RECONSTRUCTION OF A SINGULAR SOURCE IN A FRACTIONAL SUBDIFFUSION PROBLEM FROM A SINGLE POINT MEASUREMENT

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Abstract. In this paper, we reconstruct a singular time dependent source function of a fractional subdiffusion problem using observational data obtained from a single point of the boundary and inside of the domain. Specifically, the singular function under consideration is represented by the Dirac delta function which makes the analysis interesting as the temporal component of unknown source belongs to a Sobolev space of negative order. We establish the uniqueness of the examined inverse problem in both scenarios. In addition, we analyze local stability of the solution of our inverse problem. To numerically reconstruct a point-wise source, we use the techniques of topological derivatives by converting the inverse source problem in an optimization one. More precisely, we develop a second-order non-iterative reconstruction algorithm to achieve our goal. The efficacy of the proposed approach is substantiated through diverse numerical examples.

1. Introduction

Fractional systems of partial differential equations play a crucial role in understanding transport dynamics within complex systems characterized by non-exponential relaxation patterns and anomalous diffusion [46]. These equations encompass various essential formulations, including fractional advection-diffusion equations, space/time fractional diffusion equations accounting for anomalous diffusion with sources and sinks, and the fractional Fokker–Planck equation describing anomalous diffusion in an external field, among others (see, for example, [10, 21, 22]).

Time-fractional diffusion equations emerge by replacing the classical time derivative with time-fractional derivatives, providing a method to characterize the evolution of the probability density function for particles undergoing anomalous diffusion. Anomalous diffusion deviates from the conventional Fickian portrayal of Brownian motion, characterized by nonlinear growth in mean squared displacement with respect to time, as illustrated by $\left\langle x^2(t)\right\rangle \sim t^\alpha$. In particular, the time-fractional diffusion equation considered in this paper pertains to anomalous subdiffusion corresponding to $0 < \alpha < 1$. Instances of subdiffusive transport include turbulent flow, dynamics of a bead in polymer networks [4], NMR diffusometry in disordered materials [47], and the chaotic dynamics of charge transport in amorphous semiconductors [56]. Mainardi [44] highlighted the applicability of the time-fractional diffusion equation in modeling the propagation of mechanical diffusive waves in viscoelastic media. In [48], Nigmatullin employed the fractional diffusion equation to describe diffusion in media exhibiting fractal geometry.

For the mathematical formulation of the subdiffusion problem considered here, let us take an open and bounded domain Ω in $\mathbb{R}^n$ (with $n \in \{1, 2, 3\}$) with a sufficiently smooth boundary $\partial\Omega$. Furthermore, for a fixed terminal time $T > 0$, we consider the diffusion process in Ω governed by the boundary value problem

$$\left\{\begin{array}{rcll} \mathrm{d}_t^\alpha u + \mathcal{L}u & = & f^* & \text{in} \quad \Omega \times (0, T), \\ u & = & 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u & = & 0 & \text{in} \quad \Omega \times \{0\}, \end{array}\right. \tag{1}$$

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where d_t^α denotes the pointwise Caputo fractional derivative of order $0 < \alpha < 1$ in time, defined as (see, for example, Podlubny [51])

$$\mathrm{d}_t^\alpha \vartheta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \vartheta'(s) \mathrm{d}s, \quad \text{for } \vartheta \in W^{1,1}(0, T).$$

Here Γ denotes the Euler's Gamma function, which is defined on each complex number $z \in \mathbb{C}$ with positive real part (i.e. $\Re\{z\} > 0$), by

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} \, \mathrm{d}s.$$

Moreover, the operator $\mathcal{L}$ is defined by

$$\mathcal{L}u(x,t) = -\sum_{k=1}^n \frac{\partial}{\partial x_k}\Big(\sum_{\ell=1}^n a_{k\ell}(x) \frac{\partial}{\partial x_\ell} u(x,t)\Big) + \sum_{\ell=1}^n d_\ell(x) \frac{\partial}{\partial x_\ell} u(x,t) + c(x)\, u(x,t), \quad x \in \Omega, \ 0 < t < T,$$

with symmetry and uniform coercivity

$$a_{k\ell} = a_{\ell k}, \quad 1 \le k, \ell \le n, \quad a_{k\ell}, d_\ell \in \mathcal{C}^1(\overline{\Omega}), \quad c(x) \in \mathcal{C}(\overline{\Omega}),$$
$$a_0 \sum_{k=1}^n \beta_k^2 \le \sum_{k,\ell=1}^n a_{k\ell}(x) \beta_k \beta_\ell, \quad x \in \overline{\Omega}, \quad \beta_1, \cdots, \beta_n \in \mathbb{R},$$

where $a_0 > 0$ is a constant independent of x and $\beta = (\beta_1, \cdots, \beta_n)$.

If the source term f^* and the order α are appropriately specified, problem (1) can be considered as a well-posed direct problem. During the last decade, this direct problem has attracted considerable attention in the literature, with comprehensive theoretical analyses [18, 40, 43, 54, 62, 64] and the application of various numerical methods [30, 31, 37, 38, 45]. However, practical scenarios may pose challenges where the order α or the source term is unknown. Our objective is to deduce these parameters from additional measurement data, leading to the formulation of fractional diffusion inverse problems. More precisely, this paper delves into the investigation of an inverse source problem, wherein the order α is given, but the source term f^* is unknown.

The primary challenge in tackling this type of inverse problem stems from the inherent non-identifiability of the source term f^* in its abstract form (refer, for instance, to [35]). To address this issue, researchers in the literature typically make the crucial assumption of having some a priori information about the source f^*. One common approach involves expressing the source term as $f^*(x,t) = \mu^*(t) g(x)$, where g denotes the known spatial distribution of the source, and μ^* is the unknown temporal change factor. In the context of determining the t-dependent factor $\mu^* \in C[0, T]$ from measurements of the solution at a single spatial point $x_0 \in \mathrm{supp}(g)$ over the interval $(0, T)$, Sakamoto and Yamamoto [54] provided a stability estimate for this inverse problem. On the other hand, in [42], the reconstruction of $\mu^* \in C^1[0, T]$ was analyzed through observations at a single point $x_0 \notin \mathrm{supp}(g)$. The obtained Lipschitz stability estimate (see [54]) was further developed in [16] for the reconstruction of $\mu^* \in L^\infty(0, T)$ from observations of the solution at one point $x_0 \in \overline{\Omega} = (\partial\Omega \cup \Omega)$ over $(0, T)$. In that paper, Fujishiro and Kian conducted an analysis of a more general case where the factor g exhibits both temporal and spatial dependencies. Meanwhile, Liu et al. [40] established a uniqueness result for reconstructing $\mu^* \in C^1[0, T]$ using an observation of the solution at a solitary monitoring point $x_0 \in \Omega$ over $(0, T)$. Wei and Zhang [61] proposed a stable and accurate numerical approximation to reconstruct the time-dependent source term μ^* from additional measurements obtained at an interior point $x_0 \in \Omega$. They achieved this by combining the boundary element method and the first-order Tikhonov regularization. In [59], the authors employed the conjugate gradient method along with Morozov's discrepancy principle to effectively recover the time-dependent factor $\mu^* \in C^1[0, T]$ from boundary Cauchy data. They also established uniqueness and a stability estimate for this inverse time-dependent source problem. More recently, in [20], the authors analyzed a reconstruction problem aimed at detecting the time limit at which unknown sources in the time-fractional diffusion problem become inactive. They specifically focused on the time-dependent source $\mu^* = \chi_{(0,T^*)}$, where $T^* \in (0,T)$ is the unknown time at which the source f^* becomes inactive, and $\chi_{(0,T^*)}$ denotes the characteristic function of the time interval $(0,T^*)$. This work contributes to the understanding of inverse problems in the context of time-fractional diffusion equations, providing insights into the detection of inactive sources and the reconstruction of time-dependent source terms.

