ON TOPOLOGICAL DERIVATIVES FOR ELASTIC SOLIDS WITH UNCERTAIN INPUT DATA

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ABSTRACT. In this paper a new approach to the derivation of the worst scenario and the maximum range scenario methods is proposed. The derivation is based on the topological derivative concept for the boundary value problems of elasticity in two and three spatial dimensions. It is shown that the topological derivatives can be applied to the shape and topology optimization problems within a certain range of input data including the Lamé coefficients and the boundary tractions. In other words, the topological derivatives are *stable functions* and the concept of topological sensitivity is *robust* with respect to the imperfections caused by uncertain data. Two classes of integral shape functionals are considered, the first for the displacement field and the second for the stresses. For such classes, the form of topological derivatives is given and for the second class some restrictions on the shape functionals are introduced in order to assure the existence of topological derivatives. The results on topological derivatives are used for the mathematical analysis of the worst scenario and the maximum range scenario methods. The presented results can be extended to more realistic methods for some uncertain material parameters and with the optimality criteria including the shape and topological derivatives for broad classe of shape functionals.

1. INTRODUCTION

Topological derivatives are now of common use in numerical procedures of resolution for optimal design problems in structural mechanics, as well as for control and inverse problems. The concept is based on the asymptotic approximation of solutions to the elliptic boundary value problems in geometrical domains with singularly perturbed boundaries, it means that the concept includes the asymptotics due to the creation of small holes or cavities inside of the domain, or singular perturbations of the existing boundary. For integral shape functionals in the linear elasticity the associated mathematical construction is fully developed in [1] and it is used in many papers, we refer the reader e.g., to [2, 3, 4, 5, 6, 7, 8] for the results on the topological derivatives as well as numerical exemples. In particular, our results generalize and cover all formulae of topological derivatives presented in papers [3, 4, 6, 7, 8], the main extension is just the precise formula of topological derivatives for shape functionals depending on stresses. We point out that there are some restrictions on the applicability of topological derivatives for the shape functionals depending on the stresses, which are already adressed in [1], and which are explicitely given in our paper, see Remark 1 for details.

The main issue of the present paper is the robustness of topological derivatives, which is very natural question in applied science, since the actual model parameters are usually slightly different from the theoretical parameters of the mathematical model. In other words, we would like to be sure that the design obtained by the numerical procedure is not very sensitive to the model imperfections. Such a property of the function called topological derivative is not obvious for the specific mathematical object which is obtained by a complicated approximation procedure of asymptotic analysis, the procedure being in a sense arbitrary since the asymptotic ansatz for the approximation with respect to small parameter, the size of the imperfection, is selected a priori and requires the verification of its asymptotic exactness. We consider the particular case of the worst scenario method, which seems to be relatively easy to apply, however the results obtained can be rather pessimistic. The more relevant probabilistic approach to the model robustness seems to be expensive and difficult to apply, in any case it requires the knowledge of probabilistic distributions for the model parameters in question.

Key words and phrases. Topological derivative, elasticity system, uncertain input, asymptotic analysis.

In order to explain briefly the significance of the topological derivative in shape optimization we present a simple example, with the well known solution obtained by numerical methods. We start with the description of the model in hold-all-domain and compute the topological derivative of the compliance to be minimized, and finally present the optimal shape. It is clear from the numerical experiments that the topological derivative reflects an optimal topology for the shape optimization problem. Thus, in three figures below we present an example of the structure to be optimized, the contour plot of the topological derivative of the goal functional obtained for the hold-all-geometrical domain, and finally the optimal shape in the form of the bridge structure, which is well known from the literature on the subject.

Example 1. *Let us present an application of the topological derivative in the context of structural design. The compliance minimization, or equivalently the energy maximization, is considered, so the shape functional to be minimized is negative elastic energy functional, its topological derivative is given by formula* (2.39)*, and the displacement field is evaluated by solving problem* (2.1)*. In the numerical example shown in Fig. 1, the initial domain is represented by a rectangular panel* $\Omega = (0,\, 180) \times (0,\, 60)\ m^2$*, with thickness* $h = 0.3m$*, Young modulus* $E = 210 \times 10^9 N/m^2$ *and Poisson ratio* $\nu = 1/3$*, clamped on the region* $a = 9m$ *and submitted to a uniformly distributed traffic loading* $\bar{q} = 250 \times 10^3\ N/m^2$*. This load is applied on the grey strip of height* $b = 3m$ *which is positioned at the distance* $c = 30m$ *from the top of the design domain.*

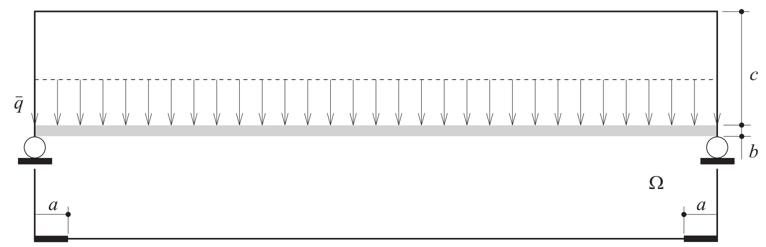

FIGURE 1. Hold-all-domain with the framework for the compliance minimization or energy maximization with respect to the shape and topology

The topological derivative of the compliance shape functional which is obtained in the first iteration of the shape and topology optimization numerical procedure is shown in Fig. 2, where white to black levels mean smaller to higher values.

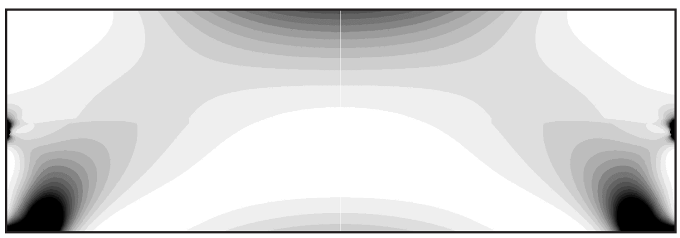

FIGURE 2. Topological derivative of the energy or compliance shape functional in hold-all-domain.

The resulting shape and topology numerical solution in the form of a well-known tie-arch bridge structure, which is acceptable from practical point of view, is computed for a volume constraint given by $0.25\ |\Omega|$*, and it is shown in Fig. 3.*

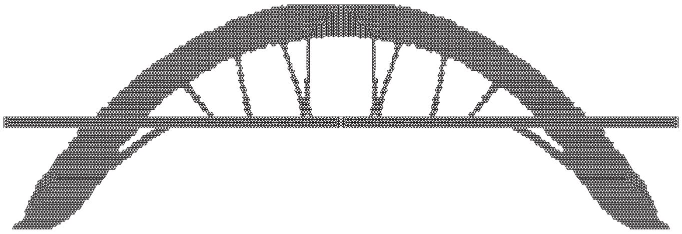

FIGURE 3. Optimal shape design, usually it is a local minimizer obtained numerically for the compliance minimization, there is a lack of sufficient optimality conditions for such shape optimization problems.

In the paper we consider topological derivative of shape functionals for elasticity, which is used to derive the worst and also the maximum range scenarios for behavior of elastic body in case of uncertain material parameters. It turns out that both problems are connected, because the criteria describing this behavior have form of functionals depending on topological derivative of elastic energy. Therefore, in the first part we describe the methodology of computing the topological derivative with some new additional conditions for shape functionals depending on stresses. For the sake of fullness of presentation the explicit formulae for stress distribution around cavities are provided. Then we present the worst scenario and maximum range scenario framework for the energy-based topological derivative. Finally, we observe that the classical methods associated to the elastic energy shape functional can be extended to cover problems with uncertain input data for the criteria based on, for instance, kinematic constraints or constraints on stresses.

2. Topological Derivative

The topological derivative $\mathcal{T}_\Omega$ of a shape functional $\mathcal{J}(\Omega)$ is introduced in [7] in order to characterize the infinitesimal variation of $\mathcal{J}(\Omega)$ with respect to the infinitesimal variation of the topology of the domain Ω. The topological derivative allows us to derive the new optimality condition for the shape optimization problem:

$$\mathcal{J}(\Omega^*) = \inf_{\Omega} \mathcal{J}(\Omega) \ .$$

The optimal domain Ω^* is characterized by the first order condition [9] defined on the boundary of the optimal domain Ω^*, $dJ(\Omega^*; V) \geq 0$ for all admissible vector fields V, and by the following optimality condition defined in the interior of the domain Ω^*:

$$\mathcal{T}_{\Omega^*}(x) \geq 0 \quad \text{in} \quad \Omega^* \ .$$

We point out, that the rigorous derivation of necessary conditions for optimality given above requires some a priori regularity of an optimal shape, since we cannot expect that a specific shape functional is shape differentiable or admits a topological derivative for any admissible geometrical domain, otherwise the optimality conditions are only formal. Since the energy type shape functionals are shape differentaible under relatively week assumptions, the compliance is a main optimality criteria in the optimal design of structural mechanics. Asymptotic analysis of eigenvalues is also an important issue for applications, the same analysis can be performed in such a case, and we refer the reader to [10] for the precise mathematical analysis of spectral problems and the topological derivatives of simple and multiple eigenvalues.

The other use of the topological derivative is connected with approximating the influence of the holes in the domain on the values of integral functionals of solutions, what allows us to solve a class of shape inverse problems.

