ON THE ROBUSTNESS WITH RESPECT TO TOPOLOGICAL PERTURBATION IN FLUID MECHANICS

M. ABDELBARIA, K. NACHI, J. SOKOLOWSKI, AND A.A. NOVOTNY

Abstract. In this paper, we study the issue of robustness with respect to topological perturbations in geometrical domains filled by a fluid flowing in Stokes-Darcy regime. We consider a cost functional given by the energy dissipation for the fluid. The topological perturbation is carried out by the nucleation of an infinitesimal circular obstacle, which can be considered as a small measurement device. Our approach is based on the topological derivative method, which has been previously employed in shape and topology optimization problems. The topological derivative (TD) is designed to provide information on the sensitivity of a given shape functional with respect to topological domain perturbations. Our main idea is to reduce the TD where the small device will be placed, and that is through a distributed control. By taking into account the effect of disturbance term or uncertain input data in TD expression, the problem of robustness with respect to topological perturbation for the energy functional can be formulated as minimax problem with pointwise observation. Numerical examples illustrate the efficiency of the proposed strategy.

1. Introduction

Problems of shape and topological sensitivity in fluids mechanics has been a topic of interest of several studies, see for example [2, 4, 18, 27]. Based on these works, we can conclude that the geometric design has a significant impact on relevant quantities describing the fluid flow, like vorticity, energy of system and drag forces. In this paper, our objective is to desensitize the energy functional with respect to a topological perturbation, which in this context represents a small circular obstacle within the fluid flow. Instead of studying the problem in domain with topological singularity, our approach consist of acting on the topological derivative by a distributed control, and taking in consideration the effect of a disturbance source term in the system. This approach leads to a minimax problem defined on the unperturbed domain. Our model is governed by the Stokes-Darcy system which describes fluid flow with slow motion in porous media. The topological perturbation is performed by inserting a small inclusion in geometrical domain, also known by the volume penalization method, see for example [23].

Closer works to our problem have been presented in the framework of insensitizing (or desensitizing) control, which goes back to the book by J. L. Lions from 1992 [25], where the notion of sentinel was introduced. Intuitively, an insensitizing control serves to neutralize a perturbation in some system according to a given cost functional. The sentinel method was generalized in several directions. We mention here the works of [10], [16] and [17], where they analyzed the problem of insensitizing for the wave equation, Stokes and Navier-Stokes systems, respectively. All

Date: September 19, 2024.

Key words and phrases. Topological derivative, shape optimization, optimal control, robust control.

these papers have dealt with perturbations in initial or boundary conditions. The case where the perturbation is prescribed on the boundary was discussed recently by [12], for the parabolic case. The authors applied the shape derivative to describe the sensitivity with respect to boundary variation, see [33]. We point out that the existence of insensitizing control is equivalent to the problem of controllability of a coupled system which involves the state and its adjoint state. In this paper, we introduce a relaxed problem in the sense that instead of looking for an exact desensitizing control for the energy functional with respect to topological perturbations, we seek a control such that the topological derivative evaluated where the obstacle will be created becomes as close as possible to zero.

Since our approach is based on the topological derivative concept, let us recall some results in topological sensitivity analysis that are available in the literature. The topological derivative was rigorously introduced by [32]. Considering its applications in shape optimization, it has been the subject of study for many models. For semilinear elliptic problems, the reader is referred to the works of [2], [21] and more recently the paper [34]. The topological sensitivity for compliance functional in linear elasticity was discussed by Garreau, Guillaume and Masmoudi in [14]. For variational inequalities and contact problems, the reader may refer to the paper [15] by Giusti, Sokołowski and Stebel. In fluid mechanics, we mention the papers [1], [19, 18] where the authors derive the TD expression for Navier-Stokes, Stokes and Quasi-Stokes systems, respectively. Applications of TD in inverse problems of detecting an unknown geometric object from a given measurements are discussed by [9], [22]. Concerning the theoretical development of the topological asymptotic analysis, the reader is referred to the monograph [29]. More recently, in [30], the numerical methods and applications of the topological derivative for several problems are discussed.

It is well know that the topological derivative depends on initial data of the system in consideration. Therefore, the question of robustness with respect to parameters arise naturally in numerical method of shape optimization. Let us mention some works in this context. In [20], the authors discuss the continuity of the topological derivative for elasticity system, with respect to Lamé coefficients and traction forces. Recently, in [24], the authors show that the TD for Helmholtz equation is robust with respect to frequency and boundary conditions. The upper and lower bounds of the TD with respect to initials data can be interpreted respectively as the worst-case design and the maximum-range design. Therefore, in this paper we deal with robustness issues with respect to topological perturbations in geometrical domains filled by a fluid flowing in Stokes-Darcy regime.

The rest of this paper is structured as follows. In Section 2, we introduce some notation and we describe our model of optimal control. The existence of robust control is discussed in Section 3. In Section 4, we derive the optimality conditions in terms of the adjoint state for robust control. Numerical results are presented in Section 5. Finally, some conclusions are drawn in Section 6.

2. Problem formulation

Let us recall briefly the definition of topological gradient and some preliminary results in topology optimization. Suppose that $\mathcal{J}(\Omega)$ is an integral functional depending on the solution of the boundary value problem defined in $\Omega \subset \mathbb{R}^N$. For a small parameter $\varrho > 0$, consider the perforated domain $\Omega_\varrho := \Omega \setminus \overline{B_\varrho(x_0)}$, where $\overline{B_\varrho(x_0)}$ is the closed ball of radius ϱ and center x_0, and with the boundary Γ_ϱ. The topological derivative of the functional $\mathcal{J}(\Omega)$ is defined by the following asymptotic expansion :

$$\mathcal{J}(\Omega_\varrho) = \mathcal{J}(\Omega) + f(\varrho)\mathcal{T}(x_0) + o(f(\varrho)), \tag{2.1}$$

where $\mathcal{J}(\Omega)$ is the functional evaluated for the given original domain and $\mathcal{J}(\Omega_\varrho)$ for a perturbed domain obtained by introducing a topological perturbation of size ϱ. The term $f(\varrho) > 0$ is a regularizing function which depends on dimension N. The remainder $o(f(\varrho))$ contains all terms of higher order than $f(\varrho)$, i.e.

$$\lim_{\varrho \to 0} \frac{o(f(\varrho))}{f(\varrho)} = 0.$$

The function $x_0 \mapsto \mathcal{T}(x_0)$ is called the topological derivative of $\mathcal{J}$ at x_0. the topological derivative $\mathcal{T}(x_0)$ provides information for creating a small hole located at x_0. Actually, if $\mathcal{T}(x_0) < 0$ then $\mathcal{J}(\Omega_\varrho) < \mathcal{J}(\Omega)$, for ϱ sufficiently small. therefore, in order to decrease the functional $\mathcal{J}$, we have to create a hole (or an obstacle) inside the geometrical domain where $\mathcal{T}$ is most negative. More generally, the function $\mathcal{T}$ can be used as a descent direction in topology optimization and, unlike to classical shape optimization, it allows us to modify the topology of domain during the optimization process. See, for example, the works of Garreau and Masmoudi [14, 19] for applications in elasticity and fluid mechanics, and [9] for applications in inverse problems of detecting an obstacle immersed in a fluid. Through the previous analysis, we deduce that the best position to place a hole B_ϱ in Ω, regarding the shape functional $\mathcal{J}$, corresponds to

$$\overline{x} = arg \min_{x \in \Omega} \mathcal{T}(x).$$

Suppose that we have an arbitrary infinitesimal topological singularity $B_\varrho(x_0)$ not necessary located at $\overline{x}$, $(x_0 \neq \overline{x})$, we can take for example

$$x_0 = arg \max_{x \in \Omega} \mathcal{T}(x),$$

which is the most undesirable situation. The issue that we will address in our paper is as follows:

> *How can we reduce the impact of the topological perturbation $B_\varrho(x_0)$ in the cost functional $\mathcal{J}$?*

In other words, we look for a control that makes the shape functional $\mathcal{J}$ less sensitive with respect to the topological singularity at x_0. Note that the problem of insensitizing control has been the subject of several research, specially when the perturbation is prescribed in the initial or boundary conditions, see for example [12, 16, 17, 25]. The abstract problem given above will become clearer after fixing the boundary value problem, which is in our case the stationary Stokes-Darcy system.