For the sake of comprehensiveness, it is essential to note that inverse source problems akin to those mentioned above have surfaced in various critical applications and garnered significant attention in recent times. An illustrative example is the challenge of identifying the space-dependent source term g within the inhomogeneous term $f^*(x,t) = \mu^*(t)g(x)$ in (1), where the temporal component μ^* is known. This problem has been addressed by numerous authors in the literature. In [52], the authors tackled a geometric inverse source problem governed by two-dimensional time-fractional subdiffusion. Specifically, they reconstructed a space-dependent source g supported in an unknown subdomain $\mathcal{S} \subset \Omega$ (i.e., $g = \chi_{\mathcal{S}}$) from partial boundary measurements of the potential field using a noniterative algorithm involving topological derivatives. Conversely, in [26], a similar inverse source problem was discussed from partial domain observation, presenting a novel computational algorithmic approach compared to those reported in [52]. Furthermore, in [27], the authors proposed a novel stable reconstruction method utilizing the coupled complex boundary method to solve the same identification problem as in [52]. For a complete bibliography on inverse source problems for time-fractional diffusion equations, readers are referred to references such as [28, 29, 55, 60, 65], as well as topical review articles such as [32, 39].

In all of the aforementioned works, the focus was on determining regular temporal components μ^*, such as those in $C[0,T]$, $C^1[0,T]$, or $L^2(0,T)$. In the present paper, we shift our attention to the reconstruction of a singular function in time from single-point observational data, either on the boundary $\partial\Omega$ or within the domain Ω. This involves data of the form $\partial_\nu u(x_0,t)$ for $0 < t < T$, where the spatial point $x_0 \in \partial\Omega$ and ν denotes the outward unit normal to $\partial\Omega$, or alternatively, from interior observations $u(x_0,t)$ for $0 < t < T$, where the spatial point x_0 lies within Ω. More precisely, the considered singular function here is represented by the Dirac delta function, corresponding to point sources in practical scenarios. This implies that the temporal component μ^* exists in a Sobolev space of negative order, specifically $\mu^* \notin L^2(0,T)$. In general, the solution u is interpreted in a weak sense, where the evaluation of $u(x_0,t)$ or $\partial_\nu u(x_0,t)$ may not always be well-defined. Therefore, the practical significance of assuming the availability of $u(x_0,t)$ or $\partial_\nu u(x_0,t)$ is misleading and needs extra regularity or appropriate approximation for the meaningful analysis and application of the solution in relevant contexts.

Recently, Liu and Yamamoto, in [41], addressed the same inverse source problem aiming to identify a singular function in (1) from interior observations $u(x_0,t)$ for $0 < t < T$, where $x_0 \in \Omega$. They considered the assumptions $d_\ell = 0$ $(\ell = 1, \cdots, n)$, $c \geq 0$ in Ω, and the space-dependent source term $g \geq 0$ in Ω (or equivalently $g \leq 0$). They proved a uniqueness result for this inverse problem based on the existence of an inverse operator of $\mathcal{L}$ and the strong maximum principle. It is important to note the critical role of conditions $c \geq 0$ and $g \geq 0$ (or equivalently $g \leq 0$) in their entire analysis. Additionally, the main novelty of current paper is that we address the identification and stability issues which is not the case in [41].

In this paper, we prove the uniqueness inverse problem under consideration in both cases, i.e., when $x_0 \in \partial\Omega$ or $x_0 \in \Omega$, without the conditions $d_\ell = 0$, $c \geq 0$, and $g \geq 0$ (or equivalently $g \leq 0$). Moreover, we establish a local stability estimate for the solution of our inverse problem. To reconstruct a singular source in time, we reformulate the inverse source problem as an optimization problem which minimizes a tracking-type functional with respect to a set of admissible pointwise sources. The necessary optimality conditions are derived in the spirit of the topological derivative method [49] which, in this context, consists in exposing the perturbation of the functional as a quadratic function of the source intensities. Then, the resulting expansion is trivially minimized with respect to the sought source parameters, leading to a non-iterative reconstruction algorithm that is initial guess-free and robust with respect to perturbations of sensory data. More precisely, utilizing the sensitivity analysis method, a second-order noniterative reconstruction algorithm is devised, enabling us to determine the number, time locations, and intensities of the unknown pointwise sources. The efficiency of the proposed approach is validated through various numerical examples. The topological sensitivity analysis method, initially proposed for shape optimization by Eschenauer et al. [15] and first mathematically justified in [17, 57], was further developed in the book by Novotny and Sokołowski [49]. This method can be seen as a particular instance of the broader class of asymptotic methods extensively elaborated in the books by Ammari and Kang [8] and Ammari et al. [6], for example. The stability and resolution analysis for a topological-derivative-based imaging functional has been presented by Ammari et al. [7, 5], demonstrating its effectiveness in the context of inverse scattering or elasticity problems. See also related work [19]. The experimental verification of the method is presented in [58] for the elastic-wave imaging. To the best of our knowledge, this paper presents the first numerical approach implemented for the recovery of a singular source in time within the context of fractional models. For completeness, it is worth noting the work of Hrizi et al. [25], who extensively discussed a similar inverse pointwise source problem for (1). They focused on recovering a Dirac delta function in space from interior measurements of the potential field. However, the mathematical analysis to establish the uniqueness and stability issues of that paper (as well as in [20, 26, 27, 52]) cannot be adopted in the context of this article due to the lack of regularity in the time-dependent source of our problem. Furthermore, in [25] (and similarly in [20, 26, 27, 52]), the authors employed the sensitivity analysis method to reconstruct the unknown source. Specifically, the sensitivity analysis was derived through an asymptotic expansion of the solution to a perturbed time-fractional diffusion problem. In their approach, since the time-dependent source μ^* was known (as in [20, 25, 26, 27, 52]), the perturbed solution was expressed as the sum of the solution to the time-fractional diffusion problem and an explicit solution to an elliptic problem. This explicit solution played a crucial role in the numerical implementation. In contrast, the same technique (used in [20, 25, 26, 27, 52] to establish the asymptotic expansion) cannot be applied here, as the time-dependent source is unknown.

Finally, let us provide a concise overview of the paper's structure. In Section 2, recalling key ingredients from the theoretical counterpart, we redefine the classical Caputo fractional derivative to present the mathematical formulation of the considered inverse problem. Section 3 deals with the uniqueness of the solution to inverse problem. The proof of the uniqueness results, mentioned in Section 3, is presented in Appendix A. In Section 4, we present a local stabilty estimate and propose a noniterative identification procedure to recover an unknown pointwise source in (1). In Section 5, we present a set of numerical examples, showcasing various features of the proposed non-iterative reconstruction algorithm, including its robustness in the presence of noisy data, whereas the closing section is devoted to some comments.

2. Preliminaries

As highlighted in the introduction, our focus in the theoretical part of this paper revolves around addressing the uniqueness question of the considered inverse problem, particularly when dealing with a less regular time-dependent source μ^* represented by the Dirac delta function (in time). In such instances, the differentiability of $u(x, \cdot)$ in time cannot be presumed, necessitating a redefinition of the Caputo derivative d_t^α in (1). To achieve this, we will begin by introducing relevant function spaces.

2.1. **Introduction of function spaces.** For $1 \le p < \infty$, let $L^p(\Omega)$, $H_0^1(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$ be the usual classical Lebesgue and Sobolev spaces. Additionally, we define the following spaces:

$$ {}_0C^1[0,T] := \Big\{\vartheta \in C^1[0,T] : \vartheta(0) = 0\Big\} \text{ and } {}^0C^1[0,T] := \Big\{\vartheta \in C^1[0,T] : \vartheta(T) = 0\Big\}. $$

For $0 < \alpha < 1$, we introduce the Sobolev-Slobodecki space $H^\alpha(0,T)$ equipped with the norm $\|\cdot\|_{H^\alpha(0,T)}$ defined as follows (e.g., Adams [1, Chapter VII])

$$ \|\vartheta\|_{H^\alpha(0,T)} = \Big(\|\vartheta\|^2_{L^2(0,T)} + \int_0^T\int_0^T \frac{|\vartheta(t)-\vartheta(\tau)|^2}{|t-\tau|^{1+2\alpha}}\,\mathrm{d}t\mathrm{d}\tau\Big)^{1/2}. $$

Further, we introduce the following function spaces (e.g., [36])

$$ {}^\alpha H(0,\,T) := \overline{{}^0C^1[0,T]}^{H^\alpha(0,T)} \text{ and } H_\alpha(0,\,T) := \overline{{}_0C^1[0,T]}^{H^\alpha(0,T)}. $$

Let $\mathcal{Y}$ be a Hilbert space over $\mathbb{R}$. Consider $\mathcal{X}$ as a dense subspace of $\mathcal{Y}$ with a continuous embedding $\mathcal{X} \longrightarrow \mathcal{Y}$. We define the dual space $\mathcal{Y}'$ of $\mathcal{Y}$ as the set of all bounded linear functionals defined on $\mathcal{Y}$. By identifying $\mathcal{Y}'$ with itself, we establish that $\mathcal{Y}$ is a dense subspace of the dual space $\mathcal{X}'$ of $\mathcal{X}$, as indicated by the inclusion

$$ \mathcal{X} \subset \mathcal{Y} \subset \mathcal{X}'. $$

We denote the value of $f \in \mathcal{X}'$ at $v \in \mathcal{X}$ by ${}_{\mathcal{X}'}\Big\langle f, v\Big\rangle_{\mathcal{X}}$. It is important to note that

$$ {}_{\mathcal{X}'}\Big\langle f, v\Big\rangle_{\mathcal{X}} = \Big(f, v\Big)_{\mathcal{Y}} \text{ if } f \in \mathcal{Y}, $$

where $\Big(f, v\Big)_{\mathcal{Y}}$ represents the scalar product in the space $\mathcal{Y}$.

