FIRST AND SECOND ORDER TOPOLOGICAL SENSITIVITY ANALYSIS FOR INCLUSIONS

J. ROCHA DE FARIA, A.A. NOVOTNY, R.A. FEIJÓO, AND E. TAROCO

Abstract. The topological derivative provides the sensitivity of a given shape functional with respect to an infinitesimal domain perturbation. Classically, this derivative comes from the second term of the topological asymptotic expansion, dealing only with infinitesimal perturbations. Therefore, as a natural extension of this concept, we can consider higher order terms in the expansion. In particular, the next one we recognize as the second order topological derivative, which allows to deal with perturbations of finite sizes. This term depends explicitly on higher-order gradients of the solution associated to the non-perturbed problem and also implicitly through the solution of an auxiliary variational problem. In this paper, we calculate the explicit as well as implicit terms of the second order topological asymptotic expansion for the total potential energy associated to the Laplace equation in two-dimensional domain. The domain perturbation is done by the insertion of a small inclusion with thermal conductivity coefficient value different from the bulk material. Finally, we present some numerical experiments showing the influence of the second order term in the topological asymptotic expansion for several values of the thermal conductivity coefficient of the inclusion.

1. Introduction

The topological sensitivity analysis gives the topological asymptotic expansion of a shape functional with respect to an infinitesimal domain perturbation, like the insertion of holes, inclusions or source term [9, 29]. The second term of this expansion provides the topological derivative, which has been applied in several problems, such as topology optimization, image processing and inverse problems [2, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 25, 26, 28, 30, 31]. Such a technique has also been applied on noninvasive medical diagnosis and nondestructive evaluation of materials, for instance.

However, for practical applications, it is necessary to consider perturbations of finite size. In general, the first order correction term provides a good estimation for the shape functional only for infinitesimal perturbations. In this case, we need to consider higher order terms in the topological asymptotic expansion. The next one is recognized as the second order topological derivative. These issues were addressed in our previous work [12], where we have shown that the second order topological derivative provides a good estimation for the shape functional even for very large holes. Furthermore, we observe that in general the second order topological derivative depends explicitly on higher-order gradients of the solution associated to the non-perturbed problem and also implicitly through the solution of an auxiliary variational problem. However, in [12] we have only calculated the explicit term of the second order topological derivative. Therefore, as a natural extension of our work, in the present paper we calculate the explicit as well as implicit terms of the second order topological asymptotic expansion for the total potential energy associated to the Laplace equation in two-dimensional domain. In addition, the domain perturbation is done by the insertion of a small inclusion, stead of a hole, with thermal conductivity coefficient value different from the bulk material. In summary, the main contribution of this paper is the calculation of the complete second order topological asymptotic expansion for inclusions, whose result is very useful for applications in the context of inverse problems, where we seek to identify a set of inclusions from boundary measurements (see, for instance [1, 10]).

In order to fix the basics ideas, let us consider an open bounded domain $\Omega \subset \mathbb{R}^2$, with a smooth boundary $\partial\Omega$. If the domain Ω is perturbed by introducing a small inclusion represented by $\mathcal{I}_\varepsilon$, which is a ball of radius ε centered at point $\hat{\mathbf{x}} \in \Omega$, we have a perturbed domain $\Omega_\varepsilon = (\Omega \backslash \mathcal{H}_\varepsilon) \cup \mathcal{I}_\varepsilon$.

From these elements, the topological asymptotic expansion of a given shape functional ψ, when exists, may be expressed as

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f_1(\varepsilon) D_T\psi + f_2(\varepsilon) D_T^2\psi + \mathcal{R}(f_2(\varepsilon)) , \tag{1.1}$$

where $f_1(\varepsilon)$ and $f_2(\varepsilon)$ are positive and smooth functions that decreases monotonically such that $f_1(\varepsilon) \to 0$, $f_2(\varepsilon) \to 0$ when $\varepsilon \to 0^+$ and

$$\lim_{\varepsilon\to 0} \frac{f_2(\varepsilon)}{f_1(\varepsilon)} = 0 , \qquad \lim_{\varepsilon\to 0} \frac{\mathcal{R}(f_2(\varepsilon))}{f_2(\varepsilon)} = 0 . \tag{1.2}$$

In addition, $D_T\psi$ and $D_T^2\psi$ are the first and second order topological derivatives of ψ, respectively. In fact, dividing eq. (1.1) by $f_1(\varepsilon)$ and taking the limit $\varepsilon \to 0$ we obtain

$$D_T\psi = \lim_{\varepsilon\to 0} \frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f_1(\varepsilon)} . \tag{1.3}$$

Moreover, dividing eq. (1.1) by $f_2(\varepsilon)$ and taking the limit $\varepsilon \to 0$ we have

$$D_T^2\psi = \lim_{\varepsilon\to 0} \frac{\psi(\Omega_\varepsilon) - \psi(\Omega) - f_1(\varepsilon) D_T\psi}{f_2(\varepsilon)} . \tag{1.4}$$

In this work, we apply the Topological-Shape Sensitivity Method developed in [27] to calculate $D_T\psi$ and $D_T^2\psi$ for the total potential energy associated to the Laplace equation in two-dimensional domain.

Finally, we present some numerical experiments showing the influence of the second order term in the topological asymptotic expansion by taking different values for the parameter ε and several values of the thermal conductivity coefficient of the inclusion.