Let Ω be the fluid domain in $\mathbb{R}^N$ $(N = 2 \,\text{or}\, 3)$, with C^2 smooth boundary $\Gamma := \partial\Omega$. The fluid is described by its velocity y and pressure p satisfying the Stokes-Darcy equations:

$$\begin{cases} -\mu\Delta y + \nabla p + \eta y &= h\chi_\omega \\ &+ \; u\chi_{\omega_1} + \tau\chi_{\omega_2} \quad \text{in} \quad \Omega, \\ \text{div } y &= 0 \qquad\qquad\quad \text{in} \quad \Omega. \\ y &= 0 \qquad\qquad\quad \text{on} \quad \Gamma, \end{cases} \tag{2.2}$$

where μ stands for the kinematic viscosity coefficient, η is the inverse permeability and h is a given source term. The control and the disturbance terms are given by u, τ, respectively. The characteristic functions of ω, ω_1, $\omega_2 \subset \Omega$ are respectively denoted by χ_ω, χ_{ω_1}, χ_{ω_2}. For simplicity the following notations are used for functional spaces,

$$\begin{aligned} &\mathbf{L}^2(\Omega) := L^2(\Omega)^N,\ \mathbf{H}_0^1(\Omega) := H_0^1(\Omega)^N,\ \mathbf{C}(\overline{\Omega}) := C(\overline{\Omega})^N, \\ &\mathbf{H}_{\text{div}}(\Omega) := \{\varphi \in \mathbf{H}_0^1(\Omega),\ \ \text{div}\,\varphi = 0\} \\ &L_0^2(\Omega) := \{\varphi \in L^2(\Omega),\ \int_\Omega \varphi\, dx = 0\}. \end{aligned}$$

The existence of solutions for Stokes-Darcy system is well known, one can check that for all h, u, $\tau \in \mathbf{L}^2(\Omega)$, there exists a unique pair $(y, p) \in \mathbf{H}_{\text{div}}(\Omega) \times L_0^2(\Omega)$ which is a solution of (2.2), see for instance [7] or [13]. The energy dissipation functional of the system (2.2) is defined as follows

$$\mathcal{E}_{u,\tau}(\Omega) = \mu \int_\Omega |\nabla y|^2\, dx + \int_\Omega \eta |y|^2\, dx. \tag{2.3}$$

The topological perturbation of the geometrical domain is defined by inserting an inclusion $\overline{B_\varrho(x_0)}$ in Ω, where $\overline{B_\varrho(x_0)} \Subset \Omega \setminus (\omega \cup \omega_1 \cup \omega_2)$ is the closed ball of radius ϱ and center x_0, and with the boundary Γ_ϱ. More precisely Ω_ϱ is defined through the penalization coefficient $\eta_\varrho := \eta\gamma_\varrho$, see [23], where γ_ϱ is a piecewise constant function given by

$$\gamma_\varrho(x) = \begin{cases} 1 & \text{in} \quad \Omega \setminus \overline{B_\varrho}, \\ \gamma & \text{in} \quad B_\varrho. \end{cases} \tag{2.4}$$

Here $\gamma > 0$ is the contrast parameter. Therefore, when $\gamma \to +\infty$, we have $|y|_{|B_\varrho} \equiv 0$, see Fig. 1.

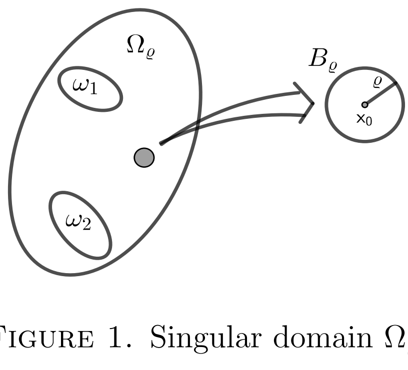


FIGURE 1. Singular domain Ω_ϱ.

By (y_ϱ, p_ϱ), we denote the unique solution of the system (2.2) with the perturbation η_ϱ :

$$\begin{cases} -\mu\Delta y_\varrho + \nabla p_\varrho + \eta_\varrho y_\varrho &= h\chi_\omega \\ &+ \; u\chi_{\omega_1} + \tau\chi_{\omega_2} \quad \text{in} \quad \Omega, \\ \text{div } y_\varrho &= 0 \qquad\qquad\quad \text{in} \quad \Omega, \\ y_\varrho &= 0 \qquad\qquad\quad \text{on} \quad \Gamma. \end{cases} \tag{2.5}$$

The topological derivative for the shape functional $\mathcal{E}$ is derived in [28]. To be more clear, we will briefly present in Appendix the proof of the following asymptotic expansion

$$\mathcal{E}_{u,\tau}(\Omega_\varrho) - \mathcal{E}_{u,\tau}(\Omega) = f(\varrho)\mathcal{T}_{u,\tau}(x_0) + o(f(\varrho)), \tag{2.6}$$

where the topological derivative $\mathcal{T}_{u,\tau}$ and the function $f(\varrho)$ are given by

$$\mathcal{T}_{u,\tau}(x_0) = (1-\gamma)\eta|y(x_0)|^2, \; f(\varrho) = meas(B_\varrho) \tag{2.7}$$

For $\gamma \geq 1$, we have $\mathcal{T}_{u,\tau} \leq 0$, which mean that creating an infinitesimal inclusion inside Ω will decrease the energy functional $\mathcal{E}_{u,\tau}$. Suppose that τ is fixed in $\mathbf{L}^2(\omega_2)$, our aim is to reduce the effect of singularity in geometrical domain for energy dissipation $\mathcal{E}_{u,\tau}$, i.e., minimize the gap between $\mathcal{E}_{u,\tau}(\Omega_\varrho)$ and $\mathcal{E}_{u,\tau}(\Omega)$ by a distributed control u. this problem can be formulated as follow :

$$\min_{u \in L^2(\omega_1)} |(\mathcal{E}_{u,\tau}(\Omega_\varrho) - \mathcal{E}_{u,\tau}(\Omega))|$$

Observe that if $|\mathcal{T}_{\overline{u},\tau}(x_0)| \leq |\mathcal{T}_{u,\tau}(x_0)|$, $\forall u \in L^2(\omega_1)$, then for ϱ small enough, we have

$$|\mathcal{E}_{\overline{u},\tau}(\Omega_\varrho) - \mathcal{E}_{\overline{u},\tau}(\Omega)| \leq |\mathcal{E}_{u,\tau}(\Omega_\varrho) - \mathcal{E}_{u,\tau}(\Omega)| \tag{2.8}$$

The last remark motivates us to introduce the following cost functional

$$\mathcal{J}(u,\tau) = \frac{1}{2}|y(x_0)|^2 + \frac{\alpha}{2}\int_{\omega_1} |u|^2\,dx - \frac{\beta}{2}\int_{\omega_2} |\tau|^2\,dx, \tag{2.9}$$

where α, $\beta > 0$ are the regularization parameters. The cost functional $\mathcal{J}$ is simultaneously minimized with respect to the control u and maximized with respect to the disturbance term τ. The control u serves to influence the topological derivative at x_0 and make it as close as possible to zero, while τ is considered to increase the robustness of the control. Therefore, the worst case disturbance corresponds to the maximum of $\mathcal{J}$ with respect to τ. The rest of this article is devoted to the analysis of the following minimax problem

$$\begin{cases} \min\limits_{u \in \mathbf{L}^2(\Omega)} \max\limits_{\tau \in \mathbf{L}^2(\Omega)} \mathcal{J}(u,\tau) \\ \text{subject to (2.2)} \end{cases} \tag{2.10}$$

Remark 1. *The regularity results for the Stokes system can be easily adapted to the system (2.2), see for example* [13] *Thm. IV .6.1., thus according to the assumptions on* Ω*, for each* $h \in \boldsymbol{L}^2(\omega)$*,* $u \in \boldsymbol{L}^2(\omega_1)$*, and* $\tau \in \boldsymbol{L}^2(\omega_2)$*, we have* $y \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$*, and by embedding theorem we conclude that* $y \in \boldsymbol{C}(\overline{\Omega})$*. Therefore the topological derivative* $\mathcal{T}$ *which is the pointwise term in the cost functional (2.9) is well defined.*

It is well know that minimax problems are closely related to the existence of a saddle points, more precisely, we have the following definition for robust control.

