega,$$

$$(\sigma(u), \varepsilon(\overline{u} - u))_{\Omega_\gamma \setminus \overline{\omega}} \geq (F, \overline{u} - u)_{\Omega_\gamma} \ \ \forall\, \overline{u} \in K_\omega,$$

what is precisely (2.14)-(2.15). Hence a passage from the elastic inclusion to the rigid inclusion is shown. We formulate the obtained result as follows

Theorem 1. *The solution u^λ of problem (2.34)-(2.35) weakly converge in $H^1_\Gamma(\Omega_\gamma)^3$ to the solution u of problem (2.14)-(2.15).*

Observe that there is no limit in ω for the stress tensor σ^λ with $\lambda \to 0$. It is interesting to compare the above passage to the limit with the fictitious domain approach in contact problems, see [18].

3. Shape differentiability of the energy functional

We are going to present the framework for shape sensitivity analysis of the variational inequality with the crack on the boundary of rigid inclusion. We cannot expect that the same limit passage with $\lambda \to 0$ can be used for the differentiability purposes, if we want to determine the energy derivative with respect to the crack length. It means that the differentiability of solutions of auxiliary variational inequality for the parameter $\lambda > 0$, in general does not imply the differentiability for the limit problem obtained with $\lambda \to 0$. However, even for $\lambda > 0$ the only directional differentiability can be expected, and the result follows from the general method described in [34]. Therefore, we are going to adapt the approach proposed in [34] to our specific problem with the crack on the boundary of a rigid inclusion, and with the nonpenetration condition prescribed on the crack lips in two spatial dimensions, and on the crack surfaces in three spatial dimensions. In particular, we need to introduce the appropriate boundary variations framework within of the speed method in such a way that the rigid inclusion is not deformed by the associated mapping, and, at the same time the crack changes its length in the tangent direction on the boundary of the inclusion. Once, this issue is settled in a satisfactory way, we can apply the general result on the sensitivity analysis of variational inequalities with unilateral constraints in order to prove that the energy functional is directionally differentiable with respect to the crack length. How to use the obtained result to evaluate the derivative for possible applications in structural mechanics is a possible subject of further studies since in the specific situation the associated stress intensity factor at the crack tips or at the crack front in two and three spatial dimensions, respectively, is still to be determined, and we can expect only that the energy derivative can be given by a path independent integral which characterize the singularity at the crack front in three spatial dimensions or at the crack tip in two spatial dimensions. From mathematical point of view, our approach is based on the polyhedricity of the admissible cone and on the conical differentiability in the sense of Hadamard of the associated metric projection onto the polyhedric cone.

First of all, we recall an abstract result on the Hadamard differentiability of the metric projection on polyhedric convex sets in Hilbert spaces due to F. Mignot and A. Haraux, which is adapted to the weak formulation of our problem ([34, 9]), given by variational inequality (2.14)-(2.15). We point out that such an abstract result is valid in particular for the unilateral constraints in the so-called Dirichlet spaces of scalar functions, so in the case of the elasticity the appropriate analysis to apply the result for the displacement fields is performed in [34, 9]. This is possible from mathematical point of view, since the constraints on the crack are imposed in the scalar fractional Sobolev space of traces of the type $H^{1/2}$ which turns out to be a Dirichlet space.

Theorem 2. *Let there be given the right-hand side $F_t = F + th$ of variational inequality* (2.14)-(2.15)*, then the unique solution $u_t \in K_\omega$ is Lipschitz continuous*

$$\|u_t - u\|_{H^1(\Omega_\gamma)} \le Ct$$

and conically differentiable in $H^1(\Omega_\gamma)$, that is, for $t > 0$, t small enough,

$$u_t = u + tQ + o(t)$$

where the conical differential solves the variational inequality

$$Q \in S_K(u), \tag{3.1}$$

$$(\sigma(O), \varepsilon(\overline{u} - Q))_{\Omega_\gamma \setminus \varpi} \ge (h, \overline{u} - Q)_{\Omega_\gamma} \quad \forall\, \overline{u} \in S_K(u). \tag{3.2}$$

The remainder converges to zero

$$\frac{1}{t}\|o(t)\|_{H^1(\Omega_\gamma)} \to 0$$

uniformly with respect to the direction h on the compact sets of the dual space $(H^{1,\omega}_\Gamma(\Omega_\gamma))^*$, i.e., Q is the Hadamard directional derivative of the solution to the variational inequality with respect to the right-hand side.

To complete the above statement, we need the description of the convex cone $S_K(u)$,

$$S_K(u) = \{v \in H^{1,\omega}_\Gamma(\Omega_\gamma) | (v^+ - v^-) \cdot \nu \ge 0 \text{ on } \gamma_0; (\sigma(u), \varepsilon(v))_{\Omega_\gamma \setminus \varpi} = (F, v)_{\Omega_\gamma}\}$$

where $\gamma_0 = \{x \in \gamma | (u^+ - \rho_0) \cdot \nu = 0\}$.

We show that the energy functional is shape differentiable with respect to the crack length, which seems to be a new result in the fracture mechanics for the specific problem. To this end we need the following framework and notation.

Shape sensitivity analysis in the hold-all-domain. Now, the domain Ω_γ is going to change in function of a small parameter $t > 0$, the evolution of the domain is governed by the mapping T_t, the construction is briefly described below, all the details can be found in [34]. We denote $\Omega_0 := \Omega_\gamma$, for $t = 0$, and $\mathcal{D} = D \cup \omega$, so D is our hold-all-domain for velocity vector fields, in addition we have the inclusion $\overline{\Omega}_0 \subset \mathcal{D}$. We assume that the boundary $\partial D = \partial \mathcal{D} \cup \partial \omega$ is smooth, and that admissible vector fields for the velocity method of shape optimization satisfy the Nagumo condition

$$V \cdot n = 0 \quad \text{on} \quad \partial D\ . \tag{3.3}$$

The above condition guarantee the following properties of the mapping $T_t(V)$, the support of any admissible vector field V is disjoint with ω, V is tangent on the boundary $\partial\omega$, we can also deform $\gamma \subset \partial\Omega$ in the tangential direction, therefore, to move the crack boundary on the surface $\partial\omega$.