Furthermore, it's worth mentioning that both $H_\alpha(0,T)$ and ${}^\alpha H(0,T)$ are dense in $L^2(0,T)$. Consequently, we can define $(H_\alpha(0,T))'$ and $({}^\alpha H(0,T))'$ by identifying the dual space $L^2(0,T)'$ with itself, following the same approach as described above. This leads us to the following inclusions (the Gel'fand triples)

$$ H_\alpha(0,T) \subset L^2(0,T) \subset \big(H_\alpha(0,T)\big)' := H_{-\alpha}(0,T), \tag{2} $$

$$ {}^\alpha H(0,T) \subset L^2(0,T) \subset \big({}^\alpha H(0,T)\big)' := {}^{-\alpha}H(0,T). \tag{3} $$

For a more comprehensive understanding, you may refer to the detailed exposition in [63].

2.2. **Definition of the extended derivative of** d_t^α**.** In this section, drawing upon the methodologies outlined in [36, 63], we define a fractional derivative for functions in $L^2(0,T)$, thereby extending the domain of the classical Caputo derivative d_t^α. To accomplish this extension, we first expand d_t^α within the space $H_\alpha(0,T)$ and then proceed to extend this operator to the broader space $L^2(0,T)$. To initiate this process, we introduce the forward and backward Riemann-Liouville integral operators as follows

$$ \Big(J^\alpha\vartheta\Big)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\vartheta(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau, \quad 0<t<T, \quad \vartheta \in \mathcal{D}(J^\alpha) = L^2(0,\,T), \tag{4} $$

$$ \Big(J_\alpha\vartheta\Big)(t) = \frac{1}{\Gamma(\alpha)}\int_t^T \frac{\vartheta(\tau)}{(\tau-t)^{1-\alpha}}\mathrm{d}\tau, \quad 0<t<T, \quad \vartheta \in \mathcal{D}(J_\alpha) = L^2(0,\,T), \tag{5} $$

where $\mathcal{D}(\mathbf{A})$ denotes the domain of the operator $\mathbf{A}$. According to references [36], we can state the following lemma.

Lemma 1. *For $0 < \alpha < 1$, the forward Riemann-Liouville integral operator $J^\alpha : L^2(0,\,T) \longrightarrow H_\alpha(0,\,T)$ is bijective and isomorphism. In particular, $H_\alpha(0,\,T) = J^\alpha L^2(0,\,T)$.*

This lemma establishes the bijectivity of the forward Riemann-Liouville integral operator J^α within the space $L^2(0,T)$, consequently leading to the existence of its inverse, denoted as $(J^\alpha)^{-1}$ and referred to as $J^{-\alpha}$. With the introduction of this inverse operator, Kubica, Ryszewska, and Yamamoto [36, Definition 2.1] define $\tilde{\partial}_t^\alpha$ for functions within the fractional space $H_\alpha(0,T)$ as follows:

Definition 2. *(Extension of* d_t^α *to* $H_\alpha(0,\,T)$*). For* $0 \le \alpha \le 1$*, we define*

$$\tilde{\partial}_t^\alpha := J^{-\alpha} = (J^\alpha)^{-1} \quad \textit{with } \mathcal{D}(\tilde{\partial}_t^\alpha) = H_\alpha(0,\,T) = J^\alpha L^2(0,\,T).$$

Based on this definition, the operator $\widetilde{\partial}_t^\alpha$ is understood as an extension of the classical Caputo derivative d_t^α in the following sense:

$$\tilde{\partial}_t^\alpha \vartheta = \lim_{n\to\infty} \mathrm{d}_t^\alpha \vartheta_n \ \text{ in } \ L^2(0,T), \ \text{ for all } \ \vartheta \in H_\alpha(0,\,T),$$

where $\vartheta_n \in {}_0C^1[0,T]$ and $\vartheta_n \longrightarrow \vartheta$ in $H^\alpha(0,\,T)$ as $n \longrightarrow \infty$.

Consider the Hilbert spaces $\mathcal{Y}$ and $\mathcal{Z}$. Let $\mathbf{A} : \mathcal{Y} \longrightarrow \mathcal{Z}$ be a bounded linear operator with its domain defined as $\mathcal{D}(\mathbf{A}) = \mathcal{Y}$. We recall that the dual operator $\mathbf{A}'$ is the maximal operator among $\widehat{\mathbf{A}} : \mathcal{Z}' \longrightarrow \mathcal{Y}'$, where $\mathcal{D}(\widehat{\mathbf{A}}) \subset \mathcal{Z}'$, and it satisfies the relation

$${}_{\mathcal{Y}'}\Big\langle \widehat{\mathbf{A}}y, z\Big\rangle_{\mathcal{Y}} = {}_{\mathcal{Z}'}\Big\langle y, \mathbf{A}z\Big\rangle_{\mathcal{Z}}, \quad \forall z \in \mathcal{Y}, \forall y \in \mathcal{D}(\widehat{\mathbf{A}}) \subset \mathcal{Z}'.$$

In what follows, we consider the dual operator $(J_\alpha)' := J_\alpha'$ of $J_\alpha : \ L^2(0,T) \longrightarrow {}^\alpha H(0,T)$ by setting $\mathcal{Y} = L^2(0,T)$ and $\mathcal{Z} = {}^\alpha H(0,T)$. With this in mind, we can now state the following lemma.

Lemma 3. *(see* [63, Proposition 9]*) For* $0 < \alpha < 1$*, the operator*

$$J_\alpha' : \ {}^{-\alpha}H(0,T) \longrightarrow L^2(0,T)$$

is bijective and isomorphism. Moreover, we have $J_\alpha' v = J^\alpha v$ *for* $v \in L^2(0,T)$*.*

Building upon Lemma 3, it follows that $(J_\alpha')^{-1}$ exists, and we denote it as $J_{-\alpha}'$. With this inverse operator in place, we are now equipped to complete the extension of the Caputo derivative to functions in $L^2(0,\,T)$.

Definition 4. *(see* [63, Definition 1]*). For* $0 < \alpha < 1$*, we define*

$$\partial_t^\alpha := J_{-\alpha}' \quad \textit{with} \quad \mathcal{D}(\partial_t^\alpha) = L^2(0,T). \tag{6}$$

In light of Lemma 3, we have

$$\tilde{\partial}_t^\alpha \vartheta = \partial_t^\alpha \vartheta = \lim_{n\to\infty} \mathrm{d}_t^\alpha \vartheta_n \ \text{ in } \ L^2(0,T), \ \text{ for all } \ \vartheta \in H_\alpha(0,\,T),$$

where $\vartheta_n \in {}_0C^1[0,T]$ and $\vartheta_n \longrightarrow \vartheta$ in $H^\alpha(0,\,T)$ as $n \longrightarrow \infty$. Consequently, ∂_t^α stands as an extension of $\widetilde{\partial}_t^\alpha$. Throughout the remainder of the paper, for the sake of simplicity in presentation, the Caputo derivative ∂_t^α is defined as follows:

$$\partial_t^\alpha := \begin{cases} J^{-\alpha} & \text{if } \mathcal{D}(\partial_t^\alpha) = H_\alpha(0,T), \\ J_{-\alpha}' & \text{if } \mathcal{D}(\partial_t^\alpha) = L^2(0,T). \end{cases}$$