In general terms the notion of the *topological* derivative (TD) has the following meaning. Assume that $\Omega \subset \mathbb{R}^N$ is an open set and that there is given a shape functional

$$\mathcal{J} \ : \ \Omega \setminus K \to \mathbb{R}$$

for any compact subset $K \subset \overline{\Omega}$. We denote by $B_\rho(x), x \in \Omega$, the ball of radius $\rho > 0$, $B_\rho(x) = \{y \in \mathbb{R}^N | \|y - x\| < \rho\}$, $\overline{B_\rho(x)}$ is the closure of $B_\rho(x)$, and assume that there exists the following limit

$$\mathfrak{T}(x) = \lim_{\rho \downarrow 0} \frac{\mathcal{J}(\Omega \setminus \overline{B_\rho(x)}) - \mathcal{J}(\Omega)}{|\overline{B_\rho(x)}|}$$

which can be defined in an equivalent way by

$$\tilde{\mathfrak{T}}(x) = \lim_{\rho \downarrow 0} \frac{\mathcal{J}(\Omega \setminus \overline{B_\rho(x)}) - \mathcal{J}(\Omega)}{\rho^N}$$

The function $\mathfrak{T}(x), x \in \Omega$, is called the topological derivative of $\mathcal{J}(\Omega)$, and provides the information on the infinitesimal variation of the shape functional $\mathcal{J}$ if a small hole is created at $x \in \Omega$. This definition is suitable for Neumann–type boundary conditions on ∂B_ρ.

In several cases this characterization is constructive, i.e. TD can be evaluated for shape functionals depending on solutions of partial differential equations defined in the domain Ω.

For instance, TD may be computed for the 3D elliptic Laplace type equation, as well as for extremal values of cost functionals for a class of optimal control problems. All these examples have one common feature: the expression for TD may be calculated in the closed functional form.

As we shall see below, the 3D elasticity case is more difficult, since it requires evaluation of integrals on the unit sphere with the integrands which can be computed at any point, but the resulting functions have no explicit functional form. In the particular case of energy functional we obtain the closed formula.

2.1. **Problem Setting for Elasticity Systems.** We introduce elasticity system in the form convenient for the evaluation of topological derivatives. Let us consider the elasticity equations in $\mathbb{R}^N$, where $N = 2$ for 2D and $N = 3$ for 3D,

$$\begin{aligned} \operatorname{div} \sigma(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \Gamma_D, \\ \sigma(u)n &= T \quad \text{on} \quad \Gamma_N, \end{aligned} \tag{2.1}$$

and the same system in the domain with the spherical cavity $B_\rho(x_0) \subset \Omega$ centered at $x_0 \in \Omega$, $\Omega_\rho = \Omega \setminus \overline{B_\rho(x_0)}$,

$$\begin{aligned} \operatorname{div} \sigma(u_\rho) &= 0 \quad \text{in} \quad \Omega_\rho, \\ u_\rho &= g \quad \text{on} \quad \Gamma_D, \\ \sigma(u_\rho)n &= T \quad \text{on} \quad \Gamma_N, \\ \sigma(u_\rho)n &= 0 \quad \text{on} \quad \partial B_\rho(x_0), \end{aligned} \tag{2.2}$$

where n is the unit outward normal vector on $\partial\Omega_\rho = \partial\Omega \cup \partial B_\rho(x_0)$. Assuming that $0 \in \Omega$, we can consider the case $x_0 = 0$.

Here u and u_ρ denote the displacement vectors fields, g is a given displacement on the fixed part Γ_D of the boundary, T is a traction prescribed on the loaded part Γ_N of the boundary. In addition, σ is the Cauchy stress tensor given, for $\xi = u$ (eq. 2.1) or $\xi = u_\rho$ (eq. 2.2), by

$$\sigma(\xi) = D\varepsilon(\xi) \ , \tag{2.3}$$

where $\varepsilon(\xi)$ is the symmetric part of the gradient of vector field ξ, that is

$$\varepsilon(\xi) = \frac{1}{2}\left(\nabla\xi + \nabla\xi^\top\right) \ , \tag{2.4}$$

and D is the elasticity tensor,

$$D = 2\mu I\!I + \lambda \left(I \otimes I\right) \ , \tag{2.5}$$

with

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \lambda = \lambda^* = \frac{\nu E}{1-\nu^2} \tag{2.6}$$

being E the Young's modulus, ν the Poisson's ratio and λ^* the particular case for plane stress. In addition, I and $\mathbb{I}$ respectively are the second and fourth order identity tensors. Thus, the inverse of D is

$$D^{-1} = \frac{1}{2\mu}\left[\mathbb{I} - \frac{\lambda}{2\mu + N\lambda}(I \otimes I)\right] ,$$

The first shape functional under consideration depends on the displacement field,

$$J_u(\rho) = \int_{\Omega_\rho} F(u_\rho)\, d\Omega , \tag{2.7}$$

where F is a C^2 function, e.g., $F(u_\rho) = (Hu_\rho \cdot u_\rho)^p$ for an integer $p \geq 2$. It is also useful for further applications in the framework of elasticity to introduce the yield functional of the form

$$J_\sigma(\rho) = \int_{\Omega_\rho} \mathcal{S}\sigma(u_\rho) \cdot \sigma(u_\rho)\, d\Omega , \tag{2.8}$$

where $\mathcal{S}$ is an isotropic fourth-order tensor, which means that $\mathcal{S}$ may be expressed as follows

$$\mathcal{S} = 2m\mathbb{I} + l\,(I \otimes I) ,$$

where l, m are real constants. Their values may vary for particular yield criteria. The following assumption assures, that J_u, J_σ are well defined for solutions of the elasticity system.

(A) The domain Ω has piecewise smooth boundary, which may have reentrant corners with $\alpha < 2\pi$ created by the intersection of two planes. In addition, g, T must be compatible with $u \in H^1(\Omega; \mathbb{R}^N)$.

The interior regularity of u in Ω is determined by the regularity of the right hand side of the elasticity system. For simplicity the following notation is used for functional spaces,

$$H^1_g(\Omega_\rho) = \{\psi \in [H^1(\Omega_\rho)]^N \mid \psi = g \quad \text{on} \quad \Gamma_D\},$$
$$H^1_{\Gamma_D}(\Omega_\rho) = \{\psi \in [H^1(\Omega_\rho)]^N \mid \psi = 0 \quad \text{on} \quad \Gamma_D\},$$
$$H^1_{\Gamma_D}(\Omega) = \{\psi \in [H^1(\Omega)]^N \mid \psi = 0 \quad \text{on} \quad \Gamma_D\},$$

here we use the convention that eg., in our notation $H^1_g(\Omega_\rho)$ stands for the Sobolev space of vector functions $[H^1_g(\Omega_\rho)]^N$.
The weak solutions to the elasticity systems are defined in the standard way.

Find $u_\rho \in H^1_g(\Omega_\rho)$ such that, for every $\phi \in H^1_{\Gamma_D}(\Omega_\rho)$,

$$\int_{\Omega_\rho} D\varepsilon(u_\rho) \cdot \varepsilon(\phi)\, d\Omega = \int_{\Gamma_N} T \cdot \phi\, dS \tag{2.9}$$

We introduce the adjoint state equations in order to simplify the form of shape derivatives of functionals J_u, J_σ [9, 11]. For the functional J_u they take on the form:

Find $w_\rho \in H^1_{\Gamma_D}(\Omega_\rho)$ such that, for every $\phi \in H^1_{\Gamma_D}(\Omega_\rho)$,

$$\int_{\Omega_\rho} D\varepsilon(w_\rho) \cdot \varepsilon(\phi)\, d\Omega = -\int_{\Omega_\rho} F'_u(u_\rho) \cdot \phi\, d\Omega, \tag{2.10}$$

whose Euler-Lagrange equation reads

$$\begin{aligned} \operatorname{div} \sigma(w_\rho) &= F'_u(u_\rho) \quad \text{in} \quad \Omega_\rho, \\ w_\rho &= 0 \quad \text{on} \quad \Gamma_D, \\ \sigma(w_\rho)n &= 0 \quad \text{on} \quad \Gamma_N, \\ \sigma(w_\rho)n &= 0 \quad \text{on} \quad \partial B_\rho(x_0), \end{aligned} \tag{2.11}$$

while $v_\rho \in H^1_{\Gamma_D}(\Omega_\rho)$ is the adjoint state for J_σ and satisfies for all test functions $\phi \in H^1_{\Gamma_D}(\Omega)$ the following integral identity:

$$\int_{\Omega_\rho} D\varepsilon(v_\rho) \cdot \varepsilon(\phi)\, d\Omega = -2\int_{\Omega_\rho} D\mathcal{S}\sigma(u_\rho) \cdot \varepsilon(\phi)\, d\Omega. \tag{2.12}$$

which associated Euler-Lagrange equation becomes

$$\begin{aligned} \operatorname{div}\sigma(v_\rho) &= -2\operatorname{div}\,(D\mathcal{S}\sigma(u_\rho)) \quad \text{in} \quad \Omega_\rho, \\ v_\rho &= 0 \quad \text{on} \quad \Gamma_D, \\ \sigma(v_\rho)n &= -2D\mathcal{S}\sigma(u_\rho)n \quad \text{on} \quad \Gamma_N, \\ \sigma(v_\rho)n &= -2D\mathcal{S}\sigma(u_\rho)n \quad \text{on} \quad S_\rho(x_0) = \partial B_\rho(x_0). \end{aligned} \tag{2.13}$$

Remark 2. *We observe that $D\mathcal{S}$ can be written as*

$$D\mathcal{S} = 4\mu m \mathbb{I} + \gamma\,(I \otimes I) \tag{2.14}$$

where

$$\gamma = \lambda l N + 2\,(\lambda m + \mu l) \tag{2.15}$$

Thus, when $\gamma = 0$, the boundary condition on $\partial B_\rho(x_0)$ in eq. (2.13) becomes homogeneous and the yield criteria must satisfy the constraint

$$\frac{m}{l} = -\left(\frac{\mu}{\lambda} + \frac{N}{2}\right), \tag{2.16}$$

which is naturally satisfied for the energy shape functional, for instance. In fact, in this particular case, tensor $\mathcal{S}$ is given by

$$\mathcal{S} = \frac{1}{2}D^{-1} \quad \Rightarrow \quad \gamma = 0 \quad \text{and} \quad 2m + l = \frac{1}{2E}\,, \tag{2.17}$$

which implies that the adjoint solution associated to J_σ can be explicitly obtained in terms of u_ρ.