2. The Topological-Shape Sensitivity Method

The Topological-Shape Sensitivity Method [27] was proposed as an alternative procedure to calculate the topological derivative. A remarkable fact of this methodology is that it can be naturally extended to calculate higher order topological derivatives. In particular, the following results holds:

Theorem 1. *Let $f_1(\varepsilon)$ be a function chosen in order to $||D_T\psi||_{L^2(\Omega)} \neq 0$, then the (first order) topological derivative given by eq. (1.3) can be written as*

$$D_T\psi = \lim_{\varepsilon\to 0} \frac{1}{f_1'(\varepsilon)} \frac{d}{d\varepsilon} \psi(\Omega_\varepsilon) . \tag{2.1}$$

Theorem 2. *Let $f_2(\varepsilon)$ be a function chosen in order to $\left|\left|D_T^2\psi\right|\right|_{L^2(\Omega)} \neq 0$, then the second order topological derivative is given by*

$$D_T^2\psi = \lim_{\varepsilon\to 0} \frac{1}{f_2'(\varepsilon)} \left(\frac{d}{d\varepsilon} \psi(\Omega_\varepsilon) - f_1'(\varepsilon) D_T\psi \right) . \tag{2.2}$$

The derivative of the shape function with respect to the parameter ε that appears in eqs. (2.1, 2.2) may be seen as its classical shape sensitivity analysis [24]. In particular, for inclusions given by a disk, the shape change velocity $\mathbf{v}$ is defined on the boundary $\partial\Omega_\varepsilon$ as

$$\begin{cases} \mathbf{v} = & -\mathbf{n} & \text{on } \partial\mathcal{I}_\varepsilon \\ \mathbf{v} = & \mathbf{0} & \text{on } \partial\Omega \end{cases} , \tag{2.3}$$

where $\mathbf{n}$ is the outward unit normal vector field. Then, the shape derivative of the shape functional results in an integral on the boundary $\partial\mathcal{I}_\varepsilon$,

$$\frac{d}{d\varepsilon}\psi(\Omega_\varepsilon) = -\int_{\partial\mathcal{I}_\varepsilon} \boldsymbol{\Sigma}_\varepsilon \mathbf{n}\cdot\mathbf{n} , \tag{2.4}$$

where tensor $\boldsymbol{\Sigma}_\varepsilon$ can be interpreted as a generalization of the Eshelby energy-momentum tensor [11]. As a consequence, this tensor plays a central role in the Topological-Shape Sensitivity Method and should be clearly identified according to the problem under consideration.

On the other hand, the topological asymptotic expansion may also be written as

$$\frac{d}{d\varepsilon}\psi(\Omega_\varepsilon) = f_1'(\varepsilon)D_T\psi + f_2'(\varepsilon)D_T^2\psi + \mathcal{R}'(f_2(\varepsilon))f_2'(\varepsilon)\ , \tag{2.5}$$

which allows a straightforward identification of all terms in the above expansion.

Before applying the above methodology for a practical problem, let us present a simple one-dimensional diffusion-reaction example given by:

Example 3. *Let be a function $u_\varepsilon \in \mathcal{U}_\varepsilon$ satisfying*

$$\int_0^1 (u_\varepsilon'\eta' + c_\varepsilon(x)\, u_\varepsilon\eta)dx = \eta(1) \quad \forall\eta \in \mathcal{V}_\varepsilon\ , \tag{2.6}$$

where $\mathcal{U}_\varepsilon = \mathcal{V}_\varepsilon = \left\{\eta \in H^1(\Omega_\varepsilon) : \eta(0) = 0\right\}$, with $\mathcal{I}_\varepsilon = [\hat{x}, \hat{x}+\varepsilon]$. Then, let us consider a perturbation on the reaction coefficient $c_\varepsilon(x)$ satisfying

$$c_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in \mathcal{I}_\varepsilon \\ 0, & \text{if } x \in \Omega\backslash\mathcal{H}_\varepsilon \end{cases} . \tag{2.7}$$

Therefore, the above problem has exact solution, which is given by the following piecewise function

- *for $0 \le x < \hat{x}$*

$$u_\varepsilon(x) = \frac{x}{\cosh\varepsilon + \hat{x}\sinh\varepsilon}\ ; \tag{2.8}$$

- *for $\hat{x} \le x \le \hat{x}+\varepsilon$*

$$u_\varepsilon(x) = \frac{\hat{x}\cosh(\hat{x}-x) - \sinh(\hat{x}-x)}{\cosh\varepsilon + \hat{x}\sinh\varepsilon}\ ; \tag{2.9}$$

- *for $\hat{x}+\varepsilon < x \le 1$*

$$u_\varepsilon(x) = x - (\hat{x}+\varepsilon) + \frac{\hat{x}\cosh\varepsilon + \sinh\varepsilon}{\cosh\varepsilon + \hat{x}\sinh\varepsilon}\ . \tag{2.10}$$

In addition, let us consider a shape function associated to the solution u_ε evaluated at point $x = 1$, that is

$$\begin{aligned}\psi(\Omega_\varepsilon) &= u_\varepsilon(1) \\ &= 1 - (\hat{x}+\varepsilon) + \frac{\hat{x}\cosh\varepsilon + \sinh\varepsilon}{\cosh\varepsilon + \hat{x}\sinh\varepsilon}\ .\end{aligned}$$

Finally, from these elements we may calculate the topological asymptotic expansion for this simple problem. In fact,