Definition 2. *The triple* $(\overline{u},\overline{\tau},\overline{y})$*, where* $\overline{y} = y(\overline{u},\overline{\tau})$*, is a solution of the robust control problem if* $(\overline{u},\overline{\tau})$ *is a saddle point of the cost functional* $\mathcal{J}$*, i.e.*

$$\mathcal{J}(\overline{u},\tau) \leq \mathcal{J}(\overline{u},\overline{\tau}) \leq \mathcal{J}(u,\overline{\tau}), \; \forall (u,\tau) \in \boldsymbol{L}^2(\Omega).$$

or equivalently

$$\begin{aligned} \mathcal{J}(\overline{u},\overline{\tau}) &= \min_{u \in \boldsymbol{L}^2(\Omega)} \max_{\tau \in \boldsymbol{L}^2(\Omega)} \mathcal{J}(u,\tau) \\ &= \max_{\tau \in \boldsymbol{L}^2(\Omega)} \min_{u \in \boldsymbol{L}^2(\Omega)} \mathcal{J}(u,\tau). \end{aligned}$$

3. Existence of robust control

In this section, we study the existence of robust control for the problem (2.10). The first works dealing with robustness of control go back to Bewely, Temam and Ziane [5], where the authors introduce a general framework for robust control for Navier-Stokes problem. For optimal control problems with poinwise observations, the reader may refer to [8, 31]. The existence of a saddle point for the functional $\mathcal{J}$ is an application of the following proposition.

Proposition 3. *Let $\mathcal{J}$ be a functional defined on $\mathcal{U}_1 \times \mathcal{U}_2$, where $\mathcal{U}_1, \mathcal{U}_2$ are two Banach spaces. If $\mathcal{J}$ satisfies the conditions*

(1) $\forall \kappa \in \mathcal{U}_2,\ \sigma \mapsto \mathcal{J}(\sigma, \kappa)$ *is convex lower semicontinuous,*
(2) $\forall \sigma \in \mathcal{U}_1,\ \kappa \mapsto \mathcal{J}(\sigma, \kappa)$ *is concave upper semicontinuous,*
(3) $\exists\, \kappa_0 \in \mathcal{U}_2,$ *such that* $\lim\limits_{\|\sigma\| \to +\infty} \mathcal{J}(\sigma, \kappa_0) = +\infty,$
(4) $\exists\, \sigma_0 \in \mathcal{U}_1,$ *such that* $\lim\limits_{\|\kappa\| \to +\infty} \mathcal{J}(\sigma_0, \kappa) = -\infty,$

then the functional $\mathcal{J}$ has at least one saddle point $(\overline{\sigma}, \overline{\kappa})$.

Proof. See [11]. p. 173. □

Our main result of this section is given by the following theorem.

Theorem 4. *For β sufficiently large, $(\beta > \beta_0 > 0)$, there exists at least one saddle point $(\overline{u}, \overline{\tau}) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ of the functional $\mathcal{J}$.*

Proof. We apply Proposition 3 with $\mathcal{U}_1 = \mathcal{U}_2 = \mathbf{L}^2(\Omega)$. We need to verify all the assumptions for the functional $\mathcal{J}$. First, we point out that the mappings $u \to y(u, \tau),\ \tau \to y(u, \tau)$ are affine, and continuous from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}^2(\Omega)$, this is a direct result of the classical energy estimate:

$$\|y\|_{\mathbf{H}^2(\Omega)} + \|p\|_{L_0^2(\Omega)} \leq K(\|u\|_{\mathbf{L}^2(\omega_2)} + \|\tau\|_{\mathbf{L}^2(\omega_2)})$$

Therefore, we deduce immediately that $\mathcal{J}$ is lower (and upper) semicontinuous with respect to u, (and τ). For convexity and concavity we can use the second derivative of $\mathcal{J}$ with respect to u, and τ, thus, we must check that $d_u^2\mathcal{J}(\xi, \xi) > 0$ and $d_\tau^2\mathcal{J}(\xi, \xi) < 0, \forall \xi \in \mathbf{L}^2(\Omega) \setminus \{0\}$. The operators $\tau \to y(u, \tau),\ u \to y(u, \tau)$ are Fréchet differentiable and their derivatives $\theta_\xi := d_u y(u, \tau)(\xi)$ and $\psi_\xi := d_\tau y(u, \tau)(\xi)$ obey the systems :

$$\begin{cases} -\mu\Delta\theta + \nabla\upsilon + \eta\,\theta &= \xi\chi_{\omega_1} \quad \text{in} \quad \Omega, \\ \text{div}\ \theta &= 0 \quad \text{in} \quad \Omega, \\ \theta &= 0 \quad \text{on} \quad \Gamma, \end{cases} \tag{3.1}$$

$$\begin{cases} -\mu\Delta\psi + \nabla\kappa + \eta\,\psi &= \xi\chi_{\omega_2} \quad \text{in} \quad \Omega, \\ \text{div}\ \psi &= 0 \quad \text{in} \quad \Omega, \\ \psi &= 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{3.2}$$

Now the first and second derivatives of the functional $\mathcal{J}$ in the direction ξ are given by

$$d_u\mathcal{J}(\xi) = y(x_0) \cdot \theta_\xi(x_0) + \alpha\langle u\,, \xi\rangle_{\mathbf{L}^2(\omega_1)}, \tag{3.3}$$

$$d_u^2\mathcal{J}(\xi,\, \xi) = |\theta_\xi(x_0)|^2 + \alpha\|\xi\|^2_{\mathbf{L}^2(\omega_1)}, \tag{3.4}$$

$$d_\tau\mathcal{J}(\xi) = y(x_0) \cdot \psi_\xi(x_0) - \beta\langle \tau\,, \xi\rangle_{\mathbf{L}^2(\omega_2)}, \tag{3.5}$$

$$d_\tau^2\mathcal{J}(\xi,\, \xi) = |\psi_\xi(x_0)|^2 - \beta\|\xi\|^2_{\mathbf{L}^2(\omega_2)}. \tag{3.6}$$

From (3.4) we conclude that $\mathcal{J}$ is convex with respect to u. On the other hand, (ψ, κ) is the solution of first Stokes system in (3.1), so we have the $\mathbf{H}^2$-regularity for the velocity ψ, see for example [7, 13]. Moreover, there exists $C_1 > 0$ depending only on Ω, such that

$$\|\psi_\xi\|_{\mathbf{H}^2(\Omega)} + \|\kappa\|_{L^2_0(\Omega)} \leq C_1 \|\xi\|_{\mathbf{L}^2(\omega_2)}. \tag{3.7}$$

Recall that the space $\mathbf{H}^2(\Omega)$ is continuously embedded in $\mathbf{C}(\overline{\Omega})$. Thus, the following estimate holds

$$|\psi_\xi(x_0)| \leq \|\psi_\xi\|_{\mathbf{C}(\overline{\Omega})} \leq C \|\xi\|_{\mathbf{L}^2(\omega_2)}, \tag{3.8}$$

where $C = C_1 \cdot C_2$, C_1 is given in (3.7) and C_2 is the embedding constant. By the expression of $d^2_\tau \mathcal{J}(\xi, \xi)$ from (3.6) and the estimate (3.8), we deduce that

$$d^2_\tau \mathcal{J}(\xi, \xi) \leq (C^2 - \beta) \|\xi\|^2_{\mathbf{L}^2(\omega_2)}.$$

Therefore, $d^2_\tau \mathcal{J}(\xi, \xi) < 0$, for $\beta > \beta_1$, with $\beta_1 > C^2$. Taking $\tau = 0$, the coercivity of $\mathcal{J}(u, 0)$ is a consequence of the following estimate

$$\mathcal{J}(u, 0) \geq \frac{\alpha}{2} \|u\|^2_{\mathbf{L}^2(\omega_1)}.$$

For the last condition, observe that we have the same estimate (3.8) for the state y, i.e.