2.2. **Main Result .** We shall define the topological derivative of the functionals J_u, J_σ at the point x_0 as [5, 7]:

$$\mathcal{T}J_u(x_0) = \lim_{\rho \downarrow 0} \frac{dJ_u(\rho)}{d(|B_\rho(x_0)|)}\,, \tag{2.18}$$

$$\mathcal{T}J_\sigma(x_0) = \lim_{\rho \downarrow 0} \frac{dJ_\sigma(\rho)}{d(|B_\rho(x_0)|)}\,. \tag{2.19}$$

Now we may formulate the following result, giving the constructive method for computing the topological derivatives:

Theorem 3. *Assume that (A) is satisfied, then*

$$\mathcal{T}J_u(x_0) = -\frac{1}{2(N-1)\pi}\,[\,2(N-1)\pi F(u) + \Psi(D^{-1}; \sigma(u), \sigma(w))]_{x=x_0}, \tag{2.20}$$

$$\mathcal{T}J_\sigma(x_0) = -\frac{1}{2(N-1)\pi}\,[\,\Psi(\mathcal{S}; \sigma(u), \sigma(u)) + \Psi(D^{-1}; \sigma(u), \sigma(v))]_{x=x_0}, \tag{2.21}$$

where $w,\, v \in H^1_{\Gamma_D}(\Omega)$ are adjoint variables satisfying the integral identities (2.10) and (2.12) for $\rho = 0$, i.e. in the whole domain Ω instead of Ω_ρ, that is

$$\int_\Omega D\varepsilon(w) \cdot \varepsilon(\phi)\, d\Omega = -\int_\Omega F'_u(u) \cdot \phi\, d\Omega. \tag{2.22}$$

$$\int_\Omega D\varepsilon(v) \cdot \varepsilon(\phi)\, d\Omega = -2\int_\Omega D\mathcal{S}\sigma(u) \cdot \varepsilon(\phi)\, d\Omega. \tag{2.23}$$

for all test functions $\phi \in H^1_{\Gamma_D}(\Omega)$.

Some of the terms in (2.20), (2.21) require explanation. The function Ψ is defined as an integral over the unit sphere $S_1(0) = \{x \in \mathbb{R}^N \mid \|x\| = 1\}$ of the following functions:

$$\Psi(\mathcal{S}; \sigma(u(x_0)), \sigma(u(x_0))) \;=\; \int_{S_1(0)} \mathcal{S}\sigma^\infty(u(x_0); x) \cdot \sigma^\infty(u(x_0); x)\, dS \tag{2.24}$$

$$\Psi(D^{-1}; \sigma(u(x_0)), \sigma(v(x_0))) \;=\; \int_{S_1(0)} \sigma^\infty(u(x_0); x) \cdot D^{-1}\sigma^\infty(v(x_0); x)\, dS \tag{2.25}$$

$$\Psi(D^{-1};\sigma(u(x_0)),\sigma(w(x_0))) \;=\; \int_{S_1(0)} \sigma^\infty(u(x_0);x)\cdot D^{-1}\sigma^\infty(w(x_0);x)\,dS \tag{2.26}$$

The symbol $\sigma^\infty(u(x_0);x)$ denotes the stresses for the solution of the elasticity system (2.2) in the infinite domain $\mathbb{R}^N \setminus \overline{B_1(0)}$ with the following boundary conditions:

- no tractions are applied on the surface of the ball, $S_1(0) = \partial B_1(0)$;
- the stresses $\sigma^\infty(u(x_0);x)$ tend to the constant value $\sigma(u(x_0))$ as $\|x\| \to \infty$.

In this notation $\sigma^\infty(u(x_0);x)$ is a function of space variables depending on the functional parameter $u(x_0)$, while $\sigma(u(x_0))$ is a value of the stress tensor computed in the point x_0 for the solution u. The dependence between them results from the boundary condition at infinity listed above. The method for obtaining such solutions (and u^∞), based on [12], is discussed in the next section.

In order to derive the above formulae (2.20), (2.21) we calculate the derivatives of the functional $J_u(\rho)$ with respect to the parameter ρ, which determines the size of the hole $B_\rho(x_0)$, by using the material derivative method [9]. Then we pass to the limit $\rho \downarrow 0$ using the asymptotic expansions of u_ρ with respect to ρ. For the functional J_u the shape derivative with respect to ρ is given by

$$J_u'(\rho) = \int_{\Omega_\rho} F_u'(u_\rho)\cdot u_\rho'\,d\Omega - \int_{S_\rho(x_0)} F(u_\rho)\,dS, \tag{2.27}$$

and in the same way for the state equation:

$$\int_{\Omega_\rho} D\varepsilon(u_\rho')\cdot\varepsilon(\phi)\,d\Omega - \int_{S_\rho(x_0)} D\varepsilon(u_\rho)\cdot\varepsilon(\phi)\,dS = 0, \tag{2.28}$$

where u_ρ' is the shape derivative, i.e. the derivative of u_ρ with respect to ρ, [9].

After substitution of the test functions $\phi = w_\rho$ in the derivative of the state equation, $\phi = u_\rho'$ in the adjoint equation, we get

$$\begin{aligned} J_u'(\rho) &= -\int_{S_\rho(x_0)} [F(u_\rho) + D\varepsilon(u_\rho)\cdot\varepsilon(w_\rho)]\,dS \\ &= -\int_{S_\rho(x_0)} [F(u_\rho) + \sigma(u_\rho)\cdot D^{-1}\sigma(w_\rho)]\,dS, \end{aligned} \tag{2.29}$$

and similarly for J_σ

$$\begin{aligned} J_\sigma'(\rho) &= -\int_{S_\rho(x_0)} [\mathcal{S}\sigma(u_\rho)\cdot\sigma(u_\rho) + D\varepsilon(u_\rho)\cdot\varepsilon(v_\rho)]\,dS \\ &= -\int_{S_\rho(x_0)} [\mathcal{S}\sigma(u_\rho)\cdot\sigma(u_\rho) + \sigma(u_\rho)\cdot D^{-1}\sigma(v_\rho)]\,dS. \end{aligned} \tag{2.30}$$

Observe, that both matrices D^{-1} and $\mathcal{S}$ are isotropic, and therefore, the corresponding bilinear forms in terms of stresses are invariant with respect to the rotations of the coordinate system.

Now we exploit the fact, that

$$\frac{dJ_u(\rho)}{d(|B_\rho(x_0)|)} = \frac{1}{2(N-1)\pi\rho^{N-1}}\frac{dJ_u}{d\rho},$$

and use the existence of the asymptotic expansions for u_ρ in the neighborhood of $B_\rho(x_0)$, namely

$$u_\rho = u(x_0) + u^\infty + O(\rho^2). \tag{2.31}$$

In addition, u^∞ is proportional to ρ, $\|u^\infty\|_{\mathbb{R}^N} = O(\rho)$, on the surface $S_\rho(x_0)$ of the ball. The expansion of $\sigma(u_\rho)$ corresponding to (2.31) has the form

$$\sigma(u_\rho) = \sigma^\infty(u(x_0);x) + O(\rho). \tag{2.32}$$

It can be proved, that w_ρ and v_ρ have similar expansions.

Using the formulae (2.31),(2.32) the following passages to the limit hold:

$$\lim_{\rho\downarrow 0}\frac{1}{\rho^{N-1}}\int_{S_\rho(x_0)}\sigma(u_\rho)\cdot D^{-1}\sigma(v_\rho)\,dS=\Psi(D^{-1};\sigma(u(x_0)),\sigma(v(x_0))),$$

$$\lim_{\rho\downarrow 0}\frac{1}{\rho^{N-1}}\int_{S_\rho(x_0)}\sigma(u_\rho)\cdot D^{-1}\sigma(w_\rho)\,dS=\Psi(D^{-1};\sigma(u(x_0)),\sigma(w(x_0))),$$

$$\lim_{\rho\downarrow 0}\frac{1}{\rho^{N-1}}\int_{S_\rho(x_0)}\mathcal{S}\sigma(u_\rho)\cdot \sigma(u_\rho)\,dS=\Psi(\mathcal{S};\sigma(u(x_0)),\sigma(u(x_0))),$$

$$\lim_{\rho\downarrow 0}\frac{1}{\rho^{N-1}}\int_{S_\rho(x_0)}F(u_\rho)\,dS=2(N-1)\pi F(u(x_0)).$$

This completes the proof of the theorem.

The main difficulty lies in the computation of the values of the functions denoted above as

$$\Psi(\mathcal{S};\sigma(u(x_0)),\sigma(u(x_0))),\ \Psi(D^{-1};\sigma(u(x_0)),\sigma(w(x_0)))\text{ and }\Psi(D^{-1};\sigma(u(x_0)),\sigma(v(x_0)))\ ,$$

which, in general, are difficult to obtain in the closed form, in contrast with the two dimensional case. Therefore, we can approximate them using numerical quadrature. It is possible, because we may calculate the values of integrands at any point on the sphere. This makes the computations more involved, but does not increase the numerical complexity in comparison to evaluating single closed form expression.