3. Uniqueness results

In this section, we establish the unique continuation property for the time-fractional diffusion problem (1), incorporating a source term from a Sobolev space of negative order. We initiate the discussion by introducing the functional space ${}^{-\alpha}H(0,T;\mathcal{V})$, where $\mathcal{V}$ denotes a Banach space. This space is defined as follows (see, e.g., [63])

$${}^{-\alpha}H(0,T;\mathcal{V}) := \Big\{ \vartheta \in L^2(0,T;\mathcal{V}) : \ J_\alpha' \vartheta \in L^2(0,T;\mathcal{V}) \Big\},$$

where J'_α represents the dual operator of $J_\alpha : L^2(0,T) \longrightarrow {}^\alpha H(0,T)$. The norm in ${}^{-\alpha}H(0,T;\mathcal{V})$ is then defined as

$$\|v\|_{^{-\alpha}H(0,T;\mathcal{V})} := \|J'_\alpha v\|_{L^2(0,T;\mathcal{V})}\,.$$

With these definitions, we formulate the problem (1) as follows

$$\left(\partial_t^\alpha + \mathcal{L}\right)u = \mu(t)g(x) \quad \text{in} \quad {}^{-\sigma}H(0,T;L^2(\Omega)), \tag{7}$$

$$u \in L^2(0,T;L^2(\Omega)) \cap {}^{-\sigma}H\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right), \tag{8}$$

where

$$0<\alpha\le\sigma<1, \quad \sigma>\frac{1}{2}, \quad g\in L^2(\Omega), \quad \mu\in {}^{-\sigma}H\left(0,T\right). \tag{9}$$

According to [63, Theorem 13], it can be affirmed that the problem (7)-(9) indeed has a unique solution. Having established these foundational components, we are now prepared to present our primary result regarding the unique continuation property of the fractional parabolic equation.

Theorem 5. *Let u be the solution of the problem* (7)-(8) *and assumes additional properties of* (9)*. Let x_0 be an arbitrary point in $\overline{\Omega}$ and $g\not\equiv 0$ in Ω.*

(i) *If $x_0\in\partial\Omega$ and $g\in H^2(\Omega)$ such that $\partial_\nu g(x_0)\neq 0$, then*

$$\partial_\nu u = 0 \text{ at } \{x_0\}\times(0,T) \text{ implies } \mu=0 \text{ in } {}^{-\sigma}H\left(0,T\right).$$

(ii) *If $x_0\in\Omega$ and $g\in H^2(\Omega)$ such that $g(x_0)\neq 0$, then*

$$u = 0 \text{ at } \{x_0\}\times(0,T) \text{ implies } \mu=0 \text{ in } {}^{-\sigma}H\left(0,T\right).$$

Proof. The readers interested in the proof of this result may refer to Appendix A. ☐

Since $\sigma>\frac{1}{2}$, by Sobolev embedding (see, for instance, [1]), we have the following inclusion:

$${}^{\sigma}H\left(0,T\right)\subset H^{\sigma}\left(0,T\right)\subset C[0,T].$$

This implies that the space ${}^{-\sigma}H(0,T)$ can accommodate any linear combination of Dirac delta functions in the form:

$$\sum_{k=1}^{m}\ell_k\,\delta_{b_k}(t),$$

where $m\in\mathbb{Z}_+$, $\ell_k\in\mathbb{R}$, $b_k\in(0,T)$, and δ_{b_k} is the Dirac delta function defined as

$${}_{(C[0,T])'}\left\langle \delta_{b_k},\Phi\right\rangle_{C[0,T]} = \Phi(b_k) \quad \text{for any} \quad \Phi\in C[0,T]. \tag{10}$$

As a simple consequence of the previous theorem, we can readily deduce the following uniqueness result.

Corollary 6. *Under the same conditions as those in Theorem 5, let us define*

$$\mu^i(t) = \sum_{k=1}^{m^i}\ell_k^i\,\delta_{b_k^i}(t), \quad \ell_k^i\neq 0 \ \text{for} \ i=1,2$$

as two point-source distributions, with distinct time locations b_k^i ($k\in\{1,\,2,\cdots,m^i\}$). In addition, let u^i be the solution to (7)-(8) *with $\mu=\mu^i$ ($i=1,\,2$) such that*

$$\begin{cases} \partial_\nu u^1(x_0,t)=\partial_\nu u^2(x_0,t) \ \text{for} \ t\in(0,T) \ \text{with} \ x_0\in\partial\Omega, \\ \text{or} \\ u^1(x_0,t)=u^2(x_0,t) \ \text{for} \ t\in(0,T) \ \text{with} \ x_0\in\Omega. \end{cases}$$

Then, it follows that $\mu^1 = \mu^2$ *in* ${}^{-\sigma}H\,(0,T)$*. In other words, we have*

$$\begin{aligned} m^1 &= m^2, \\ \ell_k^1 &= \ell_k^2 \qquad \text{for all } k = 1, \cdots, m^1, \\ b_k^1 &= b_k^2 \qquad \text{for all } k = 1, \cdots, m^1. \end{aligned}$$

4. Application to an inverse source problem: Pointwise sources reconstruction

In this section, we elucidate the practical implications of our uniqueness results (refer to Theorem 5 and Corollary 6) by addressing an inverse source problem aimed at identifying singular functions in time using pointwise data. Section 4.1 presents the inverse problem, while in Section 4.2, we demonstrate a local stability estimate of the solution to our inverse problem. Finally, in Section 4.3, we propose a one-shot algorithm facilitating the estimation of the number, locations, and intensities of pointwise sources.

4.1. Problem formulation.

In this section, we delve into an inverse source problem focused on the reconstruction of unknown pointwise sources from a singular point within the domain Ω. For $\frac{1}{2} < \alpha < 1$, we introduce μ^* as a finite combination of point sources, expressed as follows

$$\mu^*(t) = \sum_{k=1}^{N^*} q_k^* \delta_{t_k^*}(t),$$

where $N^* \in \mathbb{Z}_+$, $q_k^* \in \mathbb{R} - \{0\}$, $t_k^* \in (0,T)$, and $\delta_{t_k^*}$ denotes the Dirac delta function defined in the sense of (10). We consider the time-fractional diffusion problem

$$\begin{cases} \partial_t^\alpha u + \mathcal{L}u &= \mu^* \quad \text{in} \quad \Omega \times (0, T), \\ \qquad\quad u &= 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\ \qquad\quad u &= 0 \quad \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{11}$$

The **inverse problem**, we consider, is formulated as follows: Let $x_0 \in \Omega$ be arbitrarily given, and u be the solution to (11). Find the number of sources, N^*, the time locations t_k^* and the intensities q_k^* by the single point observation data $u(x_0, t)$ $(0 < t < T)$.

Remark 7. *Based on the findings in* [63]*, it is clear that* $u \in {}^{-\alpha}H(0,T;H^2(\Omega))$*. Given the spatial dimension* $1 \le n \le 3$*, the Sobolev embedding theorem establishes that* ${}^{-\alpha}H(0,T;H^2(\Omega))$ *is contained within* ${}^{-\alpha}H(0,T;C(\overline{\Omega}))$*. Consequently, we deduce that* $u \in {}^{-\alpha}H(0,T;C(\overline{\Omega}))$*, leading to the conclusion that* $u(x_0,\cdot) \in {}^{-\alpha}H(0,T)$*, which implies that the data is well-defined.*

In the remaining part of this article, to enhance clarity and streamline the presentation, we denote by u the solution of the problem (11).

The uniqueness of this inverse source problem has been established, as demonstrated in Corollary 6. Moreover, within this framework, we assume compatibility of the observational data $u(x_0, t)$ to guarantee the existence of a solution to the above inverse source problem. In the subsequent section, we address the stability issue.