The mapping $T_t(V) : \Omega \ni X \mapsto x(t) \in \Omega_t$ which governs the boundary variations Γ_t of the boundary Γ as well as the variations γ_t of the crack γ is defined in [34] by the dynamical system

$$\frac{dx}{dt}(t) = V(t, x(t)) \ , \quad x(0) = X \ .$$

Hence, we denote $\gamma_t = T_t(V)(\gamma)$ and $\Gamma_t = T_t(V)(\Gamma)$ and we observe that the boundary of rigid inclusion is invariant for admissible transformations $T_t(V)$, it means that $\omega = T_t(\omega)$.
Variable domain setting. In order to perform the shape sensitivity analysis of the energy functional $J(\Omega_\gamma)$ we transport the problem defined in the variable domain $\Omega_t = T_t(\Omega_\gamma)$ to the fixed domain $\Omega_0 = \Omega_\gamma$. To this end we need also a change of the unknown solution to the variational inequality, in order to make the convex cone independent of the parameter t. First, we define the problem in variable domain $\Omega_t = T_t(V)(\Omega_\gamma)$, so we look for the minimizer $u_t \in K_t$ defined by the variational inequality

$$(\sigma(u_t), \varepsilon(v - u_t))_{\Omega_t \setminus \overline{\omega}} - (F, v - u_t)_{\Omega_t} + \geq 0 \ \forall v \in K_t \ , \tag{3.4}$$

where

$$K_t = \{v \in H^{1,\omega}_{\Gamma_t}(\Omega_t) \mid (v^+ - v^-) \cdot \nu_t \geq 0 \text{ on } \gamma_t\} \ . \tag{3.5}$$

Now, we perform the shape sensitivity analysis in exactly the same way as it is described in [34] for the Signorini problems. We use also the same notation which is introduced in [34] for the sensitivity analysis of variational inequalities with the polyhedric convex cones, this is the case of the convex set (3.5). However, the results presented here seem to be new, since the model of the crack with nonpenetration condition is new in this setting. We derive the form of the directional derivative of the energy functional with respect to the crack length, which a generalization of the results given in [20] for the crack located inside of an elastic body.

In order to assure the fixed domain setting for the transported problem we introduce the new unknown solution to the modified variational inequality $z^t = DT_t^{-1} \cdot u_t \circ T_t$, and we obtain that $z^t \in K$ solves the variational inequality

$$(\sigma^t(z^t), \varepsilon^t(v - z^t))_{\Omega_\gamma \setminus \overline{\omega}} - (F^t, v - z^t)_{\Omega_\gamma} \geq 0 \ \forall v \in K \ , \tag{3.6}$$

with the energy shape functional of the form

$$J(\Omega_t) = \frac{1}{2}(\sigma^t(z^t), \varepsilon^t(z^t))_{\Omega_\gamma \setminus \overline{\omega}} - (F^t, z^t)_{\Omega_\gamma} \ . \tag{3.7}$$

The expressions for F^t, $\sigma^t(v)$ and $\varepsilon^t(v)$ are given by

$$F^t = \det(DT_t)\, {}^*DT_t \cdot (F \circ T_t) \ , \tag{3.8}$$

$$\sigma^t(v) = \det(DT_t)\, A\varepsilon^t(v) \tag{3.9}$$

$$\varepsilon^t(v) = \frac{1}{2}\{D(DT_t \cdot v) \cdot DT_t^{-1} + {}^*DT_t^{-1} \cdot {}^*(D(DT_t \cdot v))\} \ , \tag{3.10}$$

where DT_t is the Jacobian matrix of the transformation T_t, ${}^*DT_t$ denotes the transposed matrix, DT_t^{-1} its inverse, and $\det(DT_t)$ its determinant.

Since the convex set K_t is a cone, it follows from the variational inequality that we have the following equality

$$(\sigma^t(z^t), \varepsilon^t(z^t))_{\Omega_\gamma \setminus \overline{\omega}} = (F^t, z^t)_{\Omega_\gamma} \tag{3.11}$$

therefore, the equivalent form for the energy shape functional looks like that

$$J(\Omega_t) = -\frac{1}{2}(\sigma^t(z^t), \varepsilon^t(z^t))_{\Omega_\gamma \setminus \overline{\omega}} = -\frac{1}{2}(F^t, z^t)_{\Omega_\gamma} \ . \tag{3.12}$$

The structure of formulae in (3.12) is useful for the differentiation with respect to the shape parameter $t > 0$ at $t = 0^+$ we point out that in general only side derivative can be obtained in such a case.

Theorem 3. *The solutions to variational inequality* (3.6) *are shape differentiable in the sense of Hadamard, and the mateial derivatives $\dot{z} \in \mathcal{S}_K(u)$ are given by the following variational inequality*

$$(\sigma(\dot{z}), \varepsilon(v - \dot{z}))_{\Omega_\gamma \setminus \overline{\omega}} + (\dot{\sigma}(u), \varepsilon(v - \dot{z}))_{\Omega_\gamma \setminus \overline{\omega}} + (\dot{\varepsilon}(u), \sigma(v - \dot{z}))_{\Omega_\gamma \setminus \overline{\omega}} \geq (\dot{F}, v - \dot{z})_{\Omega_\gamma} \tag{3.13}$$

for all test functions in the convex cone $v \in \mathcal{S}_K(u)$.