Remark 4. *The tensor $\mathcal{S}$ in the definition of J_σ may, in fact, be arbitrary, not only isotropic. However, it is difficult to imagine such a need for the isotropic material. Anyway, in the general case, we would have to transform $\mathcal{S}$ according to the known rules for the fourth order tensor, connected with the rotation of the reference frame.*

2.2.1. *Topological Derivatives in 3D Elasticity.* The shape functionals J_u, J_σ are defined in the same way as presented in section 2.2 with the exception, that J_σ is now the energy stored in a 3D elastic body (see remark 2). The weak solutions to the elasticity system as well as adjoint equations are defined also analogously to the section 2.2. Then, considering the expansions presented in Appendix A.2, we may state the following result [8] (see also [3, 4, 6]):

Theorem 5. *The expressions for the topological derivatives of the functionals J_u, J_σ have the form*

$$\mathcal{T}J_u(x_0)=-\left[F(u)+\frac{3}{2E}\frac{1-\nu}{7-5\nu}\left(10(1+\nu)\sigma(u)\cdot\sigma(w)-(1+5\nu)\mathrm{tr}\sigma(u)\mathrm{tr}\sigma(w)\right)\right]_{x=x_0}, \quad (2.33)$$

$$\mathcal{T}J_\sigma(x_0)=\frac{3}{4E}\frac{1-\nu}{7-5\nu}\left[10(1+\nu)\sigma(u)\cdot\sigma(u)-(1+5\nu)(\mathrm{tr}\sigma(u))^2\right]_{x=x_0}. \quad (2.34)$$

2.2.2. *Topological Derivatives in 2D Elasticity.* For the convenience of the reader we recall here the results derived in [7] for the 2D case. The principal stresses associated with the displacement field u are denoted by $\sigma_I(u)$, $\sigma_{II}(u)$, the trace of the stress tensor $\sigma(u)$ is denoted by $\mathrm{tr}\sigma(u)=\sigma_I(u)+\sigma_{II}(u)$. The shape functionals J_u, J_σ are defined in the same way as presented in section 2.2, with the tensor $\mathcal{S}$ isotropic (that is similar to D). The weak solutions to the elasticity system as well as adjoint equations are defined also analogously to the section 2.2. Then, from the expansions presented in Appendix A.1, we can prove the following result [7]:

Theorem 6. *The expressions for the topological derivatives of the functionals J_u,J_σ have the form*

$$\begin{aligned}\mathcal{T}J_u(x_0)&=-\left[F(u)+\frac{1}{E}\left(a_ua_w+2b_ub_w\cos 2\delta\right)\right]_{x=x_0}\\&=-\left[F(u)+\frac{1}{E}\left(4\sigma(u)\cdot\sigma(w)-\mathrm{tr}\sigma(u)\mathrm{tr}\sigma(w)\right)\right]_{x=x_0}\end{aligned}\quad (2.35)$$

$$\begin{aligned}\mathcal{T}J_\sigma(x_0) &= - \left[(\alpha+\beta)a_u^2 + 2(\alpha-\beta)b_u^2 + \frac{1}{E}\left(a_u a_v + 2b_u b_v \cos 2\delta\right)\right]_{x=x_0} \\ &= - \left[4(\alpha-\beta)\sigma(u)\cdot\sigma(u) - (\alpha-3\beta)(\mathrm{tr}\sigma(u))^2\right. \\ &+ \left.\frac{1}{E}\left(4\sigma(u)\cdot\sigma(v) - \mathrm{tr}\sigma(u)\mathrm{tr}\sigma(v)\right)\right]_{x=x_0} \qquad (2.36)\end{aligned}$$

Some of the terms in (2.35), (2.36) require explanation. According to eq. (2.15) for $N = 2$, constants α and β are given by

$$\alpha = l + 2\left(m + \gamma\frac{\nu}{E}\right) \qquad \text{and} \qquad \beta = 2\frac{\gamma}{E}. \qquad (2.37)$$

Furthermore, we denote

$$\begin{aligned} a_u &= \sigma_{I}(u) + \sigma_{II}(u), & b_u &= \sigma_{I}(u) - \sigma_{II}(u), \\ a_w &= \sigma_{I}(w) + \sigma_{II}(w), & b_w &= \sigma_{I}(w) - \sigma_{II}(w), \\ a_v &= \sigma_{I}(v) + \sigma_{II}(v), & b_v &= \sigma_{I}(v) - \sigma_{II}(v). \end{aligned} \qquad (2.38)$$

Finally, the angle δ denotes the angle between principal stress directions for displacement fields u and w in (2.35), and for displacement fields u and v in (2.36).

Remark 7. *For the energy stored in a 2D elastic body, tensor $\mathcal{S}$ is given by eq. (2.17), $\gamma = 0$, $\alpha = 1/(2E)$ and $\beta = 0$. Thus, we obtain the following well-known result*

$$\mathcal{T}J_\sigma(x_0) = \frac{1}{2E}\left[4\sigma(u)\cdot\sigma(u) - (\mathrm{tr}\sigma(u))^2\right]_{x=x_0} \qquad (2.39)$$

Compare these expressions to the 3D case. Their simplicity comes from the fact, that on the plane the rotation of one coordinate system with respect to the other is defined by the single value of the angle (here δ). This is a purely 2D phenomenon and it makes the explicit computations possible.

3. Uncertain Input Data

The methods proposed in this section can be used in the presence of model imperfections. In order to analyse the methods, we need the continuity of topological derivatives with respect to the parameters of the model, including the Lamé coefficients and the tractions. The Lamé coefficients are given in some compact sets i.e., the closed intervals, and the tractions are given in a convex subset of a functional space. The critical case is the topological derivative of the shape functional for stresses, so that case is analysed in details, and Theorem 8 shows the continuity of the pointwise values of stresses with respect to the Lamé coefficients and tractions. This result allows us, in fact, to establish the well-posedness of both methods in section 3.1.2. The elasticity problems in two spatial dimensions are investigated in section 3.2.

In reality, the values of input data (loading, material parameters) are guaranteed only in some given intervals. One of the simplest remedy is to apply the worst scenario or maximum range scenario method [13]. In what follows, we present the methods for the traction problem (2.1) with $\partial\Omega = \Gamma_N$ and the criterion corresponding to the topological derivatives (2.34) or (2.39), respectively.

3.1. Traction Problem in 3D Elasticity.

Let us consider a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega \equiv \Gamma$, occupied by a homogeneous and isotropic elastic body. Let the body be loaded by surface forces $T \in [L^\infty(\Gamma)]^3$ and the body forces be zero.

We introduce sets of admissible uncertain input data as follows :

(i): Lamé coefficients

$$\lambda \in \mathcal{U}_{ad}^{\lambda} = \left[\underline{\lambda}, \overline{\lambda}\right], \ 0 \le \underline{\lambda} < \overline{\lambda} < \infty,$$
$$\mu \in \mathcal{U}_{ad}^{\mu} = \left[\underline{\mu}, \overline{\mu}\right], \ 0 < \underline{\mu} < \overline{\mu} < \infty;$$

(ii): surface loading forces

$T_i \in \mathcal{U}_{ad}^T$
$= \left\{\tau \in L^\infty(\Gamma) \ : \ \tau\left|_{\Gamma_p} \in C^{(0),1}\left(\overline{\Gamma_p}\right), |\tau| \le \overline{C_1}, \ |\partial\tau/\partial s_j| \le \overline{C_2} \ a.e. \ on \ \Gamma, \ j=1,2\right.\right\}$,
where
$\Gamma = \bigcup_{p=1}^P \Gamma_p, \ \Gamma_k \cap \Gamma_m = \emptyset$ for $k \neq m, \ i=1,2,3,$
s_j are local coordinates of the surface Γ_p and $\overline{C}_1, \overline{C}_2$ are given constants,
$T \equiv (T_1, T_2, T_3) \in \mathcal{U}_{ad}^T = \{T_i \in \mathcal{U}_{ad}^T, \ i=1,2,3$ and $\int_\Gamma T \mathrm{dS} = 0, \ \int_\Gamma x \times T \mathrm{dS} = 0\}$.

Finally, we define

$$\mathcal{U}_{ad} = \mathcal{U}_{ad}^\lambda \times \mathcal{U}_{ad}^\mu \times \mathcal{U}_{ad}^T \ and \ A \equiv \{\mathcal{A}, T\}, \ \mathcal{A} = \{\lambda, \mu\}.$$

We will consider the following criterion-functional based on the topological derivative associated to the energy shape functional (2.34)

$$\Phi(\mathcal{A}, \sigma) = \sigma^\top \mathcal{H}(\mathcal{A})\sigma$$

where $\sigma \equiv \sigma(y)$ is the stress tensor of a full body at the center $y \in \Omega$ of a spherical cavity,

$$\mathcal{H}(\mathcal{A}) = \frac{3}{4E}\frac{1-\nu}{7-5\nu}\left[10(1+\nu)I\!I - (1+5\nu)I \otimes I\right]. \tag{3.1}$$

Note that $\nu = \frac{\lambda}{2(\lambda+\mu)}$, $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$.

3.1.1. *Continuous Dependence of the Criterion on the Input Data.* Our main result of the present section is given by the following theorem

Theorem 8. *Let $A_n \in \mathcal{U}_{ad}$, $A_n \to A$ in $\mathbb{R}^2 \times [L^\infty(\Gamma)]^3$ as $n \to \infty$. Then*

$$\Phi(\mathcal{A}_n, \sigma(A_n)) \to \Phi(\mathcal{A}, \sigma(A)).$$

Proof is based on the formulae ([14]-Theorem 10.1.1)

$$\frac{\partial u_i}{\partial y_j}(y) = \int_\Gamma T \cdot G_y^{ij} \mathrm{dS}\ , \quad i,j = 1,2,3 \tag{3.2}$$

and

$$G_y^{ij} = G_y^{ij}(\mathcal{A}) = u^{*ij}(\mathcal{A}) - \overline{u}^{ij}(\mathcal{A}), \tag{3.3}$$

where, for $k = 1,2,3$,

$$\begin{aligned} \overline{u}_k^{ij}(\mathcal{A}) &= \frac{1}{\varkappa}\overline{u}_k^{ij0}, \quad \varkappa(\mathcal{A}) = 16\pi\mu(1-\nu) \\ \overline{u}_k^{ij0}(\mathcal{A}) &= |r|^{-3}(r_k\delta_{ij} + r_i\delta_{jk} - 3r_jr_ir_k|r|^{-2} - (3-4\nu)r_j\delta_{ik}), \end{aligned} \tag{3.4}$$

and $r = x - y$. Since

$$(\varkappa(\mathcal{A}_n))^{-1} \longrightarrow (\varkappa(\mathcal{A}))^{-1}$$

and the components $\overline{u}_k^{ij0}$ are bounded on Γ,

$$\overline{u}^{ij}(\mathcal{A}_n) \longrightarrow \overline{u}_{ij}(\mathcal{A}) \ in \ [L^2(\Gamma)]^3. \tag{3.5}$$

The vector field $u^{*ij}(\mathcal{A})$ is the displacement solving the first boundary value problem with zero body forces and the equilibriated surface loading

$$T^{*ij} = \overline{s}^{ij} + w^{ij},$$

where

$$\overline{\mathrm{s}}_k^{ij} = (\lambda\delta_{km}\mathrm{div}\overline{u}^{ij} + 2\mu\varepsilon_{km}(\overline{u}^{ij}))n_m \tag{3.6}$$

and

$$w^{ij} = a^{ij} + b^{ij} \times x, \qquad a^{ij}, b^{ij} \in \mathbb{R}^3$$

represents a rigid body displacement such that

$$\int_\Gamma w^{ij} \mathrm{dS} = 0, \quad \int_\Gamma w^{ij} \times x \, \mathrm{dS} = e_i \times e_j,$$

(e_i denote unit vectors in the directions of Cartesian coordinates). The field w^{ij} is uniquely determined by the conditions shown.