- ***first order topological derivative:*** *applying theorem 1, we have for $f_1(\varepsilon) = \varepsilon$*

$$D_T\psi = -\hat{x}^2 \qquad \forall\hat{x} \in (0,1)\ ; \tag{2.11}$$

- ***second order topological derivative:*** *applying theorem 2, with $f_2(\varepsilon) = \varepsilon^2$, we obtain*

$$D_T^2\psi = \hat{x}^3 - \hat{x} \qquad \forall\hat{x} \in (0,1)\ . \tag{2.12}$$

From the above results, we may build the following topological asymptotic expansion of the shape function

$$u_\varepsilon(1) = u(1) - \varepsilon\hat{x}^2 + \varepsilon^2\left(\hat{x}^3 - \hat{x}\right) + o\left(\varepsilon^2\right)\ . \tag{2.13}$$

Let us fix the size of the inclusion, then we observe that the point which minimizes the shape function is given by $\hat{x} = 1-\varepsilon$ for both approximations

$$u_\varepsilon(1) \approx u(1) - \varepsilon\hat{x}^2\ , \tag{2.14}$$

$$u_\varepsilon(1) \approx u(1) - \varepsilon\hat{x}^2 + \varepsilon^2\left(\hat{x}^3 - \hat{x}\right)\ , \tag{2.15}$$

where function u is solution of eq. (2.6) for $\varepsilon = 0$. On the other hand, let us now fix $\hat{x} = 0.5$. Then we take the size of the inclusion $\varepsilon \in \{0, 0.01, 0.02, ..., 0.4\}$. Considering both approximations given by eqs. (2.14,2.15), we can build the curves shown in fig. 1. From analysis of this graphic, we observe that the second order topological derivative plays an important role in the estimation of the shape function, allowing a good approximation even for very large (finite) inclusions.

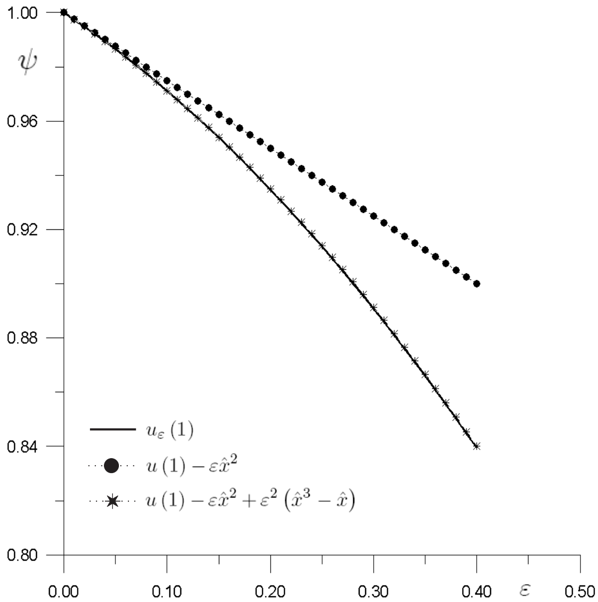

FIGURE 1. estimation of $u_\varepsilon(1)$ considering first and second order terms in the topological asymptotic expansion.

3. Topological Derivative for the Laplace Equation

As already mentioned, in this Section we will calculate first as well as second order topological derivative for inclusions taking the total potential energy associated to the Laplace equation in two-dimensional domain as the shape functional.

The variational formulation for the Laplace equation associated to the perturbed domain Ω_ε can be stated as: find $u_\varepsilon \in \mathcal{U}_\varepsilon$, such that

$$\int_{\Omega_\varepsilon} k_\delta \nabla u_\varepsilon \cdot \nabla \eta + \int_{\Gamma_N} \bar{q}\eta = 0 \quad \forall \eta \in \mathcal{U}_\varepsilon \ , \tag{3.1}$$

where the set $\mathcal{U}_\varepsilon$ is defined as

$$\mathcal{U}_\varepsilon = \{u_\varepsilon \in H^2(\Omega_\varepsilon) : u_\varepsilon|_{\Gamma_D} = 0\} \ ,$$

and Γ_D and Γ_N are the Dirichlet and Neumann boundaries, such that $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}$, with $\Gamma_D \cap \Gamma_N = \emptyset$. In addition, $\bar{q}$ is the heat flux prescribed on Γ_N and the material property k_δ is defined, for $\delta \in \mathbb{R}^+$, as

$$k_\delta := \begin{cases} k & \forall \mathbf{x} \in \Omega \backslash \mathcal{H}_\varepsilon \\ \delta k & \forall \mathbf{x} \in \mathcal{I}_\varepsilon \end{cases} \ . \tag{3.2}$$

The shape functional adopted will be the associated total potential energy, that is

$$\psi(\Omega_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} k_\delta \left|\nabla u_\varepsilon\right|^2 - \int_{\Gamma_N} \bar{q} u_\varepsilon \ . \tag{3.3}$$

For this case, the Eshelby energy-momentum tensor is given by

$$\mathbf{\Sigma}_\varepsilon = \frac{k_\delta}{2}\left|\nabla u_\varepsilon\right|^2 \mathbf{I} - k_\delta \left(\nabla u_\varepsilon \otimes \nabla u_\varepsilon\right) \ . \tag{3.4}$$

Adopting an orthonormal coordinate system $(\mathbf{t}, \mathbf{n})$ on $\partial\mathcal{I}_\varepsilon$, the continuity condition of the solution u_ε on the boundary of the inclusion $\partial\mathcal{I}_\varepsilon$ gives

$$\frac{\partial u_\varepsilon^-}{\partial t} = \frac{\partial u_\varepsilon^+}{\partial t} \quad \text{and} \quad \frac{\partial u_\varepsilon^-}{\partial n} = \frac{1}{\delta}\frac{\partial u_\varepsilon^+}{\partial n} \ , \tag{3.5}$$

where $(\cdot)^+ = (\cdot)|_{\Omega\backslash\mathcal{H}_\varepsilon}$, $(\cdot)^- = (\cdot)|_{\mathcal{I}_\varepsilon}$ and the jump condition associated to the normal derivative of u_ε comes from its respective variational formulation (eq. 3.1). Considering eq. (3.4) in eq.