$$|y(0, \tau)(x_0)| \leq \|y(0, \tau)\|_{\mathbf{C}(\overline{\Omega})} \leq C_0 (\|\tau\|_{\mathbf{L}^2(\omega_2)} + \|h\|_{\mathbf{L}^2(\omega)}),$$

which leads to

$$\mathcal{J}(0, \tau) \leq \left(\frac{C_0^2}{2} - \frac{\beta}{2} \right) \|\tau\|^2_{\mathbf{L}^2(\omega_2)} + C_0^2 \|\tau\|_{\mathbf{L}^2(\omega_2)} \cdot \|h\|_{\mathbf{L}^2(\omega)} + \frac{C_0^2}{2} \|h\|^2_{\mathbf{L}^2(\omega)}.$$

Thus for $\beta > \beta_2$, with $\beta_2 > C_0^2$, the last condition in Proposition (3) follows immediately. Finally, by setting $\beta_0 = \max\{\beta_1, \beta_2\}$, we recover both concavity and coercivity of the cost functional $\mathcal{J}$, with respect to τ. □

4. Optimality conditions

In this section, we formulate the first order optimality conditions in terms of the adjoint state. Since we have no constraints on control u and disturbance term τ, the couple $(\overline{\tau}, \overline{u})$ is characterized by the Euler-Lagrange equation:

$$d_{\overline{u}} \mathcal{J}(\xi) = 0 \text{ and } d_{\overline{\tau}} \mathcal{J}(\xi) = 0.$$

The derivatives $d_{\overline{u}} \mathcal{J}$, and $d_{\overline{\tau}} \mathcal{J}$ are given by (3.3) and (3.5). The pointwise observation in the cost functional $\mathcal{J}$ leads to singular sources on the right-hand side of the adjoint equation

$$\begin{cases} -\mu \Delta v + \nabla q + \eta v &= y \delta_{x_0} \quad \text{in} \quad \Omega, \\ \text{div } v &= 0 \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{4.1}$$

Here δ_{x_0} represents the Dirac measure concentrated at x_0. If we test the adjoint problem with the function θ, defined by the weak solution of the auxiliary system

(3.1), we get

$$\int_\Omega \theta \cdot y \, d\delta_{x_0} = \int_{\omega_1} v \cdot \xi \, dx, \tag{4.2}$$

$$\int_\Omega \psi \cdot y \, d\delta_{x_0} = \int_{\omega_2} v \cdot \xi \, dx. \tag{4.3}$$

Replacing (4.2) and (4.3) in expressions of (3.3) and (3.5), respectively, we find

$$d_{\,u}\mathcal{J} = (v + \alpha u)_{|\omega_1}, \tag{4.4}$$

$$d_{\,\tau}\mathcal{J} = (v - \beta\tau)_{|\omega_2}. \tag{4.5}$$

The source term of the adjoint state (4.1) belong to the space of bounded Borel measures in Ω, denoted by $\mathcal{M}(\Omega)^N$, which can be identified with the dual space of continuous functions. By the Sobolev embedding theorem it follows that $\delta_{x_0} \in \mathbf{W}^{-1,s}(\Omega)$ for $s < \frac{N}{N-1}$, where $\mathbf{W}^{-1,s}(\Omega)$ is the dual space of $\mathbf{W}_0^{1,s'}(\Omega)$, and s' is the Hölder conjugate of s, i.e., $\frac{1}{s} + \frac{1}{s'} = 1$. The Stokes problem with $\mathbf{W}^{-1,s}(\Omega)$ source term is discussed in the monograph of Galdi [13], Chapter 4. The result can immediately be generalized to Stokes-Darcy system. Therefore, existence and uniqueness for the adjoint state follow from the lemma :

Lemma 5. *Assume that* $1 < s < \infty$ *and* $f \in \boldsymbol{W}^{-1,s}(\Omega)$*, then the problem*

$$\langle \nabla v : \nabla \varphi \rangle + \eta \langle v, \varphi \rangle - \langle q, \operatorname{div} \varphi \rangle + \langle \operatorname{div} v, \pi \rangle = \langle f, \varphi \rangle \tag{4.6}$$

has a unique weak solution $(v, q) \in \boldsymbol{W}_0^{1,s}(\Omega) \times L_0^s(\Omega)$*, for all* $(\varphi, \pi) \in \boldsymbol{W}^{1,s'}(\Omega) \times L^{s'}(\Omega)$.

Proof. See [13]. Page 284. □

Now, we can formulate the first order necessary and sufficient optimality conditions as follow :

Proposition 6. *Suppose that* α, β *are sufficiently large and* $s \in \left(\frac{2N}{N+2}, \frac{N}{N-1}\right)$. *If* $(\overline{\tau}, \overline{u})$ *is a solution to the robust control problem (2.10). Then*

$$\overline{u} = -\frac{1}{\alpha} v \chi_{\omega_1}, \quad \overline{\tau} = \frac{1}{\beta} v \chi_{\omega_2}, \tag{4.7}$$

where $((y, p), (v, q)) \in \left[\left(\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)\right) \times L_0^2(\Omega)\right] \times [\boldsymbol{W}_0^{1,s}(\Omega) \times L_0^s(\Omega)]$*, is the unique solution of the following coupled system*

$$\left\{ \begin{array}{rcll} -\mu\Delta y + \nabla p + \eta y & = & h\chi_\omega & \\ & - & \frac{1}{\alpha} v \chi_{\omega_1} + \frac{1}{\beta} v \chi_{\omega_2} & in \;\; \Omega, \\ -\mu\Delta v + \nabla q + \eta v & = & y\delta_{x_0} & in \;\; \Omega, \\ \operatorname{div} y = 0, \; \operatorname{div} v & = & 0 & in \;\; \Omega, \\ y = v & = & 0 & on \;\; \Gamma. \end{array} \right. \tag{4.8}$$

Proof. From Euler-Lagrange equations (4.4-4.5) and the embedding $\mathbf{W}^{1,s}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$, $s \in \left(\frac{2N}{N+2}, \frac{N}{N-1}\right)$, we can deduce the expression of robust control given in (4.7). The existence of solution for the optimality system (4.8) follows directly from Theorem 4. In addition, for α, β large enough, this solution is unique. Indeed, suppose that u_i, τ_i , $i = 1, 2$, be two solutions of problem (2.10), and (y_i, v_i) be the associated solution of system (4.8). By linearity, the difference $(y_1 - y_2, v_1 - v_2)$ also solve the optimality system for $h \equiv 0$, moreover, by the stability estimate and (4.7) we get

$$\begin{aligned}\|y_1 - y_2\|_{\mathbf{H}^2(\Omega)} &\leq C(\|u_1 - u_2\|_{\mathbf{L}^2(\omega_1)} + \|\tau_1 - \tau_2\|_{\mathbf{L}^2(\omega_2)}) \\ &\leq C(\frac{1}{\alpha} + \frac{1}{\beta})\|v_1 - v_2\|_{\mathbf{L}^2(\Omega)} \\ &\leq C(\frac{1}{\alpha} + \frac{1}{\beta})\|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)} \end{aligned} \tag{4.9}$$

where C denote a generic positive constant. For $v_1 - v_2$, we have the following estimates hold

$$\begin{aligned}\|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)} &\leq C|y_1(x_0) - y_2(x_0)| \\ &\leq C\|y_1 - y_2\|_{\mathbf{C}(\Omega)} \\ &\leq C\|y_1 - y_2\|_{\mathbf{H}^2(\Omega)} \end{aligned} \tag{4.10}$$

Combining this with (4.9) we get

$$\|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)} \leq C(\frac{1}{\alpha} + \frac{1}{\beta})\|v_1 - v_2\|_{\mathbf{W}_0^{1,s}(\Omega)} \tag{4.11}$$

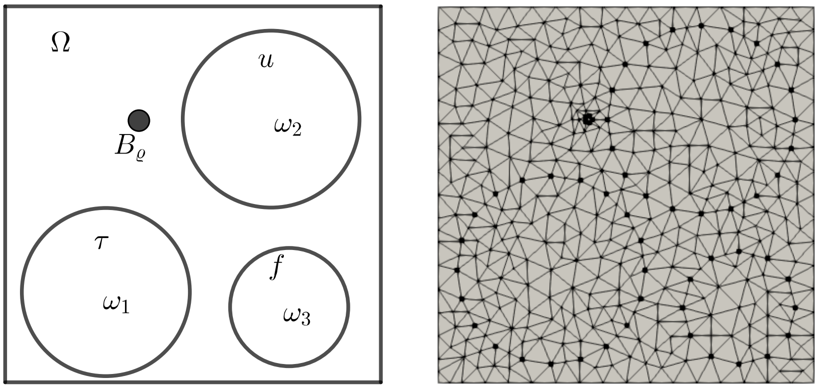
So for α, β large enough such that $C(\frac{1}{\alpha} + \frac{1}{\beta}) < 1$, the solution of problem (4.8) is unique. □