Inserting (3.4) in (3.6), we observe that

$$\overline{s}_k^{ij} = \left(\frac{\lambda}{\varkappa} \delta_{km} \mathrm{div} \overline{u}^{ij0} + 2\frac{\mu}{\varkappa} \varepsilon_{km}(\overline{u}^{ij0}) \right) n_m = \overline{s}_k^{ij}(\nu) \tag{3.7}$$

since $\overline{u}^{ij0}$, $\frac{\lambda}{\varkappa}$ and $\frac{\mu}{\varkappa}$ are independent of the modulus E.

Lemma 9. *Let us define*

$$a(\mathcal{A}; u, v) = \int_\Omega (\lambda \, \mathrm{div} u \, \mathrm{div} v + 2\mu \, \varepsilon_{ij}(u) \varepsilon_{ij}(v)) \mathrm{d}x.$$

If $\mathcal{A} \in \mathcal{U}_{ad}^\lambda \times \mathcal{U}_{ad}^\mu$, then positive constants C, c_0 exist, independent of $\mathcal{A}$ and such that

$$|a(\mathcal{A}; u, v)| \leq C \|u\|_{1,\Omega} \|v\|_{1,\Omega} \quad \forall u, v \in [H^1(\Omega)]^3, \tag{3.8}$$

$$a(\mathcal{A}; u, u) \geq c_0 \|u\|_{1,\Omega}^2 \quad \forall u \in V_0, \tag{3.9}$$

where

$$V_0 = \{ v \in [H^1(\Omega)]^3 : \int_\Gamma v \, \mathrm{dS} = 0, \int_\Gamma v \times x \, \mathrm{dS} = 0 \}.$$

Proof. The estimate (3.8) follows from the Cauchy-Schwartz inequality and the boundedness of sets U_{ad}^λ, U_{ad}^μ. To justify (3.9), we write

$$a(\mathcal{A}; u, u) \geq 2\underline{\mu} \int_\Omega \varepsilon_{ij}(u) \varepsilon_{ij}(u) \mathrm{d}x$$

and use the Korn's inequality

$$\int_\Omega \varepsilon_{ij}(u) \varepsilon_{ij}(u) \mathrm{d}x \geq c \|u\|_{1,\Omega}^2 \quad \forall u \in V_0$$

(see e.g. [14]-Lemma 7.3.3).

Lemma 10. *Let $\lambda_n \in \mathcal{U}_{ad}^\lambda$, $\mu_n \in \mathcal{U}_{ad}^\mu$, $\lambda_n \to \lambda$ and $\mu_n \to \mu$ as $n \to \infty$. Then $\nu_n \to \nu$ and*

$$T^{*ij}(\nu_n) \to T^{*ij}(\nu) \quad \text{in } [L^2(\Gamma)]^3. \tag{3.10}$$

Proof. Since $\lambda_n + \mu_n \geq \underline{\lambda} + \underline{\mu} > 0$,

$$\nu_n = \frac{\lambda_n}{2(\lambda_n + \mu_n)} \to \frac{\lambda}{2(\lambda + \mu)} = \nu.$$

We infer that

$$\overline{s}_k^{ij}(\nu_n) \to \overline{s}_k^{ij}(\nu) \quad \text{in } L^2(\Gamma), \quad k = 1, 2, 3. \tag{3.11}$$

Indeed, we have $\lambda_n/\varkappa_n \to \lambda/\varkappa$, $\mu_n/\varkappa_n \to \mu/\varkappa$ and

$$\|\overline{u}^{ij0}(\nu_n) - \overline{u}^{ij0}(\nu)\|_{H^1(\Gamma)} \leq C|\nu_n - \nu| \to 0,$$

so that (3.11) holds.

Since the field w^{ij} is independent of $\mathcal{A}$, we arrive at (3.10).

Lemma 11. *Let $\lambda_n \in \mathcal{U}_{ad}^\lambda$, $\mu_n \in \mathcal{U}_{ad}^\mu$, $\lambda_n \to \lambda$ and $\mu_n \to \mu$ as $n \to \infty$ and $u^{*ij}(\mathcal{A}_n) \in V_0$. Then*

$$u^{*ij}(\mathcal{A}_n) \to u^{*ij}(\mathcal{A}) \quad \text{in } [H^1(\Omega)]^3.$$

Proof. For brevity, let us denote $T_n^* = T^{*ij}(\nu_n)$, $T^* = T^{*ij}(\nu)$, $u_n^* = u^{*ij}(\mathcal{A}_n)$, $u^* = u^{*ij}(\mathcal{A})$. By definition, we have

$$a(\mathcal{A}_n; u_n^*, v) = \int_\Gamma T_n^* v \, \mathrm{dS} \tag{3.12}$$

$$a(\mathcal{A}; u^*, v) = \int_\Gamma T^* v \, \mathrm{dS} \tag{3.13}$$

for all $v \in [H^1(\Omega)]^3$. Let us consider also solutions $\hat{u}_n \in V_0$ of the following problem

$$a(\mathcal{A}; \hat{u}_n, v) = \int_\Gamma T_n^* v \, \mathrm{dS} \quad \forall v \in [H^1(\Omega)]^3. \tag{3.14}$$

From (3.14) and (3.13) we obtain

$$a(\mathcal{A}; \hat{u}_n - u^*, v) = \int_\Gamma (T_n^* - T^*) v \, \mathrm{dS}.$$

Inserting $v := \hat{u}_n - u^*$ and using Lemma 9, we infer that

$$c_0 \|\hat{u}_n - u^*\|_{1,\Omega} \le C \|T_n^* - T^*\|_{L^2(\Gamma)} \tag{3.15}$$

so that $\|\hat{u}_n - u^*\|_{1,\Omega} \to 0$ follows from Lemma 10.

We can show that

$$\|u_n^*\|_{1,\Omega} \le C_1 \quad \forall n \; . \tag{3.16}$$

Indeed, (3.12) and Lemma 9 yield that

$$c_0 \|u_n^*\|_{1,\Omega}^2 \le C \|T_n^*\|_{L^2(\Gamma)} \|u_n^*\|_{1,\Omega},$$

so that (3.16) follows from Lemma 10.

We can use (3.12), (3.14) and Lemma 9 to obtain

$$\begin{aligned} c_0 \|u_n^* - \hat{u}_n\|_{1,\Omega}^2 &\le a(\mathcal{A}; u_n^* - \hat{u}_n, u_n^* - \hat{u}_n) \\ &= [a(\mathcal{A}; u_n^*, u_n^* - \hat{u}_n) - a(\mathcal{A}_n; u_n^*, u_n^* - \hat{u}_n)] \\ &+ [a(\mathcal{A}_n; u_n^*, u_n^* - \hat{u}_n) - a(\mathcal{A}; \hat{u}, u_n^* - \hat{u}_n)] \\ &= a(\mathcal{A}; u_n^*, u_n^* - \hat{u}_n) - a(\mathcal{A}_n; u_n^*, u_n^* - \hat{u}_n) \\ &\le C \|\mathcal{A} - \mathcal{A}_n\|_{0,\infty,\Omega} \|u_n^*\|_{1,\Omega} \|u_n^* - \hat{u}_n\|_{1,\Omega}. \end{aligned} \tag{3.17}$$

Then (3.16) and (3.17) yield

$$\|u_n^* - \hat{u}_n\|_{1,\Omega} \le C_2 \|\mathcal{A} - \mathcal{A}_n\|_{0,\infty,\Omega} \to 0. \tag{3.18}$$

The convergence $u_n^* \to u^*$ in $[H^1(\Omega)]^3$ follows from the triangle inequality, (3.15) and (3.18).

Proposition 12. *Let $\mathcal{A}_n \in \mathcal{U}_{ad}^\lambda \times \mathcal{U}_{ad}^\mu$, $\mathcal{A}_n \to \mathcal{A}$ in $\mathbb{R}^2$. Then*

$$G_y^{ij}(\mathcal{A}_n) \to G_y^{ij}(\mathcal{A}) \quad \text{in } [L^2(\Gamma)]^3, \quad i,j \in \{1,2,3\} \tag{3.19}$$

as $n \to \infty$.

Proof. Since by (3.3) we have

$$\|G_y^{ij}(\mathcal{A}_n) - G_y^{ij}(\mathcal{A})\|_{0,\Gamma} \le \|u^{*ij}(\mathcal{A}_n) - u_{ij}^*(\mathcal{A})\|_{0,\Gamma} + \|\overline{u}^{ij}(\mathcal{A}_n) - \overline{u}^{ij}(\mathcal{A})\|_{0,\Gamma},$$

the assertion follows from Lemma 11, the Trace Theorem and (3.5).