(2.4) and taking into account the above conditions, the shape derivative of the shape functional results

$$\frac{d}{d\varepsilon}\psi(\Omega_\varepsilon) = -\frac{k\,(1-\delta)}{2}\int_{\partial\mathcal{I}_\varepsilon}\left[\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 + \frac{1}{\delta}\left(\frac{\partial u_\varepsilon^+}{\partial n}\right)^2\right] . \tag{3.6}$$

Now, we need to know the behavior of the solution u_ε in relation to ε, which, as consequence of the methodology adopted, may be obtained from an asymptotic analysis of u_ε around the neighborhood of the inclusion. In the present case, we have the following expansion for u_ε (see Appendix)

$$\begin{aligned} u_\varepsilon(\mathbf{x})|_{\Omega\backslash\mathcal{H}_\varepsilon} &= u(\mathbf{x}) + \frac{1-\delta}{1+\delta}\left[\frac{\varepsilon^2}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^2}\nabla u(\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}}) + \frac{1}{2}\frac{\varepsilon^4}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^4}\nabla\nabla u(\widehat{\mathbf{x}})(\mathbf{x}-\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}})\right] \\ &+ \varepsilon^2\widetilde{u}(\mathbf{x}) + \frac{1-\delta}{1+\delta}\frac{\varepsilon^4}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^2}\nabla\widetilde{u}(\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}}) + v_\varepsilon(\mathbf{x}) , \end{aligned} \tag{3.7}$$

where u is solution of (3.1) for $\varepsilon = 0$ (without inclusion), v_ε is such that $|v_\varepsilon|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^3$, with C independent of ε, and function $\widetilde{u}$ is solution of the following variational problem: find $\widetilde{u} \in \mathcal{V}$, such that

$$\int_\Omega \nabla\widetilde{u}\cdot\nabla\eta + \int_{\Gamma_N}\frac{\partial g}{\partial n}\eta = 0 \quad \forall\eta\in\mathcal{W} , \tag{3.8}$$

where the admissible functions set $\mathcal{V}$ and the admissible variations space $\mathcal{W}$ are defined, respectively, as

$$\mathcal{V} = \{\widetilde{u}\in H^1(\Omega) : \widetilde{u}|_{\Gamma_D} = -g\} \quad \text{and} \quad \mathcal{W} = \{\eta\in H^1(\Omega) : \eta|_{\Gamma_D} = 0\} \tag{3.9}$$

and function g is given by

$$g(\mathbf{x}) = \frac{1-\delta}{1+\delta}\nabla u(\widehat{\mathbf{x}})\cdot\frac{\mathbf{x}-\widehat{\mathbf{x}}}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^2} . \tag{3.10}$$

Considering the expansion given by eq. (3.7) in eq. (3.6), we obtain

$$\frac{d}{d\varepsilon}\psi(\Omega_\varepsilon) = -k\frac{1-\delta}{1+\delta}\left[2\pi\varepsilon\,|\nabla u(\widehat{\mathbf{x}})|^2 - 4\pi\varepsilon^3\left(\frac{1}{2}\det\nabla\nabla u(\widehat{\mathbf{x}}) - \nabla u(\widehat{\mathbf{x}})\cdot\nabla\widetilde{u}(\widehat{\mathbf{x}})\right)\right] + o\left(\varepsilon^3\right) . \tag{3.11}$$

Finally, taking into account eq. (2.5), we get the following results for $f_1(\varepsilon) = \pi\varepsilon^2$ and $f_2(\varepsilon) = \pi\varepsilon^4$

$$D_T\psi = -k\frac{1-\delta}{1+\delta}|\nabla u(\widehat{\mathbf{x}})|^2 , \tag{3.12}$$

$$D_T^2\psi = k\frac{1-\delta}{1+\delta}\left(\frac{1}{2}\det\nabla\nabla u(\widehat{\mathbf{x}}) - \nabla u(\widehat{\mathbf{x}})\cdot\nabla\widetilde{u}(\widehat{\mathbf{x}})\right) . \tag{3.13}$$

Remark 4. *From eq. (3.13) we observe that the second order topological derivative depends on higher-order gradients of the solution $u(\widehat{\mathbf{x}})$ and also through the solution $\widetilde{u}(\widehat{\mathbf{x}})$ of an auxiliary variational problem (3.8).*

Remark 5. *In the case of energy based shape functional, we can obtain the topological asymptotic expansion for homogeneous Neumann boundary condition on the hole by simply taking the limit when the material property associated to the inclusion vanishes. Thus, after compute the limit $\delta \to 0$ in eqs. (3.12, 3.13), we have*

$$D_T\psi = -k\,|\nabla u(\widehat{\mathbf{x}})|^2 , \tag{3.14}$$

$$D_T^2\psi = \frac{1}{2}k\det\nabla\nabla u(\widehat{\mathbf{x}}) - k\nabla u(\widehat{\mathbf{x}})\cdot\nabla\widetilde{u}(\widehat{\mathbf{x}}) . \tag{3.15}$$