Remark 7. *From the relations (4.7), we deduce that the robust control can be evaluated by scaling the adjoint state with the regularized parameters* $-\frac{1}{\alpha}$, $\frac{1}{\beta}$. *Therefor, if we set* $\omega_1 = \omega_2$, *the coefficient* $\frac{1}{\beta} - \frac{1}{\alpha}$ *is one that tips the balance between control and disturbance directions.*

5. Numerical examples

In this section, we present two numerical experiments to illustrate our theoretical findings. We recall that the numerical computation of the optimal control rely on two approaches, namely: *discretize-then-optimize* and *optimize-then-discretize*. See, for example, the monograph [36]. In our case, we will adopt the second path, and we will focus on the numerical solution of the optimality system (4.8). We employ the finite elements method to discretize the coupled system (4.8). The numerical simulations to be presented are conducted in dimension two using FEniCS package, see [26]. We use a $\mathbb{P}_2 - \mathbb{P}_1$ Taylor-Hood element method to solve the Stokes-Darcy equations, see [35]. For discretization, we use a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ for the geometrical domain Ω, the mixed finite element approximation for the optimality system (4.8) can reduce to the following problem

$$\begin{aligned}&\text{Find } [(\tilde{y}, \tilde{p}), (\tilde{v}, \tilde{q})] \text{ in } [\mathcal{V} \times \mathcal{Q}]^2 \text{ such that :} \\ &\langle \nabla \tilde{y}, \nabla \tilde{\varphi} \rangle + \langle \eta \tilde{y}, \tilde{\varphi} \rangle - \langle \tilde{p}, \operatorname{div} \tilde{\varphi} \rangle + \langle \operatorname{div} \tilde{y}, \tilde{r} \rangle \\ &+ \frac{1}{\alpha}\langle \tilde{v}, \tilde{\varphi} \rangle - \frac{1}{\beta}\langle \tilde{v}, \tilde{\varphi} \rangle + \langle \nabla \tilde{v}, \nabla \tilde{\varsigma} \rangle + \langle \eta \tilde{v}, \tilde{\varsigma} \rangle \\ &- \langle q_h, \operatorname{div} \tilde{\varsigma} \rangle + \langle \operatorname{div} \tilde{v}, \tilde{\lambda} \rangle + \langle \tilde{y}(x_0), \tilde{\varsigma}(x_0) \rangle \\ &= \langle h, \tilde{\varphi} \rangle \quad \forall [(\tilde{\varphi}, \tilde{r}), (\tilde{\varsigma}, \tilde{\lambda})] \in [\mathcal{V} \times \mathcal{Q}]^2\end{aligned}$$



(a) (b)

FIGURE 2. Example 1 : Geometrical domain Ω (a) , Plot of the mesh for Ω (b).

The finite dimensional subspaces $\mathcal{V}$, $\mathcal{Q}$ of $\mathbf{H}^1_0(\Omega)$, $L^2_0(\Omega)$, respectively, are defined by

$$\mathcal{V} = \left\{ y_h \in \mathbf{C}(\Omega),\ y_{h|_K} \in \mathbb{P}_2^2\,,\ \forall K \in \mathcal{T}_h \text{ and } y_{h|_\Gamma} = 0 \right\}$$

$$\mathcal{Q} = \left\{ p_h \in C(\Omega),\ p_{h|_K} \in \mathbb{P}_1,\ \forall K \in \mathcal{T}_h \text{ and } \int_\Omega p_h\, dx = 0 \right\}$$

The singular term in the right-hand-side of adjoint state is handled using the following regularization

$$\delta_{x_0} \approx \frac{\varepsilon}{\pi(\|x - x_0\|^2 + \varepsilon^2)},$$

where ε is sufficiently small. Finally, the kinematic viscosity μ and the inverse permeability η are equal to 1. In order to validate the theoretical results we have to follow these steps :

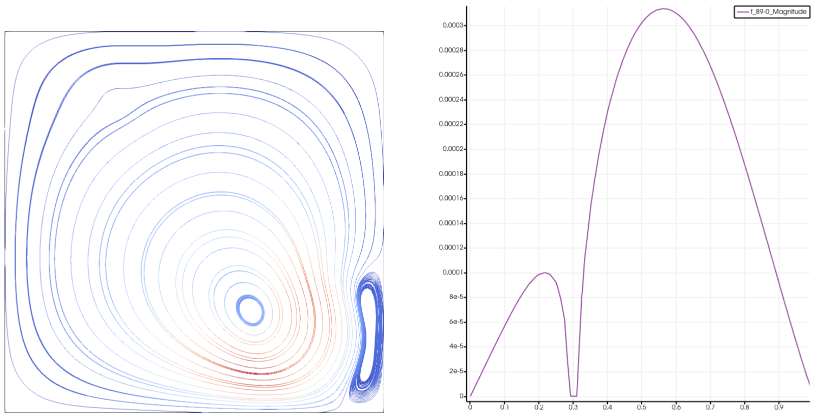
(1) Solve the uncontrolled Stokes-Darcy system in reference domain Ω and compute the energy $\mathcal{E}(\Omega)$;
(2) Solve the uncontrolled Stokes-Darcy system in perturbed domain Ω_ϱ and compute the energy $\mathcal{E}(\Omega_\varrho)$;
(3) Solve the optimality system (4.8) and evaluate the the robust control ($\overline{u}$, $\overline{\tau}$) by the relations (4.7); then compute $\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega)$
(4) Solve the Stokes-Darcy system in Ω_ϱ with robust control and compute $\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega_\varrho)$.

Example 8 (nosign)**.** In the first example, the geometrical domain is given by the unit square $\Omega =]0,1[\times]0,1[$. The control u is acting on $\omega_1 = B_{\varepsilon_1}(x_1)$, where $\varepsilon_1 = 0.25$, $x_1 = (0.7, 0.7)^\top$, and the disturbance term τ is supported in the subdomain $\omega_2 = B_{\varepsilon_2}(x_2)$, where $\varepsilon_2 = 0.25$, $x_2 = (0.3, 0.3)^\top$. The topological perturbation $B_\varrho(x_0)$ of size $\varrho = 0.01$, will be located at $x_0 = (0.3, 0.8)^\top$. The right hand-side $h = (h_1, h_2)^\top$ in system (2.2) is given by a rotational vector field of the form

$$h_1(x, y) = y \quad \text{and} \quad h_2(x, y) = -x,$$

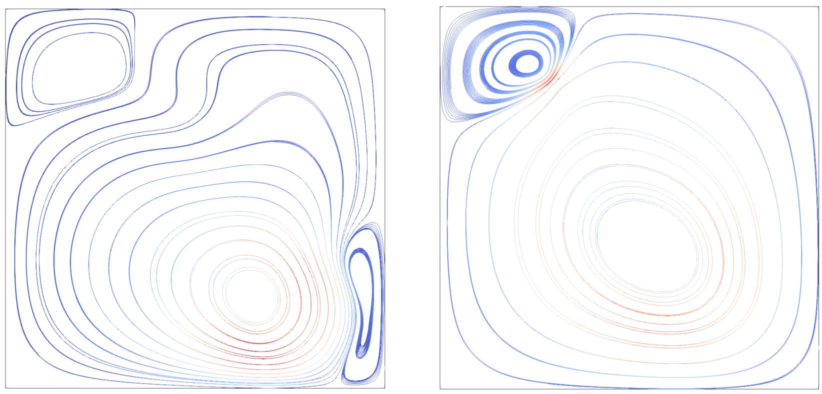
whose support is given by $\omega := \omega_3 = B_{\varepsilon_3}(x_3)$, with $x_3 = (0.8, 0.2)^\top$ and $\varepsilon_3 = 0.15$. The geometrical domain and its discretization are represented in Fig. 2. The control and the disturbance parameters are given by $\alpha = 10^8$, $\beta = 10^5$. For the contrast parameter, we test three values : $\gamma = 10^{10}$, $\gamma = 10^5$ and $\gamma = 10^2$. The graphical representations are performed in the first case $\gamma = 10^{10}$.