Proposition 13. *Let the stress components at the point y be*

$$\sigma_{kl}(\mathrm{A}) = \int_\Gamma T \cdot (c_{klij}(\mathcal{A}) G_y^{ij}(\mathcal{A})) \mathrm{d}S.$$

Assume that $A_n \in \mathcal{U}_{ad}$, $A_n \to A$ in $\mathbb{R}^2 \times [L^\infty(\Gamma)]^3$ as $n \to \infty$. Then

$$\sigma_{kl}(A_n) \to \sigma_{kl}(A).$$

Proof. We may write

$$
\begin{aligned}
|\sigma_{kl}(A_n) - \sigma_{kl}(A)| = |\textstyle\int_\Gamma T_n \cdot (c_{klij}(\mathcal{A}_n) G_y^{ij}(\mathcal{A}_n))\mathrm{d}S \\
- \textstyle\int_\Gamma T \cdot (c_{klij}(\mathcal{A}) G_y^{ij}(\mathcal{A}))\mathrm{d}S| \\
\le |\textstyle\int_\Gamma T_n \cdot ((c_{klij}(\mathcal{A}_n) - c_{klij}(\mathcal{A})) G_y^{ij}(\mathcal{A}_n))\mathrm{d}S| \\
+| \textstyle\int_\Gamma T_n \cdot (c_{klij}(\mathcal{A})(G_y^{ij}(\mathcal{A}_n) - G_y^{ij}(\mathcal{A})))\mathrm{d}S| \\
+| \textstyle\int_\Gamma (T_n - T) \cdot (c_{klij}(\mathcal{A}) G_y^{ij}(\mathcal{A}))\mathrm{d}S| \equiv I_1 + I_2 + I_3,
\end{aligned}
$$

where a positive constant C exists such that

$$I_1 \le \int_\Gamma C \|\mathcal{A}_n - \mathcal{A}\|_{0,\infty} \max_{i,j} |G_y^{ij}(\mathcal{A}_n)|\mathrm{d}S \to 0$$

and

$$I_2 \le \int_\Gamma C \max_{i,j} |G_y^{ij}(\mathcal{A}_n) - G_y^{ij}(\mathcal{A})| dS \to 0$$

due to Proposition 12 and the boundedness of T_n in $[L^\infty(\Gamma)]^3$. I_3 tend to zero by assumption.
Proof of Theorem 8. We have

$$
\begin{aligned}
|\Phi(\mathcal{A}_n, \sigma(A_n)) - \Phi(\mathcal{A}, \sigma(A))| \\
\le |\sigma(A_n)^\top \mathcal{H}(\mathcal{A}_n)(\sigma(A_n) - \sigma(A))| \\
+|\sigma(A_n)^\top (\mathcal{H}(\mathcal{A}_n) - \mathcal{H}(\mathcal{A}))\sigma(A)| \\
+|(\sigma(A_n)^\top - \sigma(A)^\top)\mathcal{H}(\mathcal{A})\sigma(A)| = J_1 + J_2 + J_3.
\end{aligned}
$$

By Proposition 13 we infer that J_1 and J_3 tend to zero. We also use the continuity of the function $\mathcal{A} \to \mathcal{H}(\mathcal{A})$, which follows from Lemma 10 and the convergence

$$E_n = 2\mu_n(1 + \nu_n) \to 2\mu(1 + \nu) = E \ge 2\underline{\mu} > 0.$$

As a consequence, J_2 tends to zero, as well.

3.1.2. *Worst Scenario and Maximum Range Scenario.* Suppose that we wish to be "on the safe side", taking uncertain input data $\mathcal{A}$ and T in consideration. Then we solve either the <u>worst scenario</u> problem

$$A^0 = arg \max_{A \in \mathcal{U}_{ad}} \Phi(\mathcal{A}, \sigma(A)) \tag{3.20}$$

or the <u>maximum range scenario</u> problem: find

: (i) A^0 according to (3.20) and
: (ii)

$$A_0 = arg \min_{A \in \mathcal{U}_{ad}} \Phi(\mathcal{A}, \sigma(A)). \tag{3.21}$$

In other words, we seek exact upper and lower bounds of the criterion functional (see the monograph [13] for applications of problem (3.21) within the frame of the fuzzy set theory).

Theorem 14. *Problems (3.20) and (3.21) have at least one solution.*

Proof. The set $\mathcal{U}_{ad}$ is compact in $\mathbb{R}^2 \times (\prod_{i=1}^{3} \prod_{p=1}^{P} C(\overline{\Gamma}_p))$, so that the assertion follows from Theorem 8.

3.2. **Traction Problem in 2D Elasticity.** Let us consider plane elasticity, i.e., either the case of plane strain or that of plane stress. It is well-known, that both cases have the same stress-strain relations, where only the coefficient λ varies. It is either λ or $\lambda^\star$, see (2.6).

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

for plane strain, whereas

$$\lambda = \lambda^* = \frac{E\nu}{1-\nu^2}$$

for plane stress.

Let us consider a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega \equiv \Gamma$, occupied by a homogeneous and isotropic elastic body, loaded only by surface loads $T \in [L^\infty(\Gamma)]^2$. Assume that $\lambda \in \mathcal{U}_{ad}^\lambda$, $\mu \in \mathcal{U}_{ad}^\mu$ and $T_i \in \mathcal{U}_{ad}^{T_i}$, $i = 1, 2$, with $\mathcal{U}_{ad}^\lambda$, $\mathcal{U}_{ad}^\mu$ and $\mathcal{U}_{ad}^{T_i}$ defined in section 3.1. Moreover, assume that the forces T are in equilibrium, i.e.

$$\int_\Gamma T \mathrm{d}S = 0, \quad \int_\Gamma (x_1 T_2 - x_2 T_1) \mathrm{d}S = 0. \tag{3.22}$$

We define

$$\mathcal{U}_{ad}^T = \{T \equiv (T_1, T_2) : T_i \in \mathcal{U}_{ad}^T, i = 1, 2, T \text{ satisfy } (3.22)\},$$
$$\mathcal{U}_{ad} = \mathcal{U}_{ad}^\lambda \times \mathcal{U}_{ad}^\mu \times \mathcal{U}_{ad}^T,$$
$$\mathcal{A} = \{\lambda, \mu\}, \quad A = \{\mathcal{A}, T\}$$

and introduce the criterion-functional based on the topological derivative associated to the energy shape functional (2.39)

$$\Phi(\mathcal{A}, \sigma) = \sigma^\top \mathcal{H}(\mathcal{A}) \sigma, \tag{3.23}$$

where $\sigma \equiv \sigma(y)$ is the stress tensor of a full body at the center $y \in \Omega$ of a circular cavity, and

$$\begin{aligned} \mathcal{H}(\mathcal{A}) &= \frac{1}{2E} (4\mathbb{I} - I \otimes I) \\ &= \frac{K + \mu}{8K\mu} (4\mathbb{I} - I \otimes I), \end{aligned} \tag{3.24}$$

where $K = \lambda + \mu$ is the bulk modulus.

3.2.1. *Continuous Dependence of the Criterion on the Input Data.* The main result of the present section will be represented by an analogue of Theorem 8 as follows.

Theorem 15. *Let $A_n \in \mathcal{U}_{ad}$, $A_n \to A$ in $\mathbb{R}^2 \times [L^\infty(\Gamma)]^2$ as $n \to \infty$. Then*

$$\Phi(\mathcal{A}_n, \sigma(A_n)) \to \Phi(\mathcal{A}, \sigma(A)).$$

For the proof we shall employ the following integral representation formula, analogous to (3.2), namely

$$\frac{\partial u_i}{\partial y_j}(y) = \int_\Gamma T \cdot G_y^{ij} \mathrm{d}S, \qquad i, j \in \{1, 2\}. \tag{3.25}$$

We can construct the vector function G_y^{ij} in a way parallel to that of the proof of Theorem 10.1.1 in [14]. First, we consider the well-known Kelvin solution

$$(u_y^i)_k = \varkappa_0^{-1}[-(K + 2\mu)\delta_{ik} \ln |r| + K r_i r_k |r|^{-2}], \tag{3.26}$$

where

$$\varkappa_0 = 4\pi\mu(K + \mu), \qquad r = x - y$$

and define

$$\overline{u}^{ij} = -\partial u_y^i / \partial y_j.$$

The corresponding surface forces on Γ are then

$$(\overline{s}^{ij})_k = [\lambda \delta_{km} \mathrm{div} \overline{u}^{ij} + 2\mu \varepsilon_{km}(\overline{u}^{ij})] n_m.$$

We can find that

$$\int_\Gamma \overline{s}^{ij} \mathrm{d}S = 0, \quad \int_\Gamma (x_1 (\overline{s}^{ij})_2 - x_2 (\overline{s}^{ij})_1) \mathrm{d}S = e_3 \cdot (e_j \times e_i). \tag{3.27}$$

Let us construct the rigid body translation

$$w^{ij} = a^{ij} + b^{ij} e_3 \times x$$

where $a^{ij} \in \mathbb{R}^2$, $b^{ij} \in \mathbb{R}$ and w^{ij} satisfies the following conditions

$$\int_\Gamma w^{ij} \mathrm{d}S = 0, \qquad \int_\Gamma (x_1 w_2^{ij} - x_2 w_1^{ij}) \mathrm{d}S = e_3 \cdot (e_i \times e_j). \tag{3.28}$$

Note that the field w^{ij} is uniquely determined by conditions (3.28). If we define

$$T^{*ij} = \overline{s}^{ij} + w^{ij},$$

the forces T^{*ij} are in equilibrium, i.e., they satisfy conditions (3.22).