4.2. Local stability.

Stability, as it pertains to our concern here, refers to the continuous dependence of the pointwise source μ^* on the measurements $u(x_0, t)$ $(0 < t < T)$. Stability is a critical consideration for numerical applications, and it has been the focus of numerous authors in various contexts (see, for example, [2, 3, 9, 11, 13, 14]). In this section, we establish a local Lipschitz stability result derived from Gâteaux differentiability (of the data $u(x_0, t)$), demonstrating that the Gâteaux derivative does not vanish. To achieve this, let $\Lambda = (q_k^*,\, t_k^*)$ for $1 \le k \le N^*$, and let $d = (\ell_k,\, b_k)$ be any vector, having the same number of point sources as Λ. For a sufficiently small step $h \neq 0$ such that $t_k^* + h\, b_k \in (0,\, T)$ for $1 \leq k \leq N^*$, we define the following perturbed source term:

$$\mu^h(t) = \sum_{k=1}^{N^*} (q_k^* + h\, \ell_k)\, \delta_{t_k^* + h\, b_k}(t). \tag{12}$$

Here, u^h denotes the solution of the perturbed problem

$$\begin{cases} \partial_t^\alpha u^h + \mathcal{L} u^h &= \mu^h \quad \text{in} \quad \Omega \times (0,T), \\ u^h &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u^h &= 0 \quad \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{13}$$

Since h is sufficiently small to ensure that the points $t_k^* + h\, b_k$ ($k \in \{1, \cdots, N^*\}$) lie in the interval $(0,T)$, the distribution μ^h is supported inside the set $\cup_{k=1}^{N^*} \{t_k^* + h\, b_k\}$. Consequently, according to [63, Theorem 13], problem (13) has a unique solution u^h such that $u^h(x_0, \cdot)$ is well-defined in ${}^{-\alpha}H(0,T)$. With these elements, we can present our stability result.

Theorem 8. *(Local Lipschitz stability). If there exists $k \in \{1, \cdots, N^*\}$ such that $\ell_k \neq 0$ or $b_k \neq 0$, then*

$$\lim_{h \to 0} \frac{1}{|h|} \Big\| u^h(x_0, \cdot) - u(x_0, \cdot) \Big\|_{{}^{-\alpha}H(0,T)} > 0. \tag{14}$$

Proof. The Taylor expansion (of order 2) applied to $\delta_{t_k^* + h\, b_k}$ shows that there exists a real $0 < \eta_k < 1$ such that (see, for example, [13])

$$\delta_{t_k^* + h\, b_k} = \delta_{t_k^*} - h\, b_k\, \delta'_{t_k^*} + \frac{h^2}{2} b_k^2\, \delta''_{t_k^* + \eta_k\, h\, b_k}.$$

Therefore,

$$\mu^h = \mu^* + h\, \mu^1 + h^2 \pi_h, \tag{15}$$

where

$$\mu^1 = \sum_{k=1}^{N^*} \Big(\ell_k\, \delta_{t_k^*} - q_k^*\, b_k\, \delta'_{t_k^*} \Big),$$

$$\pi_h = \sum_{k=1}^{N^*} \Big(\frac{b_k^2}{2} (q_k^* + h\, \ell_k) \delta''_{t_k^* + \eta_k\, h\, b_k} - \ell_k\, b_k\, \delta'_{t_k^*} \Big).$$

Consequently, from (15), the solution u^h of the problem (13) have the following expansion

$$u^h = u + h\, u^1 + h^2\, u_h^2,$$

where u^1 be the solution of

$$\begin{cases} \partial_t^\alpha u^1 + \mathcal{L} u^1 &= \mu^1 \quad \text{in} \quad \Omega \times (0,T), \\ u^1 &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u^1 &= 0 \quad \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{16}$$

and u_h^2 solves the problem

$$\begin{cases} \partial_t^\alpha u_h^2 + \mathcal{L} u_h^2 &= \pi_h \quad \text{in} \quad \Omega \times (0,T), \\ u_h^2 &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_h^2 &= 0 \quad \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{17}$$

Since the distributions μ^1 and π_h are supported in $\cup_{k=1}^{N^*} \{t_k^*\}$ and $\cup_{k=1}^{N^*} \{t_k^* + h\, b_k\}$, respectively, the problems (16) and (17) admit unique solutions (see, for example, [63]) such that the data $u^1(x_0, \cdot)$ and $u_h^2(x_0, \cdot)$ are well-defined in ${}^{-\alpha}H(0,T)$. Thus, we have

$$\Big\| \frac{u^h(x_0, \cdot) - u(x_0, \cdot)}{h} - u^1(x_0, \cdot) \Big\|_{{}^{-\alpha}H(0,T)} = |h| \Big\| u_h^2(x_0, \cdot) \Big\|_{{}^{-\alpha}H(0,T)}.$$

Hence,

$$\lim_{h\to 0} \frac{1}{\left|h\right|} \left\| u^h(x_0,\cdot) - u(x_0,\cdot) \right\|_{{}^{-\alpha}H(0,T)} = \left\| u^1(x_0,\cdot) \right\|_{{}^{-\alpha}H(0,T)}.$$

So to complete the proof of the theorem, it suffices to demonstrate that if there exists $k \in \{1, \cdots, N^*\}$ such that $\ell_k \neq 0$ or $b_k \neq 0$, then $u^1(x_0, t) \neq 0$. This will be the subject of the following lemma. □

Lemma 9. *If $u^1 = 0$ at $\{x_0\} \times (0, T)$, then $\ell_k = b_k = 0$, for $k = 1, \cdots, N^*$.*

Proof. Thanks to Theorem 5(ii), we can establish that if $u^1 = 0$ at $\{x_0\} \times (0, T)$, then $\mu^1 = 0$, implying $\ell_k = 0$ and $q_k^* b_k = 0$. Given the assumption $q_k^* \neq 0$, we deduce that $b_k = 0$, thus concluding the proof of the lemma and, as a consequence, that of Theorem 8. □

Remark 10. *From Theorem 8, it can be observed that if $\lim_{h\to 0} \frac{1}{\left|h\right|} \left\| u^h(x_0,\cdot) - u(x_0,\cdot) \right\|_{{}^{-\alpha}H(0,T)} = \ell > 0$, then there exists $\rho > 0$ and $C > 0$ such that if $|h| < \rho$, then $|h| < C \left\| u^h(x_0,\cdot) - u(x_0,\cdot) \right\|_{{}^{-\alpha}H(0,T)}$, implying the existence of $\bar{C} > 0$ such that for $\left|h\right| < \rho$,*

$$\sum_{k=1}^{N^*} |q_k^h - q_k^*| + \left|t_k^h - t_k^*\right| \leq \bar{C} \left\| u^h(x_0,\cdot) - u(x_0,\cdot) \right\|_{{}^{-\alpha}H(0,T)},$$

providing the local Lipschitz stability result for the reconstruction of the pointwise source. Here, the perturbed intensities q_k^h and time locations t_k^h are defined as follows:

$$q_k^h = q_k^* + h\,\ell_k \quad \text{and} \quad t_k^h = t_k^* + h\,b_k.$$

Next, we analyze the identification issue of our inverse problem.

4.3. **Identification.** In this section, we introduce a non-iterative reconstruction procedure based on sensitivity analysis techniques. Initially, we reframe our inverse problem as an optimization challenge.

4.3.1. **Optimization problem.** Our analysis begins by characterizing the unknown source term μ^* as the solution to a constrained optimization problem. This involves minimizing a least-squares-type functional over a set of admissible source terms defined as:

$$\mathcal{V}_\delta = \left\{ \mu : \ (0,\, T) \to \mathbb{R};\ \mu(t) = \sum_{i=1}^{m'} q_i \delta_{t_i}(t) \right\}. \tag{18}$$

Here m' is a non-negative integer, q_i are non-null scalar quantities, and $t_i \in (0,\, T)$, where $1 \leq i \leq m'$. In addition, the points t_i are assumed to be mutually distinct.