The mesh here is refined to 12681 cells. The flow in the perturbed domain is shown in Fig. 3 (a). As expected, if we plot $|y|$ over the line $x_2' = 0.8$ (x_2' is the vertical



(a) (b)

Figure 3. Example 1 : Streamline for the Stokes-Darcy flow in Ω_ϱ with $\gamma = 10^{10}$ (a) , Plot over the line $y = 0.8$ (b).



(a) (b)

Figure 4. Example 1 : Streamline for the controlled state $y(\overline{u}, \overline{\tau})$ (a) , Streamline for the adjoint state v (b).

axis in Cartesian coordinate), see Fig. 3 (b), we find $y_{|B_\varrho} \approx 0$. The optimal state $y(\overline{u}, \overline{\tau})$ and the adjoint state v given by the optimality system (4.8) are presented in Fig. 4. The control serves to bring the topological derivative as close as possible to zero at the point x_0, which is equivalent to drive the state to zero at x_0. In our case, we found the following result

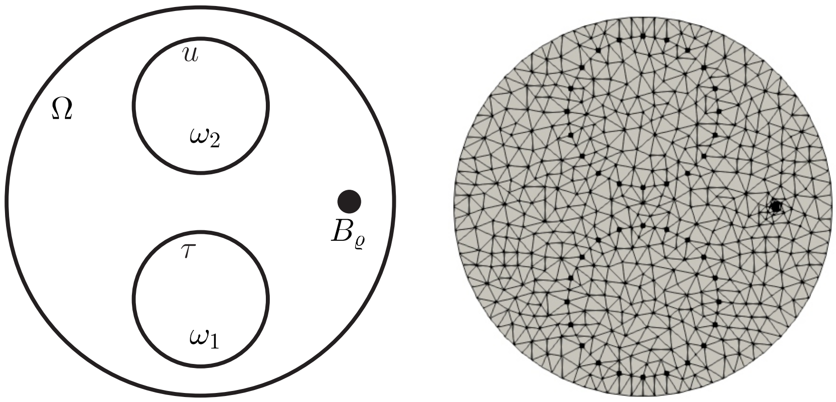
$$y(0.303, 0.809) = (3.89, 7.096)^\top \times 10^{-6}.$$

The last step consists of checking the energy gap, or the estimate (2.8). The energy functionals associated to ($u \equiv 0$, and $\tau \equiv 0$) in Ω and Ω_ϱ are respectively denoted by $\mathcal{E}(\Omega)$ and $\mathcal{E}(\Omega_\varrho)$. The quantitative results are summarized in Table 1. As intended, by a robust control $(\overline{u}, \overline{\tau})$, we can determine the minimum energy gap with respect to the topological perturbation.

Example 9 (nosign)**.** In the second example, we consider the enclosed Stokes-Darcy flow in the unit circle, $\Omega = B_R(\mathcal{O})$, $R = 1$ and $\mathcal{O} = (0,0)^\top$. The control term acts in the ball $\omega_1 = B_{\varepsilon_1}(x_1)$, where $\varepsilon_1 = 0.4$ and $x_1 = (0, 0.5)^\top$. The disturbance and the source terms are supported in $\omega_2 = B_{\varepsilon_2}(x_2)$, where $\varepsilon_2 = 0.4$ and $x_2 = (0, -0.5)^\top$. The obstacle of size $\varrho = 0.01$ is located at $x_0 = (0.7, 0)^\top$. The corresponding design domain is shown in Fig. 5. The right hand-side $h = (h_1, h_2)^\top$ of system (2.2) is

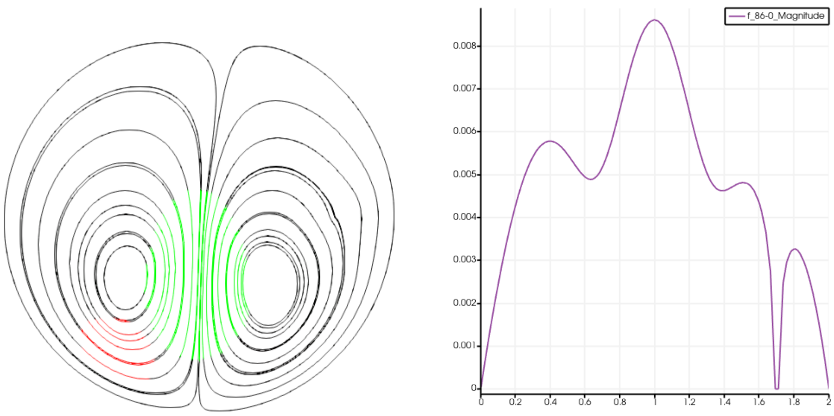
TABLE 1. Example 1 : Comparison of the energy gap for different contrast parameter.

Contrast	$\lvert\mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega)\rvert$	$\lvert\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega_\varrho) - \mathcal{E}_{\overline{u},\overline{\tau}}(\Omega)\rvert$
$\gamma = 10^{10}$	2.77452×10^{-5}	3.89782×10^{-10}
$\gamma = 10^{5}$	2.77960×10^{-5}	2.94058×10^{-10}
$\gamma = 10^{2}$	2.79900×10^{-5}	1.91397×10^{-12}



(a) (b)

FIGURE 5. Example 2 : Geometrical domain Ω (a) , Plot of the mesh for Ω (b).



(a) (b)

FIGURE 6. Example 2 : Streamline for the Stokes-Darcy flow in Ω_ϱ with $\gamma = 10^{10}$ (a) , Plot over the line $y = 0$ (b).

given by

$$h_1(x, y) = 0 \quad \text{and} \quad h_2(x, y) = -1.$$

The penalization parameters are fixed as follow : $\alpha = 10^8$, $\beta = 10^5$. Again, three values for the contrast parameter are considered : $\gamma = 10^{10}$, $\gamma = 10^5$ and $\gamma = 10^2$. The mesh in Fig. 5 is refined to 16242 elements. 6 (a) shows the streamline for the velocity field in the perturbed domain Ω_ϱ. The curve in Fig.6 (b), represents $|y|$ over the line $x_2' = 0$, we observe that at the obstacle location $x_0 = (0.7, 0)^\top$ we have $y_{|B_\varrho} \approx 0$. Fig. 7 presents the optimal state $y(\overline{u}, \overline{\tau})$ and the adjoint state v. The quantitative results obtained for the energy functional are reported in Table 2.

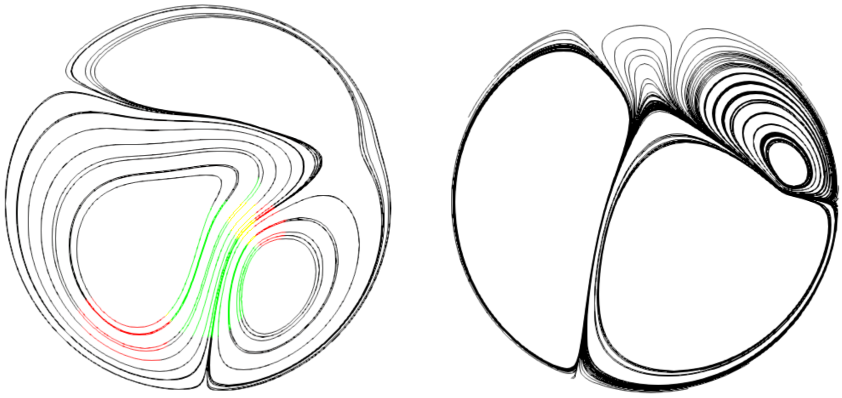


FIGURE 7. Example 2 : Streamline for the controlled state $y(\overline{u},\overline{\tau})$ (a) , Streamline for the adjoint state v (b).

TABLE 2. Example 2 : Comparison of the energy gap for different contrast parameter.