Corollary 4. *The shape derivative of the energy functional is given by the expression,*

$$dJ(\Omega_\gamma; V) = -\frac{1}{2}\left\{(\dot{\sigma}(u), \varepsilon(u))_{\Omega_\gamma \setminus \overline{\omega}} + (\sigma(u), \dot{\varepsilon}(u))_{\Omega_\gamma \setminus \overline{\omega}} + 2(\sigma(\dot{z}), \varepsilon(u))_{\Omega_\gamma \setminus \overline{\omega}}\right\} \ , \tag{3.14}$$

where $u \in K$ solves variational inequality (3.6) *, and $\dot{z} \in \mathcal{S}_K(u)$ solves the variational inequality for material derivatives* (3.13)*.*

The expressions for $\dot{\sigma}(v)$ and $\dot{\varepsilon}(v)$ are given by

$$\dot{\sigma}(v) = \mathrm{div} V A\varepsilon(v) + A\dot{\varepsilon}(v) \ , \tag{3.15}$$

$$\dot{\varepsilon}(v) = \frac{1}{2}\{D\,(DV \cdot v) + {}^{*}\,(D\,(DV \cdot v)) - Dv \cdot DV - {}^{*}DV \cdot {}^{*}Dv\} \ . \tag{3.16}$$

Therefore, we have the shape derivative of the energy functional with respect to the crack length, this formula is new and could be used in crack propagation analysis. However, the difficulty is hidden in the unknown expression for the singularity of the stress field at the crack tip. Another possibility is to define the path independent integrals to determine in an implicite way the stress intensity factor in the specific case.

4. Topological asymptotic analysis

The topological derivative introduced in [30] quantifies the sensitivity of a given shape functional with respect to the introduction of a non-smooth perturbation (hole, inclusion, source term, for instance) in a ball $B_\delta(x_0) \subset \Omega$ of radius $\delta > 0$ and center at $x_0 \in \Omega$, that is $B_\delta(x_0) = \{x \in \mathbb{R}^3 : \|x - x_0\| < \delta\}$, $\overline{B_\delta(x_0)}$ is the closure of $B_\delta(x_0)$. Therefore, this derivative can be seen as a first order correction on the shape functional $\mathcal{J}(\Omega)$ to estimate $\mathcal{J}(\Omega_\delta)$, where Ω_δ is the perturbed domain. Thus, we have the following topological asymptotic expansion for functional $\mathcal{J}$,

$$\mathcal{J}(\Omega_\delta) = \mathcal{J}(\Omega) + f(\delta) D_T(x_0) + o(f(\delta)) \ , \tag{4.1}$$

where $f(\delta)$ is a positive function that decreases monotonically such that $f(\delta) \to 0$ when $\delta \to 0^+$ and the term $D_T(x_0)$ is defined as the topological derivative of $\mathcal{J}$. Then, from (4.1) we have that the classical definition of the topological derivative is given by [27, 31]

$$D_T(x_0) = \lim_{\delta \to 0} \frac{\mathcal{J}(\Omega_\delta) - \mathcal{J}(\Omega)}{f(\delta)} = \lim_{\delta \to 0} \frac{1}{f'(\delta)} \frac{d}{d\delta} \mathcal{J}(\Omega_\delta) \ . \tag{4.2}$$

We point out, that even if formula (4.1) looks very different from the classical shape derivatives currently used in shape optimization, its nature is the same, since it is defined by [30] in the form of a singular limit of shape derivatives evaluated on boundaries of small voids with respect to the radius of the voids $\delta \to 0$. In this way the topological derivative is a generalization of the classical shape derivative in smooth case to the singular boundary perturbations. We refer the reader to [26] for the asymptotic analysis in singularly perturbed domains by means of the matched and compound asymptotic expansions which leads to the topological derivatives of shape functionals in elasticity with complete proofs in general case.

We recall here, that the topological derivative has been successfully applied in the context of topology optimization ([1, 2, 6, 28, 29, 10, 11]), inverse problems ([3, 8, 24]) and image processing ([4, 5, 16, 23]). Concerning the theoretical development of the topological asymptotic analysis, the reader may refer to [26], for instance.

In our particular case, we consider a regular perturbation of the domain given by the nucleation of a small elastic inclusion with Young modulus $E_\eta = \eta E$, where E is the Young modulus of the bulk material and $\eta \in [0, \infty)$ represents the contrast. We assume that there is a small elastic

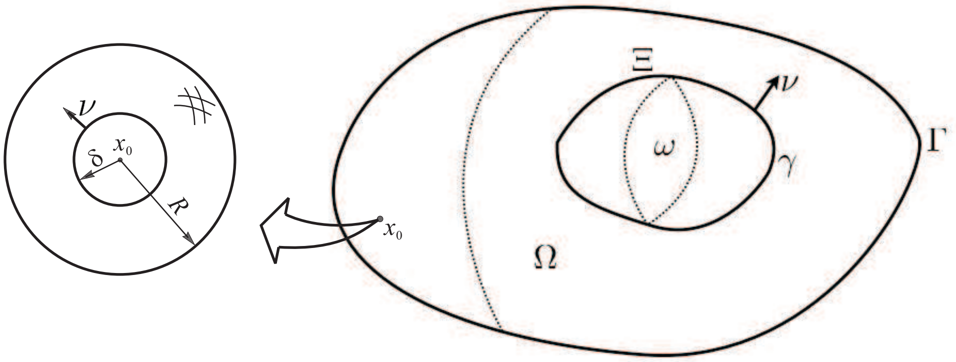

FIGURE 2. Domain Ω with rigid inclusion ω and an elastic inclusion $B_\delta(x_0)$.