In this context, the unknown source term μ^* is characterized as the solution to the following optimization problem:

$$\underset{\mu\in\mathcal{V}_\delta}{\text{Minimize}}\ \mathcal{J}(\mu), \text{ subject to (21)}, \tag{19}$$

where $\mathcal{J}$ is a least-squares cost function defined on each trial source term $\mu \in \mathcal{V}_\delta$ by

$$\mathcal{J}(\mu) := \int_0^T \left| J_\alpha' \Big(u_\mu(x_0,\cdot) - u(x_0,\cdot) \Big) \right|^2 \mathrm{d}t \tag{20}$$

with J_α' is the dual operator of J_α (refer to Section 2.2 for detailed explanations) and u_μ is the associated potential, solving the time-fractional diffusion problem:

$$\left\{ \begin{array}{rcll} \partial_t^\alpha u_\mu + \mathcal{L} u_\mu &=& \mu & \text{in} \quad \Omega \times (0,\, T), \\ u_\mu &=& 0 & \text{on} \quad \partial\Omega \times (0,\, T), \\ u_\mu &=& 0 & \text{in} \quad \Omega \times \{0\}. \end{array} \right. \tag{21}$$

As the pointwise observation data $u(x_0, t)$ is compatible, a solution $\mu \in \mathcal{V}_\delta$ exists for the considered inverse source problem. Consequently, we have $u_\mu = u$ at $\{x_0\} \times (0, T)$, implying $J'_\alpha\Big(u_\mu(x_0, \cdot) - u(x_0, \cdot)\Big) = 0$. Hence, $\mathcal{J}(\mu) = 0$ which implies that μ is a minimum of $\mathcal{J}$. On the other hand, consider another solution $\mu^1 \in \mathcal{V}_\delta$ for the optimization problem (19) with $\mathcal{J}(\mu^1) = 0$. Then, $J'_\alpha\Big(u_{\mu^1}(x_0, \cdot) - u(x_0, \cdot)\Big) = 0$ in $L^2(0,\, T)$. By the injectivity of J'_α (see Lemma 3), $\big(u_{\mu^1}(x_0, \cdot) - u(x_0, \cdot)\big) = 0$ in ${}^{-\alpha}H\,(0, T)$. Consequently, $u_{\mu^1} = u = u_\mu$ at $\{x_0\} \times (0, T)$. With the uniqueness result from Corollary 6, we conclude that $\mu = \mu^1$, solving the inverse source problem. In summary, this discussion implies that the solution of (19) is "equivalent" to the solution of the considered inverse source problem.

Remark 11. *In* (20)*, one can not opt the traditional least-squares cost function* $\mathcal{J}(\mu) = \big\|u_\mu(x_0, \cdot) - u(x_0, \cdot)\big\|^2_{L^2(0,T)}$ *as there is a lack of required regularity. This is due to the fact that* $\big(u_\mu(x_0, \cdot) - u(x_0, \cdot)\big)$ *belongs to the space* ${}^{-\alpha}H(0, T)$ *and it is well known that* $L^2(0, T)$ *is merely a subset of* ${}^{-\alpha}H(0, T)$*. In conclusion, by following all these issues, we introduced the dual operator* J'_α *with an* L^2*-norm, where* $J'_\alpha(w)$ *is in* $L^2(0, T)$ *for all* $w \in {}^{-\alpha}H(0, T)$ *(refer to Lemma 3 for further details). In the specific scenario where we approximate the Dirac delta by a function in* $L^2(0, T)$ *(see* (28)*), one can infer that the solutions of* (11) *or* (21) *belong to* $L^2(0, T; \mathcal{C}(\overline{\Omega}))$ *(see, for example,* [36]*). Hence, we are empowered to replace the tracking-type cost functional* (20) *with its classical least-squares counterpart (i.e. without* J'_α*).*

To tackle the minimization problem (19), we propose a second-order one-shot reconstruction algorithm based on the sensitivity analysis of the least-squares functional (20) with respect to the set of admissible solutions (18). The key concepts of the proposed reconstruction process are outlined in the following section.

4.3.2. **Sensitivity Analysis .** This section is devoted to minimizing the time shape functional $\mathcal{J}$ with respect to the set of admissible solutions $\mathcal{V}_\delta$. To assess the relevant sensitivities of this functional, we propose perturbing the trial singular source term $\mu \in \mathcal{V}_\delta$ using a fixed number, N, of point sources with arbitrary time locations $t_k \in (0,\, T)$ and intensities $q_k \in \mathbb{R}$. More precisely, the perturbed form of the source term μ is defined as

$$\mu_\delta(t) = \mu(t) + \sum_{k=1}^{N} q_k \delta_{t_k}(t). \tag{22}$$

Building upon (21) and (22), we can introduce the forward solution u_{μ_δ} as that solving

$$\begin{cases} \partial_t^\alpha u_{\mu_\delta} + \mathcal{L} u_{\mu_\delta} &= \mu_\delta \quad \text{in} \quad \Omega \times (0,\, T), \\ u_{\mu_\delta} &= 0 \quad \text{on} \quad \partial\Omega \times (0,\, T), \\ u_{\mu_\delta} &= 0 \quad \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{23}$$

Then the perturbed counterpart of the tracking time shape functional is written as

$$\mathcal{J}(\mu_\delta) = \int_0^T \Big| J'_\alpha\Big(u_{\mu_\delta}(x_0, \cdot) - u(x_0, \cdot)\Big)\Big|^2 \mathrm{d}t. \tag{24}$$

We propose the ansatz

$$u_{\mu_\delta}(x, t) = u_\mu(x, t) + \sum_{k=1}^{N} q_k v_k(x, t), \tag{25}$$

where v_k solves

$$\begin{cases} \partial_t^\alpha v_k + \mathcal{L} v_k &= \delta_{t_k} \quad \text{in} \quad \Omega \times (0, T), \\ v_k &= 0 \quad \text{on} \quad \partial\Omega \times (0,\, T), \\ v_k &= 0 \quad \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{26}$$

Remark 12. *According to* (26)*, in principle we have to solve a number N of auxiliary anomalous diffusion problems for each t_k. However, by assuming $t_1 < t_2 < \cdots < t_k < \cdots < t_N$ and since the initial conditions are homogeneous, then it is need to solve just one problem for t_1 and then shift the obtained solution in time by setting $v_{k+1}(x,t) = v_k(x, t - t_k)$ for $t > t_k$ and $v_{k+1}(x,t) = 0$, otherwise, with $k = 1, 2, \cdots, N$. This feature simplifies the algorithm implementation and will be explored in the numerical section.*

Let us evaluate the difference

$$
\begin{aligned}
\mathcal{J}(\mu_\delta) - \mathcal{J}(\mu) = {} & 2\sum_{i=1}^{N} q_i \int_0^T J'_\alpha\big(u_\mu(x_0,\cdot) - u(x_0,\cdot)\big) J'_\alpha\big(v_i(x_0,\cdot)\big)\,\mathrm{d}t \\
& + \sum_{i=1}^{N}\sum_{j=1}^{N} q_i q_j \int_0^T J'_\alpha\big(v_i(x_0,\cdot)\big) J'_\alpha\big(v_j(x_0,\cdot)\big)\,\mathrm{d}t.
\end{aligned}
$$

Let us now introduce the vector of intensities $q = (q_1, q_2, \cdots, q_N)$. Then, we have

$$
\mathcal{J}(\mu_\delta) - \mathcal{J}(\mu) = 2h \cdot q + Hq \cdot q. \tag{27}
$$

Moreover, h and H are the first and second order topological derivatives, respectively. The vector h and matrix H have entries

$$
h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{pmatrix},
$$

where

$$
\begin{aligned}
h_i &= \int_0^T J'_\alpha\big(u_\mu(x_0,\cdot) - u(x_0,\cdot)\big) J'_\alpha\big(v_i(x_0,\cdot)\big)\,\mathrm{d}t \\
H_{ij} &= \int_0^T J'_\alpha\big(v_i(x_0,\cdot)\big) J'_\alpha\big(v_j(x_0,\cdot)\big)\,\mathrm{d}t
\end{aligned}
$$

These derivations induce a second-order non-iterative reconstruction algorithm as the one introduced in [12] and further developed in [50]. See also [23, 24] for different approaches also based on higher-order topological derivatives.

5. Numerical results

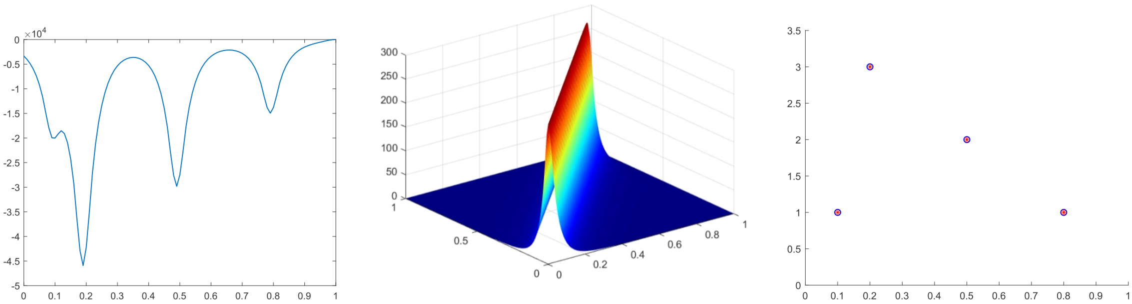
In this section, we restrict ourselves to the bidimensional case. We consider a domain $\Omega = (0,1) \times (0,1)$. The operator $\mathcal{L} = -\Delta = -\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial y^2}$ and the observable point is fixed at $x_0 = (0.6, 0.3) \in \Omega$. In addition, we consider $\alpha \in (0.1, 0.9)$. This scenario has been adopted to push the method to the limit, since the theory has been developed for $\alpha > 1/2$. The boundary value problems are discretized by using standard Finite Element Method in space and Finite Difference Method in time following the same procedure as described in [53]. In particular, the domain Ω is discretized into 6400 three-node finite elements. Finally, the final time is set as $T = 1$ and the resulting interval $(0,1)$ is discretized into 100 uniform time steps. See Remark 13.