Contrast	$\lvert\mathcal{E}(\Omega_\varrho)-\mathcal{E}(\Omega)\rvert$	$\lvert\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega_\varrho)-\mathcal{E}_{\overline{u},\overline{\tau}}(\Omega)\rvert$
$\gamma=10^{10}$	0.00149909	9.47862×10^{-6}
$\gamma=10^{5}$	0.00151717	7.52874×10^{-6}
$\gamma=10^{2}$	0.00159826	5.33838×10^{-8}

6. Conclusion

In this paper, a new method that leads to insensitivity of the energy functional with respect to topological perturbation is presented. The model problem in fluid mechanics is governed by the Stokes-Darcy equations. A penalization approach is used to deal with the no-slip boundary condition on the infinitesimal obstacle. Our approach is based on the resolution of a minimax auxiliary problem, where the cost functional involves point evaluations of the state. This study can naturally be generalized in several directions, in particular for elliptic linear problems, where the theory of topological sensitivity is successfully developed. For non linear problem, it is well know that the topological derivative depends on the direct and adjoint states and probably their gradients. In this case, our approach leads to an optimal control problem for a cascade system, i.e. a coupled system which include the non linear state and its costate, with a control term acting partially through the state equation. This issue is currently under development and will be the subject for a forthcoming work.

Appendix A. Asymptotic expansion of the energy shape functional

In this section, we will derive the asymptotic expansion (2.6). In general, several methods are devoted to calculate the topological derivative expression, for example the adjoint methods. We refer the reader to [3], where the authors give a comprehensive review about this approach. The topological derivative can also be derived as a singular limit of shape derivative. This method is widely used, especially for the energy functionals. See the monograph [29] and references therein. The common step among all of these methods is to perform the asymptotic analysis of the state with respect to the parameter ϱ. In our case, we will see that the penalization of the no-slip boundary condition enormously simplifies the asymptotic analysis for the y_ϱ and $\mathcal{E}(\Omega_\varrho)$ as well.

Lemma 10. *Let y and y_ϱ be the weak solutions to the systems (2.2) and (2.5), respectively. Then, we have the following estimate*

$$\|y - y_\varrho\|_{\boldsymbol{H}_0^1(\Omega)} \leq C\varrho^{\frac{N}{2}+s}, \tag{A.1}$$

where $s \in]0,1]$, and C is constant independent of the parameter ϱ.

Proof. The weak form for the Stokes-Darcy system (2.2) in reference domain Ω is given by

$$\mu \int_\Omega \nabla y \cdot \nabla \varphi + \int_\Omega \eta y \cdot \varphi - \int_\Omega m \cdot \varphi = 0\,,\ \forall \varphi \in \mathbf{H}_{\mathrm{div}}(\Omega) \tag{A.2}$$

where $m = h\chi_\omega + u\chi_{\omega_1} + \tau\chi_{\omega_2}$. In the perturbed domain Ω_ϱ, the weak form of the system (2.5) reads

$$\mu \int_{\Omega_\varrho} \nabla y_\varrho \cdot \nabla \varphi + \int_{\Omega_\varrho} \eta_\varrho y_\varrho \cdot \varphi - \int_{\Omega_\varrho} m \cdot \varphi = 0\,, \forall \varphi \in \mathbf{H}_{\mathrm{div}}(\Omega) \tag{A.3}$$

Let us now subtract (A.3) from (A.2), to obtain

$$\mu \int_\Omega \nabla(y_\varrho - y) \cdot \nabla \varphi + \int_\Omega (\eta_\varrho y_\varrho - \eta y) \cdot \varphi = 0\,,\ \forall \varphi \in \mathbf{H}_{\mathrm{div}}(\Omega) \tag{A.4}$$

By setting $\varphi = y_\varrho - y$ in (A.4), we get

$$\mu \int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega (\eta_\varrho y_\varrho - \eta y) \cdot (y_\varrho - y) = 0. \tag{A.5}$$

The coefficient η_ϱ is defined piecewise in Ω_ϱ, hence we have the following decomposition for the last integral

$$\begin{aligned}\int_\Omega (\eta_\varrho y_\varrho - \eta y) \cdot (y_\varrho - y) &= \int_{\Omega\setminus B_\varrho} \eta |y_\varrho - y|^2 \\ &\quad + \int_{B_\varrho} \eta(\gamma y_\varrho - y) \cdot (y_\varrho - y) \\ &= \int_\Omega \eta_\varrho |y_\varrho - y|^2 \\ &\quad - \int_{B_\varrho} (1-\gamma)\eta y \cdot (y_\varrho - y).\end{aligned} \tag{A.6}$$

After replacing in (A.5), we get

$$\int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho |y_\varrho - y|^2 = \int_{B_\varrho} (1-\gamma)\eta y \cdot (y_\varrho - y). \tag{A.7}$$

Using the Cauchy-Schwarz inequality and the Lebesgue differentiation theorem (see [6], page 351), we find the following estimate

$$\mu \int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho |y_\varrho - y|^2 \leq C\varrho^{\frac{N}{2}} \|y_\varrho - y\|_{\mathbf{L}_2(B_\varrho)}. \tag{A.8}$$

On other side, by Hölder inequality and Sobolev embedding $\mathbf{H}_0^1(B_\varrho) \hookrightarrow \mathbf{L}^2(B_\varrho)$, we get

$$\begin{aligned} \|y_\varrho - y\|_{\mathbf{L}^2(B_\varrho)} &\le C\varrho^{\frac{N}{2q}} \left(\int_{B_\varrho} |y_\varrho - y|^{2p} \right)^{\frac{1}{2p}} \\ &= C\varrho^{\frac{N}{2q}} \|y_\varrho - y\|_{\mathbf{L}^{2p}(B_\varrho)} \\ &\le C\varrho^{s} \|y_\varrho - y\|_{\mathbf{H}_0^1(\Omega)}, \end{aligned} \tag{A.9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $q \ge \frac{N}{2}$, and $s = \frac{N}{2q}$. Using (A.8) and (A.9), we obtain

$$\mu \int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho |y_\varrho - y|^2 \le C\varrho^{\frac{N}{2}+s} \|y_\varrho - y\|_{\mathbf{H}_0^1(\Omega)}. \tag{A.10}$$

Finally, the estimate (A.1) can be derived directly from the coercivity inequality, i.e,

$$c\|y_\varrho - y\|^2_{\mathbf{H}_0^1(\Omega)} \le \mu \int_\Omega |\nabla(y_\varrho - y)|^2 + \int_\Omega \eta_\varrho |y_\varrho - y|^2. \tag{A.11}$$

□

Now, let us go back to the shape functional $\mathcal{E}(\Omega)$. By setting $\varphi = y_\varrho - y$ in the weak forms (A.2) and (A.3), we obtain

$$\mathcal{E}(\Omega) = \mu \int_\Omega \nabla y_\varrho \cdot \nabla y + \int_\Omega \eta_\varrho y_\varrho \cdot y, \tag{A.12}$$

$$\mathcal{E}(\Omega_\varrho) = \mu \int_\Omega \nabla y_\varrho \cdot \nabla y + \int_\Omega \eta y_\varrho \cdot y. \tag{A.13}$$

In addition, we have

$$\begin{aligned} \mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega) &= \int_{B_\varrho} (1-\gamma)\eta y_\varrho \cdot y \\ &= \int_{B_\varrho} (1-\gamma)\eta |y|^2 + (1-\gamma)\eta (y_\varrho - y) \cdot y \end{aligned} \tag{A.14}$$

The first term gives the (TD) of $\mathcal{E}$, while the second term is a remainder of order $o(\varrho^N)$. More precisely, we have

$$\begin{aligned} \int_{B_\varrho} \eta (y_\varrho - y) \cdot y &\le C\varrho^{\frac{N}{2}} \|y_\varrho - y\|_{\mathbf{H}_0^1(B_\varrho)} \\ &\le C\varrho^{\frac{N}{2}} \|y_\varrho - y\|_{\mathbf{H}_0^1(\Omega)} = o(\varrho^N). \end{aligned} \tag{A.15}$$

Therefore,

$$\mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega) = \int_{B_\varrho} (1-\gamma)\eta |y|^2 + o(\varrho^N). \tag{A.16}$$

Again, using the Lebesgue differentiation theorem we find the topological derivative expression at x_0, namely

$$\mathcal{E}(\Omega_\varrho) - \mathcal{E}(\Omega) = meas(B_\varrho)(1-\gamma)\eta |y(x_0)|^2 + o(\varrho^N). \tag{A.17}$$

References

[1] Samuel Amstutz. The topological asymptotic for the Navier-Stokes equations. *ESAIM: Control, Optimization and Calculus of Variations*, 11(3):401–425, 2005. DOI :10.1051/cocv:2005012.