inclusion $B_\delta(x_0)$ in the elastic region $\Omega_\omega = \Omega \setminus \overline{\omega}$. If the elastic inclusion becomes a cavity, it is denoted by $\omega_\delta = B_\delta(x_0)$. The cavity can be obtained from the elastic inclusion by the limit passage $\eta \to 0$, in the limit case we have a singular perturbation of the domain. In the case of elastic inclusion the elastic region Ω_ω is decomposed into two disjoint parts $\Omega_\omega \setminus \overline{B_\delta(x_0)}$ and $B_\delta(x_0)$ with different material properties, namely E and ηE, respectively. The other limit passage with the contrast $\eta \to \infty$ results in the small rigid inclusion $\omega_\delta = B_\delta(x_0)$. See Fig. (2). We are also interested in the topological asymptotic expansion of the energy shape functional of the form

$$\Pi_\delta(v) = \frac{1}{2}(\sigma(v), \varepsilon(v))_{\Omega_\omega \setminus \overline{B_\delta(x_0)}} + \frac{1}{2}(\sigma(v), \varepsilon(v))_{B_\delta(x_0)} - (F, v)_{\Omega_\gamma}\ , \tag{4.3}$$

where we have to find function $v = u_\delta$ such that

$$\begin{aligned}
-\mathrm{div}\sigma = F \quad & \text{in} \quad \Omega \setminus \overline{\omega}, && (4.4)\\
\sigma - A\varepsilon(u_\delta) = 0 \quad & \text{in} \quad \Omega_\omega \setminus \overline{B_\delta(x_0)}, && (4.5)\\
\sigma - A_\eta\varepsilon(u_\delta) = 0 \quad & \text{in} \quad B_\delta(x_0), && (4.6)\\
[u_\delta] = 0 \quad & \text{on} \quad \partial B_\delta(x_0) && (4.7)\\
[\sigma]\nu = 0 \quad & \text{on} \quad \partial B_\delta(x_0) && (4.8)\\
u_\delta = 0 \quad & \text{on} \quad \Gamma, && (4.9)\\
(u_\delta - \rho_0)\cdot\nu \geq 0 \quad & \text{on} \quad \gamma^+, && (4.10)\\
\sigma_\tau = 0,\ \sigma_\nu \leq 0 \quad & \text{on} \quad \gamma^+, && (4.11)\\
\sigma_\nu(u_\delta - \rho_0)\cdot\nu = 0 \quad & \text{on} \quad \gamma^+, && (4.12)\\
-\int_\Xi \sigma\nu\cdot\rho = \int_\omega F\cdot\rho \quad & \forall \rho \in R(\omega). && (4.13)
\end{aligned}$$

with A such as before and $A_\eta = \eta A$ (since E_η is the Young modulus of the inclusion).

4.1. **Domain decomposition.** Since the problem is non-linear, let us introduce a domain decomposition given by $\Omega_R = \Omega_\omega \setminus \overline{B_R(x_0)}$, where $B_R(x_0)$ is a ball of radius $R > \delta$ and center at $x_0 \in \Omega$, that is $B_R(x_0) = \{x \in \mathbb{R}^3 : \|x - x_0\| < R\}$, $\overline{B_R(x_0)}$ is the closure of $B_R(x_0)$, as shown in Fig. (2). For the sake of simplicity, we assume that $F = 0$ in $B_R(x_0)$. Thus, we have the following linear elasticity system defined in $B_R(x_0)$ with an inclusion $B_\delta(x_0)$ inside

$$-\mathrm{div}\sigma = 0 \quad \text{in} \quad B_R(x_0), \tag{4.14}$$
$$\sigma - A\varepsilon(w_\delta) = 0 \quad \text{in} \quad B_R(x_0) \setminus \overline{B_\delta(x_0)}, \tag{4.15}$$
$$\sigma - A_\eta\varepsilon(w_\delta) = 0 \quad \text{in} \quad B_\delta(x_0), \tag{4.16}$$
$$w_\delta = v \quad \text{on} \quad \partial B_R(x_0), \tag{4.17}$$
$$[w_\delta] = 0 \quad \text{on} \quad \partial B_\delta(x_0), \tag{4.18}$$
$$[\sigma]\nu = 0 \quad \text{on} \quad \partial B_\delta(x_0). \tag{4.19}$$

We are interested in the Steklov-Poincaré operator on ∂B_R, that is

$$\mathcal{A}_\delta : v \in H^{1/2}(\partial B_R) \mapsto \sigma(w_\delta)\nu \in H^{-1/2}(\partial B_R)\ . \tag{4.20}$$

Then we have $\sigma(u_R)\nu = \mathcal{A}_\delta(u_R)$ on ∂B_R, where u_R is solution of the variational inequality in Ω_R, that is

$$u_R \in K_\omega : a_{\Omega_R}(u_R, \varphi - u_R) \geq (F, \varphi - u_R)_{\Omega_\gamma \setminus \overline{B_R(x_0)}} \quad \forall \varphi \in K_\omega \tag{4.21}$$

and the bilinear form a_{Ω_R} is such that

$$a_{\Omega_R}(u,\varphi) = \int_{\Omega_R} \sigma(u) \cdot \varepsilon(\varphi) + \int_{\partial B_R} \mathcal{A}_\delta(u) \cdot \varphi\ . \tag{4.22}$$