4. Numerical Experiments

Now, we shall study, through some examples, the influence of the second order topological derivative in the topological asymptotic expansion. Therefore, we will compute the estimate for the shape functional taking into account only the first order topological derivative

$$\psi(\Omega_\varepsilon) \approx \psi(\Omega) + f_1(\varepsilon) D_T \psi \ . \tag{4.1}$$

Then we will compare it with the estimate considering both first and second order topological derivatives, that is

$$\psi(\Omega_\varepsilon) \approx \psi(\Omega) + f_1(\varepsilon) D_T \psi + f_2(\varepsilon) D_T^2 \psi \ . \tag{4.2}$$

Remark 6. *Solution $\widetilde{u}(\mathbf{x})$ of the auxiliary problem (3.8) depends on the point where the inclusion is positioned. Thus, the incorporation of its contribution on the second order topological derivative calculation is a quite cumbersome task and impracticable from the computational point of view. Therefore, the influence of function $\widetilde{u}(\mathbf{x})$ will be disregarded in the numerical experiments.*

For that, let us consider a body represented by $\Omega = (0,1) \times (0,1)$, submitted to a temperature $\bar{u} = 0$ on Γ_{D_1} and Γ_{D_2}, and a heat flux $\bar{q}_1 = 1$ on Γ_{N_1}, $\bar{q}_2 = 2$ on Γ_{N_2} and $\bar{q} = 0$ on $\Gamma_N \setminus \left(\overline{\Gamma_{N_1} \cup \Gamma_{N_2}}\right)$, as shown in fig. (2), where $a = 0.2$. This body is perturbed by introducing inclusions with center at $\mathbf{x}^* = (0.5, 0.5)$, where $\delta \in \{1/16, 1/8, 1/4, 1/2, 2, 4, 8, 16\}$ and $k = 1$. Then, for each value of δ, we take $\varepsilon \in \{0.01, 0.02, 0.04, 0.08, 0.16\}$. Considering these values of ε and δ, we compute the topological asymptotic expansion associated to the domain Ω at $\mathbf{x}^*$. Then, in order to compute the shape functional values $\psi(\Omega_\varepsilon)$, we effectively insert the inclusions with center at the fixed point $\mathbf{x}^*$. Finally, from these results, we can compare the accuracy obtained from both estimates given by eqs. (4.1, 4.2).

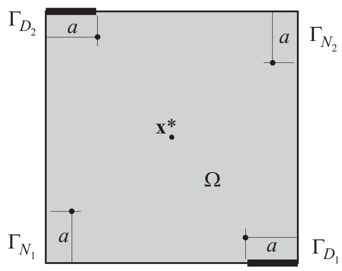

Figure 2. example.

The solutions u and u_ε associated to the original Ω and perturbed Ω_ε domains, respectively, are approximated using the standard three node triangular finite element. In particular, the meshes were constructed maintaining the same number of elements $ne = 120$ along the boundary of the inclusion for whichever value of its radius ε. Since an automatic mesh generation was used, the following expected size h^e for the elements was adopted for all meshes

$$h^e \approx \frac{2\pi}{ne} \|\mathbf{x}^* - \mathbf{x}\| \ . \tag{4.3}$$

The behavior of the topological asymptotic expansion as a function of ε, evaluated at $\mathbf{x}^*$, is shown in figs. (3-6) for different values of δ, where the label used to identify the curves is summarized in table 1.

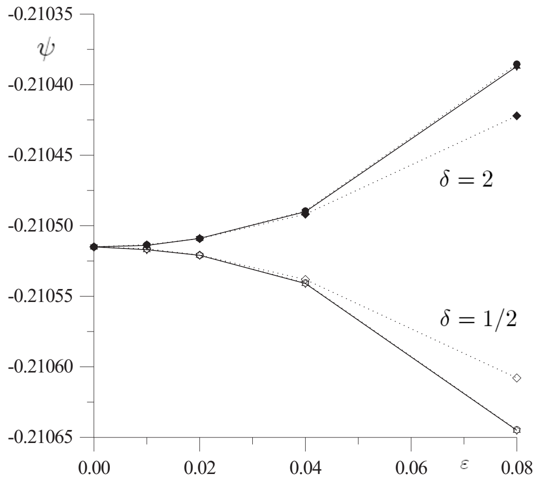

FIGURE 3. estimate of $\psi(\Omega_\varepsilon)$ for $\delta = 1/2$ and $\delta = 2$.

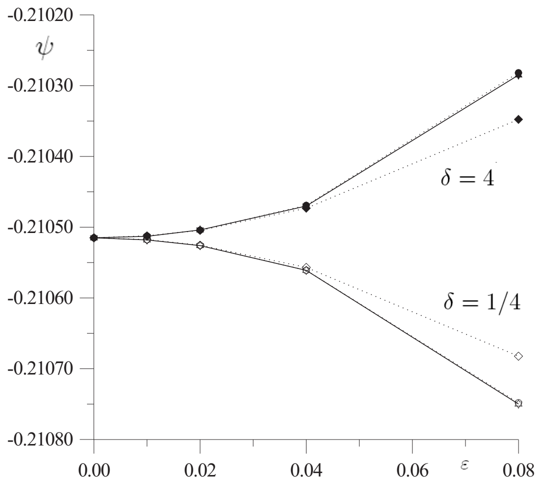

FIGURE 4. estimate of $\psi(\Omega_\varepsilon)$ for $\delta = 1/4$ and $\delta = 4$.