[2] Samuel Amstutz. Topological sensitivity analysis for some nonlinear PDE systems. *Journal de mathématiques pures et appliquées*, 85(4):540–557, 2006. DOI :10.1016/j.matpur.2005.10.008.

[3] Phillip Baumann and Kevin Sturm. Adjoint-based methods to compute higher-order topological derivatives with an application to elasticity. *Engineering Computations*, 39(1):60–114, 2022. DOI :10.1108/EC-07-2021-0407.

[4] Maitine Bergounioux and Yannick Privat. Shape optimization with Stokes constraints over the set of axisymmetric domains. *SIAM Journal on Control and Optimization*, 51(1):599–628, 2013. DOI :10.1137/100818133.

[5] Thomas R Bewley, Roger Temam, and Mohammed Ziane. A general framework for robust control in fluid mechanics. *Physica D: Nonlinear Phenomena*, 138(3-4):360–392, 2000. DOI :10.1016/S0167-2789(99)00206-7.

[6] Vladimir Igorevich Bogachev and Maria Aparecida Soares Ruas. Measure theory, volume 1. Springer, 2007.

[7] Franck Boyer and Pierre Fabrie. Eléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles, volume 52. Springer, 2005.

[8] Charles EA Brett. Optimal control and inverse problems involving point and line functionals and inequality constraints. PhD thesis, University of Warwick, 2014.

[9] Fabien Caubet and Marc Dambrine. Localization of small obstacles in Stokes flow. *Inverse Problems*, 28(10):105007, 2012. DOI :10.1088/0266-5611/28/10/105007.

[10] René Dáger. Insensitizing controls for the 1-d wave equation. *SIAM Journal on Control and Optimization*, 45(5):1758–1768, 2006. DOI :10.1137/060654372.

[11] Ivar Ekeland and Roger Temam. Convex analysis and variational problems. SIAM, 1999.

[12] Sylvain Ervedoza, Pierre Lissy, and Yannick Privat. Desensitizing control for the heat equation with respect to domain variations. *Journal de l'école polytechnique mathématiques*, 9:1397–1429, 2022. DOI :10.5802/jep.209.

[13] Giovanni Galdi. An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems. Springer, 2011.

[14] Stéphane Garreau, Philippe Guillaume, and Mohamed Masmoudi. The topological asymptotic for PDE systems: the elasticity case. *SIAM Journal on Control and Optimization*, 39(6):1756–1778, 2001. DOI :10.1137/S0363012900369538.

[15] Sebastian M Giusti, J Sokołowski, and Jan Stebel. On topological derivatives for contact problems in elasticity. *Journal of Optimization Theory and Applications*, 165:279–294, 2015. DOI :10.1007/s10957-014-0594-7.

[16] Sergio Guerrero. Controllability of systems of Stokes equations with one control force: existence of insensitizing controls. 24(6):1029–1054, 2007. DOI :10.1016/j.anihpc.2006.11.001.

[17] Mamadou Gueye. Insensitizing controls for the Navier–Stokes equations. 30(5):825–844, 2013. DOI :10.1016/j.anihpc.2012.09.005.

[18] Philippe Guillaume and Maatoug Hassine. Removing holes in topological shape optimization. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(1):160–191, 2008. DOI :10.1051/cocv:2007045.

[19] Maatoug Hassine and Mohamed Masmoudi. The topological asymptotic expansion for the Quasi-Stokes problem. *ESAIM: Control, Optimisation and Calculus of Variations*, 10(4):478–504, 2004. DOI :10.1051/cocv:2004016.

[20] I Hlaváček, AA Novotny, J Sokołowski, and A Żochowski. On topological derivatives for elastic solids with uncertain input data. *Journal of Optimization Theory and Applications*, 141(3):569–595, 2009. DOI :10.1007/s10957-008-9490-3.

[21] Mohamed Iguernane, Serguei A Nazarov, Jean-Rodolphe Roche, Jan Sokołowski, and Katarzyna Szulc. Topological derivatives for semilinear elliptic equations. 2009. DOI :10.2478/v10006-009-0016-4.

[22] Victor A Kovtunenko and Karl Kunisch. High precision identification of an object: Optimality-conditions-based concept of imaging. *SIAM Journal on Control and Optimization*, 52(1):773–796, 2014. DOI :10.1137/13091172X.

[23] Piotr Krzyżanowski, Sadokat Malikova, Piotr Bogusław Mucha, and Tomasz Piasecki. Comparative analysis of obstacle approximation strategies for the steady incompressible Navier–Stokes equations. *Applied Mathematics & Optimization*, 89(2):1–20, 2024. DOI :10.1007/s00245-024-10105-w.

[24] Günter Leugering, Antonio André Novotny, and Jan Sokołowski. On the robustness of the topological derivative for Helmholtz problems and applications. *Control and Cybernetics*, 51(2):227–248, 2022. DOI :10.2478/candc-2022-0015.

[25] JL Lions. Sentinelles pour les systmes distribués à données incompletes, recherches en mathématiques appliquées, 21, 1992.

[26] Anders Logg, Kent-Andre Mardal, and Garth Wells. Automated solution of differential equations by the finite element method: The FEniCS book, volume 84. Springer, 2012.

[27] Marwan Moubachir and Jean-Paul Zolesio. Moving shape analysis and control: applications to fluid structure interactions. Chapman and Hall/CRC, 2006.

[28] LF N. Sá, RC R. Amigo, Antonio André Novotny, and EC N. Silva. Topological derivatives applied to fluid flow channel design optimization problems. *Structural and multidisciplinary optimization*, 54:249–264, 2016. DOI :10.1007/s00158-016-1399-0.

[29] Antonio André Novotny and Jan Sokołowski. Topological derivatives in shape optimization. Springer, 2012.

[30] Antonio André Novotny, Jan Sokołowski, and Antoni Żochowski. Applications of the topological derivative method. Springer, 2019.

[31] Rolf Rannacher and Boris Vexler. A priori error estimates for the finite element discretization of elliptic parameter identification problems with pointwise measurements. *SIAM Journal on Control and Optimization*, 44(5):1844–1863, 2005. DOI :10.1137/040611100.

[32] Jan Sokołowski and Antoni Żochowski. On the topological derivative in shape optimization. *SIAM Journal on Control and Optimization*, 37(4):1251–1272, 1999. DOI :10.1137/S0363012997323230.

[33] Jan Sokołowski and Jean-Paul Zolésio. Introduction to shape optimization. Springer, 1992.

[34] Kevin Sturm. Topological sensitivities via a lagrangian approach for semilinear problems. *Nonlinearity*, 33(9):4310, 2020. DOI :10.1088/1361-6544/ab86cb.

[35] Barna Szabó and Ivo Babuška. Introduction to finite element analysis: formulation, verification and validation, volume 35. John Wiley & Sons, 2011.

[36] Fredi Tröltzsch. Optimal control of partial differential equations: theory, methods and applications, volume 112. American Mathematical Society, 2024.

(M. Abdelbaria) Laboratory of Rheology and Mechanics, Faculty of Technology, Hassiba Ben Bouali University, Hay Salem, 02000 Chlef Algeria
Email address: m.abdelbari@univ-chlef.dz

(K. Nachi) Laboratory of Mathematical Analysis and Applications, Faculty of Exact and Applied Science, University Oran1, Ahmed Ben Bella, BP 1524, El M'naouer Oran, Algeria
Email address: nachi.khadra@univ-oran1.dz

(J. Sokolowski) Institut Elie Cartan, Laboratoire de Mathématiques, Université de Lorraine-Nancy, B.P. 239, 54506, Vandoeuvre lès Nancy cedex, France
Email address: jan.sokolowski@univ-lorraine.fr

(A.A. Novotny) Laboratório Nacional de Computação Científica LNCC/MCTI, Coordenação de Métodos Matemáticos e Computacionais, Av. Getúlio Vargas 333, 25651-075 Petrópolis - RJ, Brasil
Email address: novotny@lncc.br