Finally, in the disk $B_R(x_0)$ we have

$$\int_{B_R \setminus \overline{B_\delta}} \sigma(w) \cdot \varepsilon(w) + \int_{B_\delta} \sigma(w) \cdot \varepsilon(w) = \int_{\partial B_R} \mathcal{A}_\delta(w) \cdot w\ , \tag{4.23}$$

where $w = w_\delta$ is the solution of the elasticity system in the disk (4.14)-(4.19) or equivalently solution of the following variational problem

$$w_\delta \in W : \int_{B_R \setminus \overline{B_\delta}} \sigma(w_\delta) \cdot \varepsilon(\varphi) + \int_{B_\delta} \sigma(w_\delta) \cdot \varepsilon(\varphi) = 0 \quad \forall \varphi \in W_0\ , \tag{4.24}$$

with W and W_0 such that

$$W = \{ w \in H^1(B_R)^3 \mid [w] = 0 \text{ on } \partial B_\delta, w = v \text{ on } \partial B_R \}\ , \tag{4.25}$$
$$W_0 = \{ \varphi \in H^1(B_R)^3 \mid [\varphi] = 0 \text{ on } \partial B_\delta, \varphi = 0 \text{ on } \partial B_R \}\ . \tag{4.26}$$

4.2. **Shape sensitivity analysis of the energy functional.** Let us introduced the energy-based shape functional defined in the disk $B_R(x_0)$, that is

$$\mathcal{E}_\delta(w_\delta) := \frac{1}{2} \int_{B_R \setminus \overline{B_\delta}} \sigma(w_\delta) \cdot \varepsilon(w_\delta) + \frac{1}{2} \int_{B_\delta} \sigma(w_\delta) \cdot \varepsilon(w_\delta)\ . \tag{4.27}$$

We need to calculate

$$\begin{aligned} \frac{d}{d\delta}\mathcal{E}_\delta(w_\delta) &= \int_{B_R \setminus \overline{B_\delta}} \sigma(w_\delta) \cdot \varepsilon(\dot{w}_\delta) + \int_{B_\delta} \sigma(w_\delta) \cdot \varepsilon(\dot{w}_\delta) \\ &+ \int_{B_R \setminus \overline{B_\delta}} \Sigma(w_\delta) \cdot \nabla V + \int_{B_\delta} \Sigma(w_\delta) \cdot \nabla V\ , \end{aligned} \tag{4.28}$$

which was obtained using the Reynold's transport theorem and the concept of material derivatives of spacial fields ([14, 34]). Some of the terms in (4.29) require explanation. Vector V represents the shape change velocity field defined on the disk $B_R(x_0)$ and such that $V = 0$ on ∂B_R and $V = \nu$ on ∂B_δ. Thus, $\dot{w}_\delta \in W_0$ is the material (total) derivative with respect to δ. Finally, the Eshelby energy-momentum tensor Σ takes the form ([7, 15])

$$\Sigma(w_\delta) := \frac{1}{2}\sigma(w_\delta) \cdot \varepsilon(w_\delta)\mathrm{I} - (\nabla w_\delta)^T \sigma(w_\delta)\ . \tag{4.29}$$

Since $\dot{w}_\delta \in W_0$ and considering that $\Sigma(w_\delta)$ is a free-divergence tensor field ($\text{div}\Sigma(w_\delta) = 0$), the shape derivative of the energy functional becomes

$$\frac{d}{d\delta}\mathcal{E}_\delta(w_\delta) = -\int_{\partial B_\delta} [\Sigma(w_\delta)]\nu \cdot \nu \ . \tag{4.30}$$

4.3. **Topological derivatives calculation.** By introducing (4.30) in (4.2), we have

$$D_T(x_0) = -\lim_{\delta \to 0} \frac{1}{f'(\delta)} \int_{\partial B_\delta} [\Sigma(w_\delta)]\nu \cdot \nu \ . \tag{4.31}$$

4.3.1. *Topological derivative of the energy functional in three spatial dimensions for a small cavity.* In the three spatial dimensions we consider the particular case associated to the energy change due to the nucleation of a spherical cavity. Thus, for the convenience of the reader we recall here the results derived in [12, 17, 29] for the three dimensional elasticity case.

Theorem 5. *Let us consider the contrast $\eta \to 0$. Thus, the elastic inclusion degenerates to a spherical cavity with homogeneous Neumann boundary condition. In this case, the energy shape functional admits for $\delta \to 0$ the following topological asymptotic expansion*

$$\Pi_\delta(u_\delta) = \Pi(u) + \pi\delta^3 D_T(x_0) + o(\delta^3) \ , \tag{4.32}$$

with the topological derivative $D_T(x_0)$ given by

$$D_T(x_0) = \mathbb{H}\sigma(u(x_0)) \cdot \varepsilon(u(x_0)) \quad \forall x_0 \in \Omega \setminus \overline{\omega} \ , \tag{4.33}$$

where u is solution of the variational inequality (2.14)-(2.15) and $\mathbb{H}$ is a forth-order tensor defined as

$$\mathbb{H} = \frac{1-\upsilon}{7-5\upsilon}\left(10\mathbb{I} - \frac{1-5\upsilon}{1-2\upsilon}\mathrm{I} \otimes \mathrm{I}\right) \ . \tag{4.34}$$

4.3.2. *Topological derivative of the energy functional in two spatial dimensions for a small inclusion.* In two spatial directions we derive also an exterior expansion for the solutions of the variational inequality. Therefore, the result obtained is more precise compared to the general case of the cavity in three spatial dimensions. First, we repeat the model description, and then we develop the asymptotic analysis in linear elasticity to derive the equivalent form of perturbation of the bilinear form.

Since in this Section we are dealing with a two dimensional elasticity problem, then the domain $\Omega \subset \mathbb{R}^2$. Thus, all indices introduced in the Section 2 take values from 1 to 2, instead of 1 to 3. In the particular case of plane stress, the Lamé coefficient $l = l^*$, where

$$l^* = \frac{\upsilon E}{1-\upsilon^2} \ .$$

In addition, the crack γ is represented now by a smooth 1D curve described as

$$x_i = x_i(y), \quad y \in D, \ i = 1, 2,$$

with bounded domain $D \subset \mathbb{R}$. The space $R(\omega)$ of infinitesimal rigid displacements is redefined simply by setting

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \quad \text{and} \quad C = (c^1, c^2); \quad b, c^i = const, \ i = 1, 2.$$

The displacement field $u = (u_1, u_2)$; $u = \rho_0$ in ω; $\rho_0 \in R(\omega)$; and in the domain $\Omega \setminus \overline{\omega}$ we have to find the stress tensor components $\sigma = \{\sigma_{ij}\}$, solution of (2.1)-(2.7) in $\Omega \subset \mathbb{R}^2$ for $i, j = 1, 2$. Hence, all definitions and results presented in the previous Sections hold.