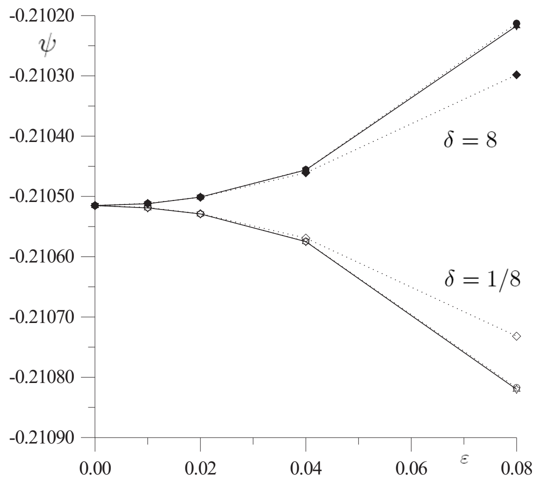

FIGURE 5. estimate of $\psi(\Omega_\varepsilon)$ for $\delta = 1/8$ and $\delta = 8$.

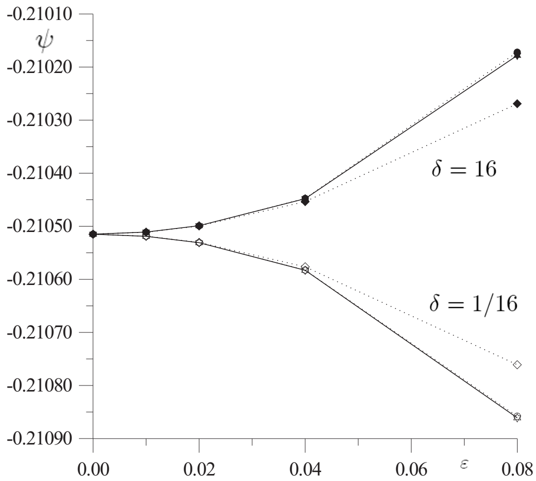

FIGURE 6. estimate of $\psi(\Omega_\varepsilon)$ for $\delta = 1/16$ and $\delta = 16$.

TABLE 1. label of the graphics in figs. (3-6).

$\delta > 1$	$\delta < 1$
$\psi(\Omega_\varepsilon)$	$\psi(\Omega_\varepsilon)$
$\psi(\Omega) + f_1(\varepsilon) D_T \psi$	$\psi(\Omega) + f_1(\varepsilon) D_T \psi$
$\psi(\Omega) + f_1(\varepsilon) D_T \psi + f_2(\varepsilon) D_T^2 \psi$	$\psi(\Omega) + f_1(\varepsilon) D_T \psi + f_2(\varepsilon) D_T^2 \psi$

According to remark 5, the limit case $\delta \to 0$ gives the topological asymptotic expansion for homogeneous Neumann boundary condition on the perturbation. Then, in fig. (7), we observe the behaviour of the topological asymptotic expansion when δ decreases to zero.

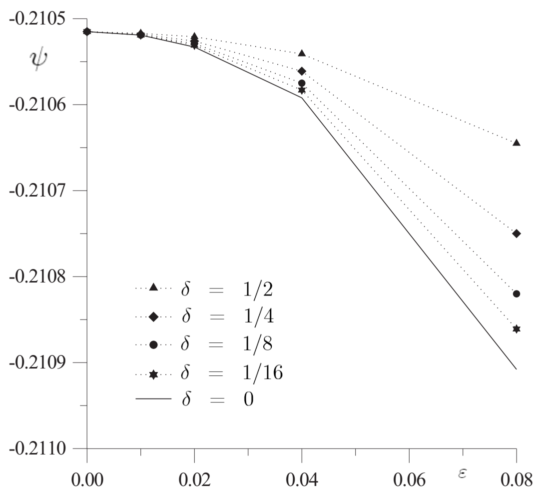

FIGURE 7. behaviour of the topological asymptotic expansion when δ decreases to 0.

Finally, taking again $\delta = 0$ and considering a larger variation of $\varepsilon \in \{0.08, 0.16, 0.24, 0.32\}$, we observe in fig. (8) that the discrepancy between the shape functional and its estimate considering first and second order topological derivatives (eq. 8) grows with ε. However, the estimation becomes bad only for very large holes ($\varepsilon > 0.16$).

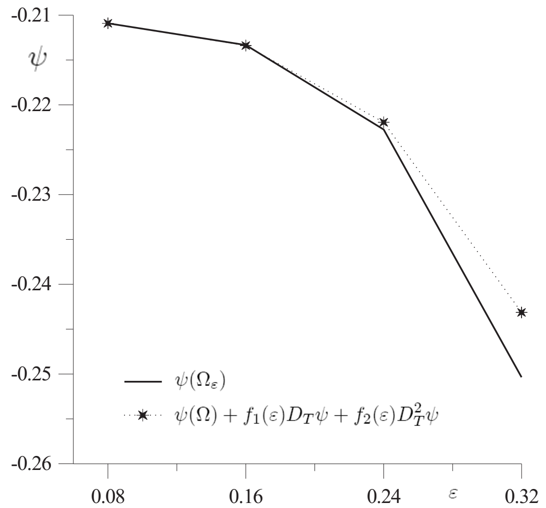

FIGURE 8. estimate of $\psi(\Omega_\varepsilon)$ considering the second order term of the topological asymptotic expansion for $\varepsilon \in \{0.08, 0.16, 0.24, 0.32\}$.