There exists a unique displacement field u^{*ij}, which solves the first boundary value problem of elasticity with zero body forces and surface loads T^{*ij} and satisfies the normalization conditions

$$\int_\Gamma u^{*ij}\mathrm{d}S = 0, \qquad \int_\Gamma (x_1 u_2^{*ij} - x_2 u_1^{*ij})\mathrm{d}S = 0. \tag{3.29}$$

Next, we assume that the field u fullfils conditions (3.29) as well and consider the so-called Love formula

$$\frac{\partial u_i}{\partial y_j}(y) = \int_\Gamma (\overline{s}^{ij} \cdot u - T \cdot \overline{u}^{ij})\mathrm{d}S, \tag{3.30}$$

which follows by differentiating the so-called Somigliana identity

$$u_i(y) = \int_\Gamma (T \cdot u_y^i - u \cdot s_y^i)\mathrm{d}S, \tag{3.31}$$

where

$$\partial s_y^i / \partial y_j = -\overline{s}^{ij}.$$

By Reciprocity Theorem, we obtain

$$\int_\Gamma (T \cdot u^{*ij} - T^{*ij} \cdot u)\mathrm{d}S = 0. \tag{3.32}$$

Then (3.30) and (3.32) yield

$$\frac{\partial u_i}{\partial y_j}(y) = \int_\Gamma T \cdot (u^{*ij} - \overline{u}^{ij})\mathrm{d}S + \int_\Gamma u \cdot (\overline{s}^{ij} - T^{*ij})\mathrm{d}S.$$

The last integral vanishes by virtue of normalization conditions, since

$$\overline{s}^{ij} - T^{*ij} = -w^{ij}.$$

As a consequence, we arrive at the formula (3.25), where

$$G_y^{ij} = u^{*ij} - \overline{u}^{ij}. \tag{3.33}$$

Now we may go on in proving Theorem 15 as in the proof of Theorem 8. We establish an analogue of Lemma 9, where the subspace V_0 is defined by

$$V_0 = \left\{ v \in [H^1(\Omega)]^2 : \int_\Gamma v\,\mathrm{d}S = 0, \int_\Gamma (x_1 v_2 - x_2 v_1)\mathrm{d}S = 0 \right\}.$$

For the Korn's inequality in V_0, see e.g. section 10.2.2 in [14].

As far as an analogue of Lemma 10 is concerned, we use the formula

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

for plane strain and

$$\nu = \frac{\lambda^*}{\lambda^* + 2\mu}$$

for plane stress.

It is readily seen that $\overline{s}^{ij} \equiv \overline{s}^{ij}(\nu)$, i.e., it does not depend on the modulus E. Then we can prove that $\lambda_n^* \to \lambda^*$, $\nu_n \to \nu$ and

$$\overline{s}^{ij}(\nu_n) \to \overline{s}^{ij}(\nu) \quad \text{in } [L^2(\Gamma)]^2 \quad \text{as} \quad \nu_n \to \nu\ ,$$

since

$$\lambda_n(K_n + 2\mu_n)/\varkappa_{0n} \to \lambda(K + 2\mu)/\varkappa_0 \tag{3.34}$$

and $\lambda_n K_n/\varkappa_{0n} \to \lambda K/\varkappa_0$ for $K_n = \lambda_n + \mu_n$, $\lambda_n \in \mathcal{U}_{ad}^\lambda$, $\mu_n \in \mathcal{U}_{ad}^\mu$, $\mathcal{A}_n \to \mathcal{A}$.

The field w^{ij} is independent of $\mathcal{A}$, so that we arrive at

$$T^{*ij}(\nu_n) \to T^{*ij}(\nu) \quad \text{in} \quad [L^2(\Gamma)]^2.$$

An analogue of Lemma 11 can be proved in the same way as Lemma 11. We infer that

$$u^{*ij}(\mathcal{A}_n) \to u^{*ij}(\mathcal{A}) \quad \text{in} \quad [H^1(\Omega)]^2. \tag{3.35}$$

Using again (3.34), we observe that

$$\overline{u}^{ij}(\mathcal{A}_n) \to \overline{u}^{ij}(\mathcal{A}) \quad \text{in} \quad [L^2(\Gamma)]^2. \tag{3.36}$$

Combining (3.33) with (3.35), the Trace Theorem and (3.36), we obtain

$$G_y^{ij}(\mathcal{A}_n) \to G_y^{ij}(\mathcal{A}) \quad \text{in} \quad [L^2(\Gamma)]^2. \tag{3.37}$$

Theorem 15 follows in a way parallel to the proof of Theorem 8, from (3.37), the uniform convergence of surface loads on Γ and the continuity of the function $\mathcal{A} \mapsto \mathcal{H}(\mathcal{A})$.

3.2.2. *Worst Scenario and Maximum Range Scenario.* Both the worst scenario problem (3.20) and the maximum range scenario problem (3.21) have at least one solution. This assertion is a consequence of Theorem 15 and the compactness of the set $\mathcal{U}_{ad}$ in $\mathbb{R}^2 \times \prod_{i=1}^{2} \prod_{p=1}^{P} C(\overline{\Gamma}_p)$.

4. Conclusions

In the present paper it is shown that the topological derivatives of shape functionals in elasticity are robust mathematical objects in shape and topology optimization. Our analysis is performed in the particular and useful for application case of the worst scenario and maximum range scenario methods. The case of probabilistic distributions of imperfections in model parameters should be still investigated. The influence of imperfections on the resulting numerical solution of optimal design problems with respect to the uncertainty of model parameters depends on the probabilistic distributions of the shape gradients and of the topological derivatives. Such studies can be performed for some specific problems, and should be based on some effective description of probabilistic model imperfections, which seems to us to be quite complicated. In any case, such investigations should be performed in order to complete our analysis.

We have seen that the worst case and maximal range scenario problems are solvable with criterions of energy-based topological derivative. Therefore, the proposed method leads to a general approach to deal with uncertain input data considering different criteria based on topological derivative of arbitrary shape functionals. In fact, the same methodology may be applied to derive similar analysis for criteria dependent for example on displacement (kinematic constraints) and yield constraints, extending the classical result obtained for the energy functional.

Appendix A. Stress Distribution Around Cavities

We present in this appendix the analytical solution for the stress distribution around a circular ($N = 2$) and spherical ($N = 3$) cavities respectively for two and three-dimensional linear elastic bodies.

A.1. **Circular Cavity .** Considering a polar coordinate system (r, θ), we have the following expansion for the stress distribution $\sigma(\xi_\rho)$ around a free boundary circular cavity ($\sigma^{rr}(\xi_\rho) = \sigma^{r\theta}(\xi_\rho) = 0$ on $\partial B_\rho(x_0)$), with $\xi_\rho = u_\rho$ or $\xi_\rho = w_\rho$

$$\sigma^{rr}(\xi_\rho) = \frac{a_\xi}{2}\left(1 - \frac{\rho^2}{r^2}\right) + \frac{b_\xi}{2}\left(1 - 4\frac{\rho^2}{r^2} + 3\frac{\rho^4}{r^4}\right)\cos 2\theta_\xi + \mathcal{O}(\rho) \ , \tag{A.1}$$

$$\sigma^{\theta\theta}(\xi_\rho) = \frac{a_\xi}{2}\left(1 + \frac{\rho^2}{r^2}\right) - \frac{b_\xi}{2}\left(1 + 3\frac{\rho^4}{r^4}\right)\cos 2\theta_\xi + \mathcal{O}(\rho) \ , \tag{A.2}$$

$$\sigma^{r\theta}(\xi_\rho) = -\frac{b_\xi}{2}\left(1 + 2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right)\sin 2\theta_\xi + \mathcal{O}(\rho) \ , \tag{A.3}$$

where the angle $\theta_u = \theta$ and $\theta_w = \theta + \delta$, with δ denoting the angle between principal stress directions for displacement fields u and w in (2.35). In addition, the following expansion for $\sigma(v_\rho)$

satisfying the boundary condition on $\partial B_\rho(x_0)$ given by $\sigma^{r\theta}(v_\rho)=0$ and $\sigma^{rr}(v_\rho)=-2\gamma\sigma^{\theta\theta}(u_\rho)$, holds

$$\begin{aligned}
\sigma^{rr}(v_\rho) &= \frac{a_v}{2}\left(1-\frac{\rho^2}{r^2}\right)+\frac{b_v}{2}\left(1-4\frac{\rho^2}{r^2}+3\frac{\rho^4}{r^4}\right)\cos 2\theta_v \\
&\quad -2\gamma a_u\frac{\rho^2}{r^2}+4\gamma b_u\left(2\frac{\rho^2}{r^2}-\frac{\rho^4}{r^4}\right)\cos 2\theta+\mathcal{O}(\rho)\ , && \text{(A.4)}\\
\sigma^{\theta\theta}(v_\rho) &= \frac{a_v}{2}\left(1+\frac{\rho^2}{r^2}\right)-\frac{b_v}{2}\left(1+3\frac{\rho^4}{r^4}\right)\cos 2\theta_v \\
&\quad +2\gamma a_u\frac{\rho^2}{r^2}+4\gamma b_u\frac{\rho^4}{r^4}\cos 2\theta+\mathcal{O}(\rho)\ , && \text{(A.5)}\\
\sigma^{r\theta}(v_\rho) &= -\frac{b_v}{2}\left(1+2\frac{\rho^2}{r^2}-3\frac{\rho^4}{r^4}\right)\sin 2\theta_v+4\gamma b_u\left(\frac{\rho^2}{r^2}-\frac{\rho^4}{r^4}\right)\sin 2\theta+\mathcal{O}(\rho)\ , && \text{(A.6)}
\end{aligned}$$

where the angle $\theta_v=\theta+\delta$, with δ denoting the angle between principal stress directions for displacement fields u and v in (2.36). Finally,

$$a_\xi=\sigma_I(\xi)+\sigma_{II}(\xi) \quad \text{and} \quad b_\xi=\sigma_I(\xi)-\sigma_{II}(\xi)\ ,$$

where $\sigma_I(\xi)$ and $\sigma_{II}(\xi)$ are the principal stress values of tensor $\sigma(\xi)$, for $\xi=u$, $\xi=w$ or $\xi=v$ associated to the original domain without hole Ω.