Remark 13. *In order to simplify the numerical computations, the Dirac delta $\delta_{t_k} = \delta(t - t_k)$ is replaced by a step function $s \in L^2(0,T)$, such that*

$$
s(t - t_k) = \begin{cases} 0, & t < t_k - \triangle t, \\ \frac{1}{2\triangle t}, & t_k - \triangle t \le t \le t_k + \triangle t, \\ 0, & t > t_k + \triangle t, \end{cases} \tag{28}
$$

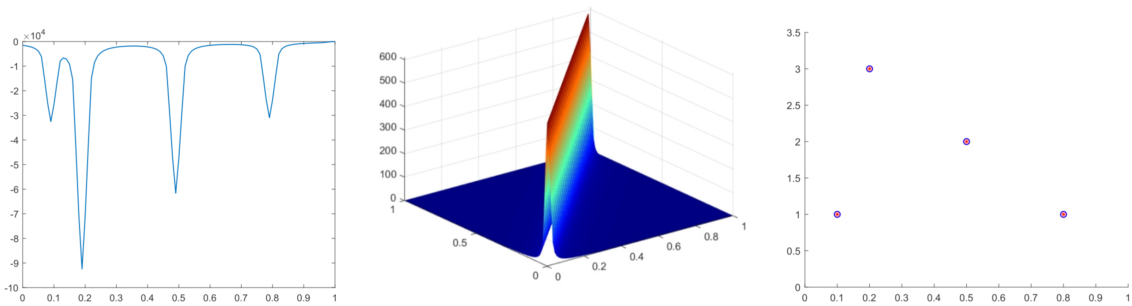
where $\triangle t$ is the time-step size. From this approximation, the tracking-type cost functional (20) *can be replaced by its standard last-square counterpart for numerical computations (cf. Remark 11), allowing for removing $J'_\alpha(\cdot)$ from the expressions in this part of the article.*

5.1. **Example 1.** In the first example, we set $\alpha \in \{0.1, 0.5, 0.9\}$. The target is given by a number $N^* = 4$ of pointwise sources described by the pairs $(q_1^*, t_1^*) = (1, 0.1)$, $(q_2^*, t_2^*) = (3, 0.2)$, $(q_3^*, t_3^*) = (2, 0.5)$ and $(q_4^*, t_4^*) = (1, 0.8)$. The obtained results are reported in Figures 1, 2 and 3 for $\alpha \in \{0.9, 0.5, 0.1\}$, respectively. In each of the figures, we show the first-order topological derivatives (a), the second-order topological derivatives (b) and the targets ($*$) together with the reconstructed (o) solutions (c). From an analysis of the figures, we observe that the first-order topological derivative gives qualitative information on the time locations of the true sources (a). The second-order topological derivative (b) is computed from the solutions to the canonical problems (26), which depend only on the domain Ω and parameter α, so that they can be solved only once and tabulated for solving any other problem endowed with different targets (q_k^*, t_k^*) and observable point x_0. Finally, we also observe that the proposed reconstruction algorithm returns the exact solutions in all cases (c), even for the situation beyond the theoretical setup, namely, for $\alpha < 1/2$.



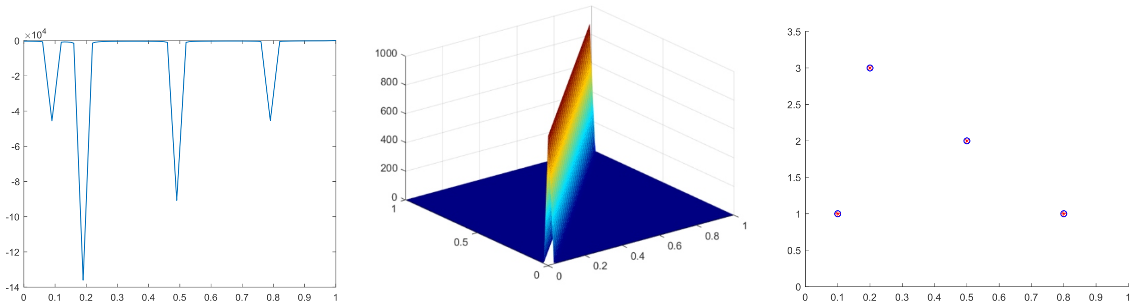
(a) (b) (c)

FIGURE 1. Example 1. Obtained results for $\alpha = 0.9$: first-order topological derivative (a), second-order topological derivative (b) and target ($*$) together with the reconstructed (o) solution (c). In the horizontal axes varies the time t, whereas in the vertical axes vary the corresponding quantities.



(a) (b) (c)

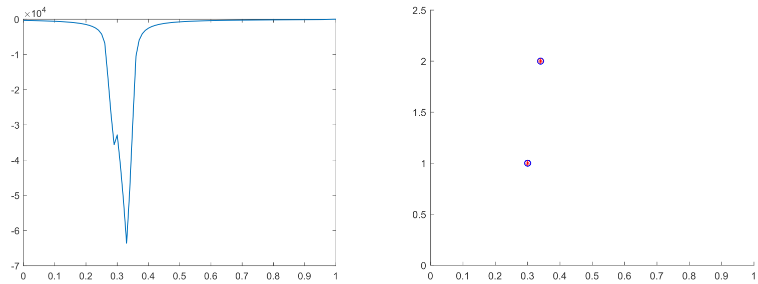
FIGURE 2. Example 1. Obtained results for $\alpha = 0.5$: first-order topological derivative (a), second-order topological derivative (b) and target ($*$) together with the reconstructed (o) solution (c). In the horizontal axes varies the time t, whereas in the vertical axes vary the corresponding quantities.



(a) (b) (c)

FIGURE 3. Example 1. Obtained results for $\alpha = 0.1$: first-order topological derivative (a), second-order topological derivative (b) and target ($*$) together with the reconstructed (o) solution (c). In the horizontal axes varies the time t, whereas in the vertical axes vary the corresponding quantities.

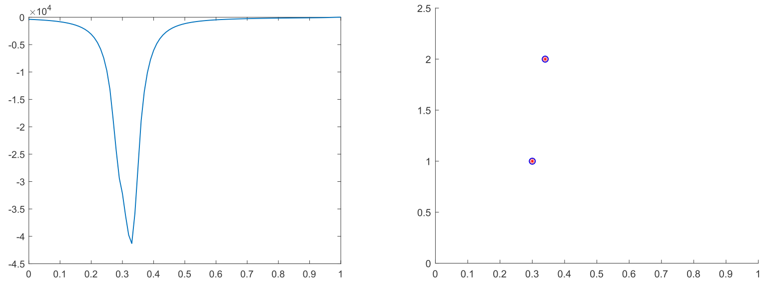
5.2. **Example 2.** In the second example, we consider $\alpha \in \{0.5, 0.8\}$. The target is given by two ($N^* = 2$) pointwise sources close to each other of the form $(q_1^*, t_1^*) = (1, 0.30)$ and $(q_2^*, t_2^*) = (2, 0.32)$. Actually, they are separated by two time steps only. The obtained results are reported in Figures 4 and 5 for $\alpha \in \{0.5, 0.8\}$, respectively. In each of the figures, we show the first-order topological derivatives (a) and the targets ($*$) together with the reconstructed (o) solutions (b). In Figure 4(a), we observe two picks in the first-order topological derivative with their tips coinciding with the true time locations. In contrast, from Figure 5(a) we observe only one pick in the first-order topological derivative with its tip coinciding with the true time location of higher intensity. However, by using the second-order topological derivative, the algorithm shows to be effective in reconstructing both sources independent of α, as can be seen in Figures 4(b) and 5(b). Finally, it is important to mention that the second-order reconstruction algorithm still works even if the distance between the sources is given just by one time step.