We use the existence of the asymptotic expansions for w_δ, solution of the elasticity system (4.14)-(4.19) now defined in the disk $B_R(x_0) \subset \mathbb{R}^2$, in the neighborhood of $B_\delta(x_0)$, namely

$$w_\delta(x) = w_0(x) + w^\infty(x) + o(\delta) \ . \tag{4.35}$$

In addition, w^∞ is proportional to δ, $\|w^\infty\|_{\mathbb{R}^2} = O(\delta)$, on the surface ∂B_ρ of the ball. The expansion of $\sigma(w_\delta)$ corresponding to (4.14)-(4.19) has the form

$$\sigma(w_\delta(x)) = \sigma^\infty(w_0(x_0), x) + O(\delta) \ . \tag{4.36}$$

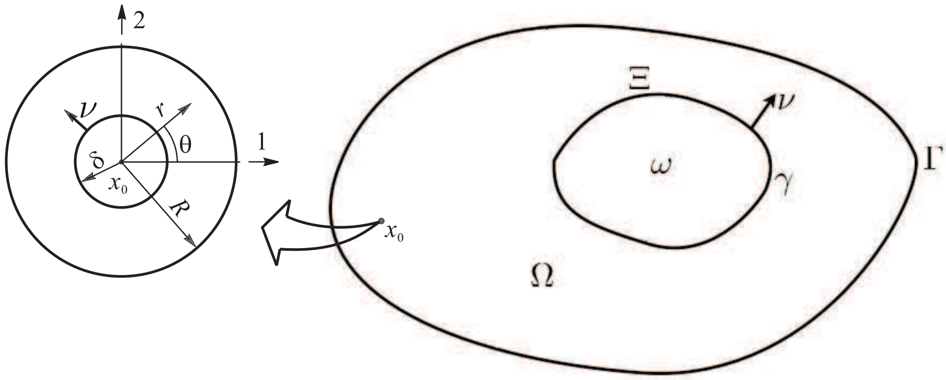

FIGURE 3. Domain Ω with rigid inclusion ω and an elastic inclusion $B_\delta(x_0)$.

where σ^∞ is the stress distribution around a circular inclusion in an infinity medium and w_0 is solution of the elasticity system (4.14)-(4.19) defined in the disk $B_R(x_0) \subset \mathbb{R}^2$ for $\delta = 0$. Thus, σ^∞ can be calculated explicitly, which is given in a polar coordinate system (r, θ) by:

- for $r \geq \delta$

$$\sigma^\infty_{rr}(r,\theta) = a\left(1 - \tfrac{1-\eta}{1+\eta\alpha}\tfrac{\delta^2}{r^2}\right) + b\left(1 - 4\tfrac{1-\eta}{1+\eta\beta}\tfrac{\delta^2}{r^2} + 3\tfrac{1-\eta}{1+\eta\beta}\tfrac{\delta^4}{r^4}\right)\cos 2\theta \qquad (4.37)$$

$$\sigma^\infty_{\theta\theta}(r,\theta) = a\left(1 + \tfrac{1-\eta}{1+\eta\alpha}\tfrac{\delta^2}{r^2}\right) - b\left(1 + 3\tfrac{1-\eta}{1+\eta\beta}\tfrac{\delta^4}{r^4}\right)\cos 2\theta \qquad (4.38)$$

$$\sigma^\infty_{r\theta}(r,\theta) = -b\left(1 + 2\tfrac{1-\eta}{1+\eta\beta}\tfrac{\delta^2}{r^2} - 3\tfrac{1-\eta}{1+\eta\beta}\tfrac{\delta^4}{r^4}\right)\sin 2\theta \qquad (4.39)$$

- for $0 < r < \delta$

$$\sigma^\infty_{rr}(r,\theta) = 2\tfrac{\eta\alpha}{1+\eta\alpha}\tfrac{a}{1-\upsilon} + 4\tfrac{\eta\beta}{1+\eta\beta}\tfrac{b}{3-\upsilon}\cos 2\theta \qquad (4.40)$$

$$\sigma^\infty_{\theta\theta}(r,\theta) = 2\tfrac{\eta\alpha}{1+\eta\alpha}\tfrac{a}{1-\upsilon} - 4\tfrac{\eta\beta}{1+\eta\beta}\tfrac{b}{3-\upsilon}\cos 2\theta \qquad (4.41)$$

$$\sigma^\infty_{r\theta}(r,\theta) = -4\tfrac{\eta\beta}{1+\eta\beta}\tfrac{b}{3-\upsilon}\sin 2\theta \qquad (4.42)$$

In the above formulas, coefficients a and b are given respectively by

$$a = \frac{1}{2}(\sigma_1 + \sigma_2) \quad \text{and} \quad b = \frac{1}{2}(\sigma_1 - \sigma_2)\ , \qquad (4.43)$$

where $\sigma_{1,2}$ are the eigenvalues of tensor $\sigma(w_0(x_0))$. In addition, constants α and β are respectively given by

$$\alpha = \frac{1+\upsilon}{1-\upsilon} \quad \text{and} \quad \beta = \frac{3-\upsilon}{1+\upsilon}\ . \qquad (4.44)$$