A.2. **Spherical Cavity .** Let us introduce a spherical coordinate system (r,θ,φ). Then, the stress distribution around the spherical cavity $B_\rho(x_0)$ is given by

$$\begin{aligned}
\sigma^{rr}(\xi_\rho) &= \sigma_1^{rr}+\sigma_2^{rr}+\sigma_3^{rr}+\mathcal{O}(\rho)\ ,\\
\sigma^{r\theta}(\xi_\rho) &= \sigma_1^{r\theta}+\sigma_2^{r\theta}+\sigma_3^{r\theta}+\mathcal{O}(\rho)\ ,\\
\sigma^{r\varphi}(\xi_\rho) &= \sigma_1^{r\varphi}+\sigma_2^{r\varphi}+\sigma_3^{r\varphi}+\mathcal{O}(\rho)\ ,\\
\sigma^{\theta\theta}(\xi_\rho) &= \sigma_1^{\theta\theta}+\sigma_2^{\theta\theta}+\sigma_3^{\theta\theta}+\mathcal{O}(\rho)\ ,\\
\sigma^{\theta\varphi}(\xi_\rho) &= \sigma_1^{\theta\varphi}+\sigma_2^{\theta\varphi}+\sigma_3^{\theta\varphi}+\mathcal{O}(\rho)\ ,\\
\sigma^{\varphi\varphi}(\xi_\rho) &= \sigma_1^{\varphi\varphi}+\sigma_2^{\varphi\varphi}+\sigma_3^{\varphi\varphi}+\mathcal{O}(\rho)\ ,
\end{aligned}$$

where $\xi_\rho=u_\rho$, $\xi_\rho=w_\rho$ or $\xi_\rho=v_\rho$; σ_i^{rr}, $\sigma_i^{r\theta}$, $\sigma_i^{r\varphi}$, $\sigma_i^{\theta\theta}$, $\sigma_i^{\theta\varphi}$ and $\sigma_i^{\varphi\varphi}$, for $i=1,2,3$, are written, as:

for $i=1$

$$\begin{aligned}
\sigma_1^{rr} &= \tfrac{\sigma_I(\xi)}{14-10\nu}\left[12\left(\tfrac{\rho^3}{r^3}-\tfrac{\rho^5}{r^5}\right)+\left(14-10\nu-10(5-\nu)\tfrac{\rho^3}{r^3}+36\tfrac{\rho^5}{r^5}\right)\sin^2\theta\sin^2\varphi\right]\ , && \text{(A.7)}\\
\sigma_1^{r\theta} &= \tfrac{\sigma_I(\xi)}{14-10\nu}\left[7-5\nu+5(1+\nu)\tfrac{\rho^3}{r^3}-12\tfrac{\rho^5}{r^5}\right]\sin 2\theta\sin^2\varphi\ , && \text{(A.8)}\\
\sigma_1^{r\varphi} &= \tfrac{\sigma_I(\xi)}{14-10\nu}\left[7-5\nu+5(1+\nu)\tfrac{\rho^3}{r^3}-12\tfrac{\rho^5}{r^5}\right]\sin\theta\sin 2\varphi\ , && \text{(A.9)}\\
\sigma_1^{\theta\theta} &= \tfrac{\sigma_I(\xi)}{56-40\nu}\Big[14-10\nu+(1+10\nu)\tfrac{\rho^3}{r^3}+3\tfrac{\rho^5}{r^5}-\left(14-10\nu+25(1-2\nu)\tfrac{\rho^3}{r^3}-9\tfrac{\rho^5}{r^5}\right)\cos 2\varphi\\
&\quad +\left(28-20\nu-10(1-2\nu)\tfrac{\rho^3}{r^3}+42\tfrac{\rho^5}{r^5}\right)\cos 2\theta\sin^2\varphi\Big]\ , && \text{(A.10)}\\
\sigma_1^{\theta\varphi} &= \tfrac{\sigma_I(\xi)}{14-10\nu}\left[7-5\nu+5(1-2\nu)\tfrac{\rho^3}{r^3}+3\tfrac{\rho^5}{r^5}\right]\cos\theta\sin 2\varphi\ , && \text{(A.11)}\\
\sigma_1^{\varphi\varphi} &= \tfrac{\sigma_I(\xi)}{56-40\nu}\Big[28-20\nu+(11-10\nu)\tfrac{\rho^3}{r^3}+9\tfrac{\rho^5}{r^5}+\left(28-20\nu+5(1-2\nu)\tfrac{\rho^3}{r^3}+27\tfrac{\rho^5}{r^5}\right)\cos 2\varphi\\
&\quad -30\left((1-2\nu)\tfrac{\rho^3}{r^3}-\tfrac{\rho^5}{r^5}\right)\cos 2\theta\sin^2\varphi\Big]\ , && \text{(A.12)}
\end{aligned}$$

for $i=2$

$$\sigma_2^{rr} = \frac{\sigma_{II}(\xi)}{14-10\nu}\left[12\left(\frac{\rho^3}{r^3}-\frac{\rho^5}{r^5}\right)+\left(14-10\nu-10(5-\nu)\frac{\rho^3}{r^3}+36\frac{\rho^5}{r^5}\right)\sin^2\theta\cos^2\varphi\right] , \tag{A.13}$$

$$\sigma_2^{r\theta} = \frac{\sigma_{II}(\xi)}{14-10\nu}\left[7-5\nu+5(1+\nu)\frac{\rho^3}{r^3}-12\frac{\rho^5}{r^5}\right]\cos^2\varphi\sin 2\theta , \tag{A.14}$$

$$\sigma_2^{r\varphi} = \frac{-\sigma_{II}(\xi)}{14-10\nu}\left[7-5\nu+5(1+\nu)\frac{\rho^3}{r^3}-12\frac{\rho^5}{r^5}\right]\sin\theta\sin 2\varphi , \tag{A.15}$$

$$\begin{aligned}\sigma_2^{\theta\theta} = \frac{\sigma_{II}(\xi)}{56-40\nu}&\left[14-10\nu+(1+10\nu)\frac{\rho^3}{r^3}+3\frac{\rho^5}{r^5}+\left(14-10\nu+25(1-2\nu)\frac{\rho^3}{r^3}-9\frac{\rho^5}{r^5}\right)\cos 2\varphi\right.\\&\left.+\left(28-20\nu-10(1-2\nu)\frac{\rho^3}{r^3}+42\frac{\rho^5}{r^5}\right)\cos 2\theta\cos^2\varphi\right] ,\end{aligned} \tag{A.16}$$

$$\sigma_2^{\theta\varphi} = \frac{-\sigma_{II}(\xi)}{14-10\nu}\left[7-5\nu+5(1-2\nu)\frac{\rho^3}{r^3}+3\frac{\rho^5}{r^5}\right]\cos\theta\sin 2\varphi , \tag{A.17}$$

$$\begin{aligned}\sigma_2^{\varphi\varphi} = \frac{\sigma_{II}(\xi)}{56-40\nu}&\left[28-20\nu+(11-10\nu)\frac{\rho^3}{r^3}+9\frac{\rho^5}{r^5}-\left(28-20\nu+5(1-2\nu)\frac{\rho^3}{r^3}+27\frac{\rho^5}{r^5}\right)\cos 2\varphi\right.\\&\left.-30\left((1-2\nu)\frac{\rho^3}{r^3}-\frac{\rho^5}{r^5}\right)\cos 2\theta\cos^2\varphi\right] ,\end{aligned} \tag{A.18}$$

for $i=3$

$$\begin{aligned}\sigma_3^{rr} = \frac{\sigma_{III}(\xi)}{14-10\nu}&\left[14-10\nu-(38-10\nu)\frac{\rho^3}{r^3}+24\frac{\rho^5}{r^5}\right.\\&\left.-\left(14-10\nu-10(5-\nu)\frac{\rho^3}{r^3}+36\frac{\rho^5}{r^5}\right)\sin^2\theta\right] ,\end{aligned} \tag{A.19}$$

$$\sigma_3^{r\theta} = \frac{-\sigma_{III}(\xi)}{14-10\nu}\left[14-10\nu+10(1+\nu)\frac{\rho^3}{r^3}-24\frac{\rho^5}{r^5}\right]\cos\theta\sin\theta , \tag{A.20}$$

$$\sigma_3^{r\varphi} = 0 , \tag{A.21}$$

$$\sigma_3^{\theta\theta} = \frac{\sigma_{III}(\xi)}{14-10\nu}\left[(9-15\nu)\frac{\rho^3}{r^3}-12\frac{\rho^5}{r^5}+\left(14-10\nu-5(1-2\nu)\frac{\rho^3}{r^3}+21\frac{\rho^5}{r^5}\right)\sin^2\theta\right] , \tag{A.22}$$

$$\sigma_3^{\theta\varphi} = 0 , \tag{A.23}$$

$$\sigma_3^{\varphi\varphi} = \frac{\sigma_{III}(\xi)}{14-10\nu}\left[(9-15\nu)\frac{\rho^3}{r^3}-12\frac{\rho^5}{r^5}-15\left((1-2\nu)\frac{\rho^3}{r^3}-\frac{\rho^5}{r^5}\right)\sin^2\theta\right] , \tag{A.24}$$

where $\sigma_I(\xi)$, $\sigma_{III}(\xi)$ and $\sigma_{III}(\xi)$ are the principal stress values of tensor $\sigma(\xi)$, for $\xi=u$, $\xi=w$ or $\xi=v$ associated to the original domain without hole Ω.

Remark 16. *It is important to mention that the stress distribution for $i=1,2$ was obtained from a rotation of the stress distribution for $i=3$. In addition, the derivation of this last result (for $i=3$) can be found in* [12]*, for instance.*

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