5. Conclusions

In this work, we have obtained an explicit formula for the topological asymptotic expansion considering first and second order approximations. In particular, we have applied the Topological-Shape Sensitivity Method to calculate first and second order topological derivatives for the total potential energy associated to the Laplace equation in two-dimensional domain, which was perturbed through the insertion of a small inclusion. Then, we have presented some numerical experiments showing the influence of the second order approximation term in the topological asymptotic expansion. From these experiments, we have observed that the estimate considering only the explicit term of the second order topological derivative remains precise even for very large inclusions or holes, allowing to deal with perturbations of finite size. This feature is very important in the development of topology optimization and reconstruction algorithms.

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Appendix: Asymptotic Analysis

In this appendix we present the derivation of the asymptotic formula given by eq. (3.7). Therefore, let us propose the following expansion

$$u_\varepsilon(\mathbf{x}) = u(\mathbf{x}) + w(\mathbf{x}/\varepsilon) + \widetilde{u}_\varepsilon(\mathbf{x}) \ , \tag{5.1}$$

where $u(\mathbf{x})$ is the solution of the problem stated in eq. (3.1) for $\delta = 1$ (without perturbation) and $w(\mathbf{x}/\varepsilon)$ is the solution of the following exterior problem

$$\left\{ \begin{array}{rcll} \Delta w^+ & = & 0 & \text{in} \quad \mathbb{R}^2 \backslash \mathcal{H}_1 \\ \Delta w^- & = & 0 & \text{in} \quad \mathcal{I}_1 \\ w & \to & 0 & \text{at} \quad \infty \\ w^+ & = & w^- & \text{on} \quad \partial\mathcal{I}_1 \\ \frac{\partial w^+}{\partial n} - \delta \frac{\partial w^-}{\partial n} & = & (\delta - 1)\left(\varepsilon \nabla u(\widehat{\mathbf{x}}) \cdot \mathbf{n} - \varepsilon^2 \nabla\nabla u(\widehat{\mathbf{x}})\, \mathbf{n} \cdot \mathbf{n}\right) & \text{on} \quad \partial\mathcal{I}_1 \end{array} \right. , \tag{5.2}$$

whose explicit solution may be written as

$$
\begin{aligned}
w^{+}(\mathbf{x}/\varepsilon) &= \frac{1-\delta}{1+\delta}\left[\frac{\varepsilon^{2}}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^{2}}\nabla u(\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}})+\frac{1}{2}\frac{\varepsilon^{4}}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^{4}}\nabla\nabla u(\widehat{\mathbf{x}})(\mathbf{x}-\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}})\right], \quad (5.3)\\
w^{-}(\mathbf{x}/\varepsilon) &= \frac{1-\delta}{1+\delta}\left[\nabla u(\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}})+\frac{1}{2}\nabla\nabla u(\widehat{\mathbf{x}})(\mathbf{x}-\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}})\right]. \quad (5.4)
\end{aligned}
$$

In addition, the remaining term of expansion (5.1) satisfies

$$
\left\{\begin{array}{rcllc}
\Delta\widetilde{u}_{\varepsilon}^{+} &=& 0 & \text{in} & \Omega\backslash\mathcal{H}_{\varepsilon}\\
\Delta\widetilde{u}_{\varepsilon}^{-} &=& 0 & \text{in} & \mathcal{I}_{\varepsilon}\\
\widetilde{u}_{\varepsilon} &=& -w & \text{on} & \Gamma_{D}\\
-\frac{\partial\widetilde{u}_{\varepsilon}}{\partial n} &=& \frac{\partial w}{\partial n} & \text{on} & \Gamma_{N}\\
\widetilde{u}_{\varepsilon}^{+} &=& \widetilde{u}_{\varepsilon}^{-} & \text{on} & \partial\mathcal{I}_{\varepsilon}\\
\frac{\partial\widetilde{u}_{\varepsilon}^{+}}{\partial n}-\delta\frac{\partial\widetilde{u}_{\varepsilon}^{-}}{\partial n} &=& \varepsilon^{3}(\delta-1)D^{3}u(\xi)(\mathbf{n})^{3} & \text{on} & \partial\mathcal{I}_{\varepsilon}
\end{array}\right., \quad (5.5)
$$

where ξ is an intermediate point between $\mathbf{x}$ and $\widehat{\mathbf{x}}$.

In the same way, we assume that $\widetilde{u}_{\varepsilon}$, solution of the boundary value problem (5.5), satisfies the expansion

$$
\widetilde{u}_{\varepsilon}(\mathbf{x}) = \varepsilon^{2}\widetilde{u}(\mathbf{x})+\widetilde{w}(\mathbf{x}/\varepsilon)+v_{\varepsilon}(\mathbf{x}), \quad (5.6)
$$

where $\widetilde{w}$ is the solution of the following exterior problem: find $\widetilde{w}$ such that

$$
\left\{\begin{array}{rcllc}
\Delta\widetilde{w}^{+} &=& 0 & \text{in} & \mathbb{R}^{2}\backslash\mathcal{H}_{1}\\
\Delta\widetilde{w}^{-} &=& 0 & \text{in} & \mathcal{I}_{1}\\
\widetilde{w} &\to& 0 & \text{at} & \infty\\
\widetilde{w}^{+} &=& \widetilde{w}^{-} & \text{on} & \partial\mathcal{I}_{1}\\
\frac{\partial\widetilde{w}^{+}}{\partial n}-\delta\frac{\partial\widetilde{w}^{-}}{\partial n} &=& (\delta-1)\varepsilon^{3}\nabla\widetilde{u}(\widehat{\mathbf{x}})\cdot\mathbf{n} & \text{on} & \partial\mathcal{I}_{1}
\end{array}\right., \quad (5.7)
$$