The jump condition of the stress field $\sigma(w_\delta)$ can be written as

$$[\sigma(w_\delta)]\nu = 0 \;\Rightarrow\; [\sigma_{rr}(w_\delta)] = 0 \quad \text{and} \quad [\sigma_{r\theta}(w_\delta)] = 0 \text{ on } \partial B_\delta\ . \qquad (4.45)$$

In the same way, the continuity condition of the displacement field w_δ implies

$$[w_\delta] = 0 \;\Rightarrow\; [\varepsilon_{\theta\theta}(w_\delta)] = 0 \text{ on } \partial B_\delta\ . \qquad (4.46)$$

The Eshelby tensor flux through the boundary of the inclusion is given by

$$\begin{aligned}\Sigma(w_\delta)\nu\cdot\nu &= \frac{1}{2}\sigma(w_\delta)\cdot\varepsilon(w_\delta) - \sigma(w_\delta)\nu\cdot(\nabla w_\delta)\,\nu \qquad (4.47)\\ &= \frac{1}{2}\left(\sigma_{\theta\theta}(w_\delta)\varepsilon_{\theta\theta}(w_\delta) - \sigma_{rr}(w_\delta)\varepsilon_{rr}(w_\delta)\right.\\ &+ \left.\sigma_{r\theta}(w_\delta)\left(\partial_\theta w^r_\delta - \partial_r w^\theta_\delta\right)\right)\ .\end{aligned}$$

From the jump and continuity conditions on the boundary ∂B_δ given by (4.45, 4.46) and considering the constitutive relation (4.5)-(4.6) for $l = l^*$, the jump of the Eshelby tensor flux in the normal direction results in (see, for instance, [13])

$$\begin{aligned} [\Sigma(w_\delta)]\nu\cdot\nu &= \frac{1}{2}\left([\sigma_{\theta\theta}(w_\delta)]\varepsilon_{\theta\theta}(w_\delta) - \sigma_{rr}(w_\delta)[\varepsilon_{rr}(w_\delta)]\right. \\ &+ \left. 2(1-\delta)\sigma_{r\theta}(w_\delta)\varepsilon_{r\theta}(w_\delta)\right) . \end{aligned} \tag{4.48}$$

Finally, considering (4.49) in (4.31) and also formulas (4.37)-(4.41) we can calculate the integral on ∂B_δ explicitly, which allows to identify function $f(\delta) = \pi\delta^2$. Then, after calculate the limit $\delta \to 0$, we obtain the following result:

Theorem 6. *The energy shape functional admits for $\delta \to 0$ the following topological asymptotic expansion*

$$\Pi_\delta(u_\delta) = \Pi(u) + \pi\delta^2 D_T(x_0) + o(\delta^2) , \tag{4.49}$$

with the topological derivative $D_T(x_0)$ given by

$$D_T(x_0) = \mathbb{H}_\eta \sigma(u(x_0)) \cdot \varepsilon(u(x_0)) \quad \forall x_0 \in \Omega \setminus \overline{\omega} , \tag{4.50}$$

where u is solution of the variational inequality (2.14)-(2.15) in $\Omega \subset \mathbb{R}^2$ and $\mathbb{H}_\eta$ is a forth-order tensor defined as

$$\mathbb{H}_\eta = \frac{1}{4}\frac{(1-\eta)^2}{1+\beta\eta}\left(2\frac{1+\beta}{1-\eta}\mathbb{I} + \frac{\alpha-\beta}{1+\alpha\eta}\mathrm{I}\otimes\mathrm{I}\right) . \tag{4.51}$$

Corollary 7. *Let us consider the contrast $\eta \to 0$. Thus, the elastic inclusion degenerates to a circular cavity with homogeneous Neumann boundary condition and the tensor $\mathbb{H}_0$ becomes*

$$\mathbb{H}_0 = \frac{1}{4}\left(2(1+\beta)\mathbb{I} + (\alpha-\beta)\mathrm{I}\otimes\mathrm{I}\right) . \tag{4.52}$$

Corollary 8. *Let us consider the contrast $\eta \to \infty$. Thus, the elastic inclusion degenerates to rigid one and the tensor $\mathbb{H}_\infty$ takes the form*

$$\mathbb{H}_\infty = -\frac{1}{4}\left(2\frac{1+\beta}{\beta}\mathbb{I} - \frac{\alpha-\beta}{\alpha\beta}\mathrm{I}\otimes\mathrm{I}\right) . \tag{4.53}$$

Remark 9. *From equality (4.23) we observe that the result given by theorem 6 represents the topological derivative of the Steklov-Poincaré operator (4.20). In addition, since solution $u \in K_\omega$ of the variational inequality (2.14)-(2.15) in $\Omega \subset \mathbb{R}^2$ is a $H^1(\Omega_\gamma)^2$ function, then it is convenient to compute the topological derivative from quantities evaluated on the boundary ∂B_R. In particular, we have the following representation for the strain tensor $\varepsilon(u(x_0))$ ([33])*

$$\begin{aligned} \varepsilon_{11} + \varepsilon_{22} &= \frac{1}{\pi R^3}\int_{\partial B_R}(u_1x_1 + u_2x_2) , & (4.54)\\ \varepsilon_{11} - \varepsilon_{22} &= \frac{1}{\pi R^3}\int_{\partial B_R}\left((1-9k)(u_1x_1 - u_2x_2) + \frac{12k}{R^2}(u_1x_1^3 - u_2x_2^3)\right) , & (4.55)\\ 2\varepsilon_{12} &= \frac{1}{\pi R^3}\int_{\partial B_R}\left((1+9k)(u_1x_2 + u_2x_1) - \frac{12k}{R^2}(u_1x_2^3 + u_2x_1^3)\right) . & (4.56) \end{aligned}$$

where

$$k = \frac{l^* + m}{l^* + 3m} .$$

Once the above integrals are evaluated e.g. numerically, then we can use the constitutive relation (4.5) to compute the stress tensor $\sigma(u(x_0))$. Finally, these results can by used to compute the topological derivative through formula (4.50).

4.4. **Approximation of solutions for variational inequalities.** We define a variational inequality for the crack problem with a perturbed bilinear form. The bilinear form is defined in the whole domain of integration, it is bounded and coercive on the energy space for the crack problem without any inclusion, and provides the first order topological sensitivity for the solutions of nonlinear elasticity boundary value problem with the nonlinear crack.

Approximation of crack problem in Ω_δ

We determine the modified bilinear form as a sum of two terms, as it is for the energy functional, the first term defines the elastic energy in the domain Ω, the second term is a correction term, determined in Section 4.3. The correction term is quite complicated to evaluate, and we provide its explicit form, such a form is actually defined by the formulae in Section 4.3. The values of the symmetric bilinear form $a(\delta;\cdot,\cdot)$ are given by the expression

$$a(\delta; v, v) = a(u, u) + \delta^2 b(v, v) \ . \tag{4.57}$$

The derivative $b(v,v)$ of the bilinear form $a(\delta;v,v)$ with respect to δ^2 at $\delta = 0+$ is given by the expression

$$b(v, v) = -2\pi e_v(0) - \frac{2\pi m}{l^* + 3m}\big(\sigma_{II}\delta_1 - \sigma_{12}\delta_2\big) \ , \tag{4.58}$$

where all the quantities are evaluated for the displacement field v according to formulae in Section 4.3 where we provide the line integrals which defines all terms in (4.54), (4.55) and (4.56). Hence, we can determine the bilinear form $a(\delta;v,w)$ for all v,w, from the equality

$$2a(\delta; v, w) = a(\delta; v + w, v + w) - a(\delta; w, w) - a(\delta; v, v) \ .$$

In the same way the bilinear form $b(v,w)$ is determined from the formula for $b(v,v)$.
The convex set is defined in this case by

$$K_\delta = \{v \in H^1_\Gamma(\Omega_\delta)^2 \,|\, [v]\nu \geq 0 \quad \text{on } \gamma\} \ . \tag{4.59}$$

Let us consider the following variational inequality which provides a sufficiently precise for our purposes approximation u_δ of the solution $u(\Omega_\delta)$ to crack problem defined in singularly perturbed domain Ω_δ,

$$u_\delta \in K_\delta \ : \quad a(\delta; u, v - u) \geq (F, v - u)_{\Omega_\delta} \quad \forall v \in K_\delta \ . \tag{4.60}$$

The result obtained is the following, for simplicity we assume that the linear form $L(\delta;\cdot)$ is independent of δ.

Theorem 10. *For δ sufficiently small we have the following expansion of the solution u_δ with respect to the parameter δ at $0+$,*

$$u_\delta = u(\Omega) + \delta^2 q + o(\delta^2) \quad \text{in } H^1(\Omega)^2 \ , \tag{4.61}$$

where the topological derivative q of the solution $u(\Omega)$ to the crack problem is given by the unique solution of the following variational inequality

$$\begin{aligned} q \in \mathcal{S}_K(u) = \{v \in (H^1_\Gamma(\Omega_\gamma))^2 \,|\, [v]\cdot\nu \geq 0 \quad \text{on } \Xi(u) \ , \quad a(0; u, v) = 0\} \\ a(q, v - q) + b(u, v - q) \geq 0 \quad \forall v \in \mathcal{S}_K(u) \ . \end{aligned} \tag{4.62}$$

The coincidence set $\Xi(u) = \{x \in \gamma \,|\, [u(x)]\cdot\nu(x) = 0\}$ is well defined ([9]) for any function $u \in H^1(\Omega)^2$, and $u \in K$ is the solution of variational inequality (4.59) for $\delta = 0$.

For the proof of theorem we refer the reader to [32].

For the convenience of the reader we provide the explicite formulae for the terms in $b(v,v)$ defined by (4.58), we refer to section 4.3 and to [32, 33] for details. We have

$$2\pi e_{\boldsymbol{v}}(0) = \frac{\pi(l^* + m)}{\pi^2 R^6}\left(\int_{\Gamma_R}(v_1x_1 + v_2x_2)\,ds\right)^2 \tag{4.63}$$
$$+\frac{m}{\pi^2R^6}\left(\int_{\Gamma_R}\left((1-9k)(v_1x_1 - v_2x_2) + \frac{12k}{R^2}(v_1x_1^3 - v_2x_2^3)\right]\,ds\right)^2$$
$$+\frac{m}{\pi^2R^6}\left(\int_{\Gamma_R}\left[(1+9k)(v_1x_2 + v_2x_1) - \frac{12k}{R^2}(v_1x_2^3 + v_2x_1^3)\right]\,ds\right)^2 ,$$

with

$$\sigma_{II} = \frac{m}{\pi R^3}\int_{\Gamma_R}\left[(1-9k)(v_1x_1 - v_2x_2) + \frac{12k}{R^2}(v_1x_1^3 - v_2x_2^3)\right]\,ds,$$
$$\sigma_{12} = \frac{m}{\pi R^3}\int_{\Gamma_R}\left[(1+9k)(v_1x_2 + v_2x_1) - \frac{12k}{R^2}(v_1x_2^3 + v_2x_1^3)\right]\,ds,$$

and

$$\delta_1 = \frac{9k}{\pi R^3}\int_{\Gamma_R}\left[(v_1x_1 - v_2x_2) - \frac{4}{3R^2}(v_1x_1^3 - v_2x_2^3)\right]\,ds,$$
$$\delta_2 = \frac{9k}{\pi R^3}\int_{\Gamma_R}\left[(v_1x_2 + v_2x_1) - \frac{4}{3R^2}(v_1x_2^3 + v_2x_1^3)\right]\,ds.$$

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