which also has explicit solution, that is

$$
\widetilde{w}^{+}(\mathbf{x}/\varepsilon) = \frac{1-\delta}{1+\delta}\frac{\varepsilon^{4}}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^{2}}\nabla\widetilde{u}(\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}}) \quad\text{and}\quad \widetilde{w}^{-}(\mathbf{x}/\varepsilon) = \frac{1-\delta}{1+\delta}\varepsilon^{2}\nabla\widetilde{u}(\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}}). \quad (5.8)
$$

Introducing the notation,

$$
g(\mathbf{x}) = \frac{1-\delta}{1+\delta}\frac{1}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^{2}}\nabla u(\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}}) \quad\text{and}\quad h(\mathbf{x}) = \frac{1}{2}\left(\frac{1-\delta}{1+\delta}\right)\frac{1}{\|\mathbf{x}-\widehat{\mathbf{x}}\|^{4}}\nabla\nabla u(\widehat{\mathbf{x}})(\mathbf{x}-\widehat{\mathbf{x}})\cdot(\mathbf{x}-\widehat{\mathbf{x}}), \quad (5.9)
$$

then

$$
w^{+}(\mathbf{x}/\varepsilon) = \varepsilon^{2}g(\mathbf{x})+\varepsilon^{4}h(\mathbf{x}). \quad (5.10)
$$

Thus, function $\widetilde{u}$ satisfies a boundary value problem stated as: find $\widetilde{u}$ such that

$$
\left\{\begin{array}{rclll}
\Delta\widetilde{u} &=& 0 & \text{in} & \Omega\\
\widetilde{u} &=& -g & \text{on} & \Gamma_{D}\\
-\frac{\partial\widetilde{u}}{\partial n} &=& \frac{\partial g}{\partial n} & \text{on} & \Gamma_{N}
\end{array}\right. \quad (5.11)
$$

and the remaining term of expansion (5.6) solves: find v_{ε} such that

$$
\left\{\begin{array}{rcllc}
\Delta v_{\varepsilon}^{+} &=& 0 & \text{in} & \Omega\backslash\mathcal{H}_{\varepsilon}\\
\Delta v_{\varepsilon}^{-} &=& 0 & \text{in} & \mathcal{I}_{\varepsilon}\\
v_{\varepsilon} &=& -\widetilde{w}-\varepsilon^{4}h & \text{on} & \Gamma_{D}\\
\frac{\partial v_{\varepsilon}}{\partial n} &=& -\frac{\partial\widetilde{w}}{\partial n}-\varepsilon^{4}\frac{\partial h}{\partial n} & \text{on} & \Gamma_{N}\\
v_{\varepsilon}^{+} &=& v_{\varepsilon}^{-} & \text{on} & \partial\mathcal{I}_{\varepsilon}\\
\frac{\partial v_{\varepsilon}^{+}}{\partial n}-\delta\frac{\partial v_{\varepsilon}^{-}}{\partial n} &=& \varepsilon^{3}(\delta-1)\left(D^{3}u(\xi)(\mathbf{n})^{3}-D^{2}\widetilde{u}(\zeta)(\mathbf{n})^{2}\right) & \text{on} & \partial\mathcal{I}_{\varepsilon}
\end{array}\right., \quad (5.12)
$$

where ζ is an intermediate point between $\mathbf{x}$ and $\widehat{\mathbf{x}}$.

Finally, we have the following expansion for solution u_{ε},

$$
u_{\varepsilon}(\mathbf{x}) = u(\mathbf{x})+w(\mathbf{x}/\varepsilon)+\varepsilon^{2}\widetilde{u}(\mathbf{x})+\widetilde{w}(\mathbf{x}/\varepsilon)+v_{\varepsilon}(\mathbf{x}). \quad (5.13)
$$

where v_{ε} satisfies the estimate below:

Proposition 7. *Let v_ε satisfying the boundary-value problem (5.12). Then, there exist a constant C, independent of ε, such that*

$$|v_\varepsilon|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^3. \tag{5.14}$$

Proof. See [1, 10, 20] $\square$

Remark 8. *The reader interested in asymptotic analysis for arbitrary shaped inclusions may refer to* [10]*, whose methodology may be extended to obtain higher order expansions like the one given by (5.13).*

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(J. Rocha de Faria) Laboratório Nacional de Computação Científica LNCC/MCT, Av. Getúlio Vargas 333, 25651-075 Petrópolis - RJ, Brasil

(A.A. Novotny) Laboratório Nacional de Computação Científica LNCC/MCT, Coordenação de Matemática Aplicada e Computacional, Av. Getúlio Vargas 333, 25651-075 Petrópolis - RJ, Brasil
E-mail address: novotny@lncc.br

(R.A. Feijóo) Laboratório Nacional de Computação Científica LNCC/MCT, Av. Getúlio Vargas 333, 25651-075 Petrópolis - RJ, Brasil

(E. Taroco) Laboratório Nacional de Computação Científica LNCC/MCT, Av. Getúlio Vargas 333, 25651-075 Petrópolis - RJ, Brasil