(a) (b)

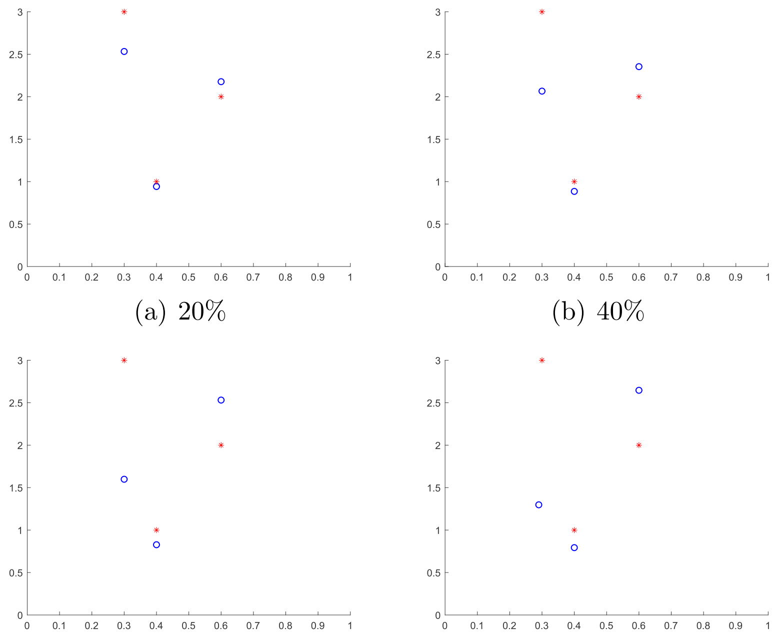
FIGURE 4. Example 2. Obtained results for $\alpha = 0.5$: first-order topological derivative (a) and target ($*$) together with the reconstructed (o) solution (b). In the horizontal axes varies the time t, whereas in the vertical axes vary the corresponding quantities.

5.3. **Example 3.** Finally, in order to verify the robustness of the reconstruction algorithm with respect to noisy data, the measurement $u(x_0, t)$ (for $0 < t < 1$) is corrupted with White Gaussian Noise (WGN) of zero mean. We set $\alpha = 0.5$. The target is given by a number $N^* = 3$ of pointwise sources characterized by the pairs $(q_1^*, t_1^*) = (3, 0.3)$, $(q_2^*, t_2^*) = (1, 0.4)$ and $(q_3^*, t_3^*) = (2, 0.6)$. The obtained reconstructions are reported in Figure 6 for varying levels of noise, namely, 20%, 40%, 60% and 73%.



(a) (b)

FIGURE 5. Example 2. Obtained results for $\alpha = 0.8$: first-order topological derivative (a) and target ($*$) together with the reconstructed (o) solution (b). In the horizontal axes varies the time t, whereas in the vertical axes vary the corresponding quantities.



(a) 20% (b) 40% (c) 60% (d) 73%

FIGURE 6. Example 3. Obtained results for varying level of noise. In the horizontal axes varies the time t, whereas in the vertical axes varies the source intensity. The target is represented by ($*$) and the reconstructed solution by (o).

In Table 1 we present the quantitative results. With no noise, the reconstruction is exact (second column). The higher is the noise level, the worse is the result, as expected. More precisely, the time locations are exact up to 60% of noise with an increasing discrepancy on the source intensity (third, fourth and fifth columns). Finally, the time location of one source gets lost for 73% of noise, leading to the worst result, as expected. From an analysis of Figure 6 and Table 1, we observe that the method is quite resilient with respect to noisy data.

Table 1. Example 3: Quantitative summary of the obtained results.

Source	0%	20%	40%	60%	73%
(q_1^*, t_1^*)	$(3.00, 0.30)$	$(2.53, 0.30)$	$(2.06, 0.30)$	$(1.60, 0.30)$	$(1.30, 0.29)$
(q_2^*, t_2^*)	$(1.00, 0.40)$	$(0.94, 0.40)$	$(0.88, 0.40)$	$(0.83, 0.40)$	$(0.79, 0.40)$
(q_3^*, t_3^*)	$(2.00, 0.60)$	$(2.18, 0.60)$	$(2.35, 0.60)$	$(2.53, 0.60)$	$(2.65, 0.60)$

6. Concluding remarks

In this study, we address the reconstruction of singular functions (in time) in a fractional subdiffusion scenario using observational data obtained from both a single boundary point and within the domain itself. We explore theoretical and numerical facets, delving into a unique continuation result in the theoretical domain. Specifically, we establish that an unknown source, belonging to a Sobolev space of negative order, can be uniquely identified from either a single boundary point or within the domain. Also, we have derived a local Lipschitz stability result. The full stability issue (for example logarithmic or Hölder or Lipschitz) is, however, remains an open question requires further profound investigation. In our numerical exploration, we present a rapid and precise reconstruction method. We define the singular functions to be reconstructed as a linear combination of Dirac delta functions in time. Subsequently, the unknown pointwise sources are determined through an optimization process aimed at minimizing a time tracking shape functional, measuring the discrepancy between simulated and measured potentials within the domain Ω. We devise a second-order noniterative reconstruction algorithm, enabling us to ascertain the number, time-locations, and intensities of concealed pointwise sources. Through several numerical examples, we validate the efficiency and accuracy of the proposed approach.

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Appendix A. Mathematical justifications

In this appendix, we present the proof for Theorem 5. To establish this result, we follow the approach outlined in [63, Section 7]. Specifically, we begin by transforming the problem (7)-(9) into a regular system, and then proceed to establish Duhamel's principle for this system.

A.1. Transfer to a Regular System.

Here, we are converting the system (7)-(9) into a regular form, as described in the following proposition:

Proposition 14. *Let $w := J'_\sigma u$, where u is the solution of the system (7)-(9). Then*

$$w \in H_\sigma\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \tag{29}$$

satisfies

$$\left(\partial_t^\alpha + \mathcal{L}\right) w = (J'_\sigma \mu)\, g \quad in \quad L^2\left(0, T; L^2(\Omega)\right) \tag{30}$$

Proof. Drawing upon the regularity properties of u (as indicated in (8)), we have

$$u \in {}^{-\sigma}H\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right).$$

Leveraging Lemma 3, we further infer:

$$w = J'_\sigma u \in L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right). \tag{31}$$

Moreover, utilizing (8), we deduce that

$$u \in L^2\big(0, T; L^2(\Omega)\big).$$

Once again employing Lemma 3, we can conclude that $w = J'_\sigma u = J^\sigma u$, which, in turn, implies, by virtue of Lemma 1, the following:

$$w \in H_\sigma\big(0, T; L^2(\Omega)\big). \tag{32}$$

As a result, combining (31) and (32), we attain the regularity properties denoted in (29).

Now, we proceed to demonstrate that w indeed satisfies (30). Utilizing Lemma 3, we are aware that the operator J'_σ is injective within the space ${}^{-\sigma}H(0,T)$. Applying J'_σ to both sides of (7) results in the following expression:

$$J'_\sigma\left(\partial_t^\alpha u\right) + \mathcal{L}\left(J'_\sigma u\right) = \left(J'_\sigma \mu\right) g \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

Next, we will establish the following equality:

$$J'_\sigma\left(\partial_t^\alpha u\right) = \partial_t^\alpha\left(J'_\sigma u\right).$$

Drawing upon Lemma 1.3(iv) in [36] and invoking [34, Problem 5.26 (p. 168)], we can conclude that

$$J'_\sigma = J'_{\sigma-\alpha} J'_\alpha = J'_\alpha J'_{\sigma-\alpha}.$$

Therefore, on one hand, using the fact that $\partial_t^\alpha = (J'_\alpha)^{-1}$ (see Definition 4), we have

$$J'_\sigma\left(\partial_t^\alpha\right) = J'_\sigma\left(J'_\alpha\right)^{-1} = J'_{\sigma-\alpha} J'_\alpha\left(J'_\alpha\right)^{-1} = J'_{\sigma-\alpha}.$$

On the other hand, we have

$$\partial_t^\alpha\left(J'_\sigma\right) = \left(J'_\alpha\right)^{-1} J'_\sigma = \left(J'_\alpha\right)^{-1} J'_\alpha J'_{\sigma-\alpha} = J'_{\sigma-\alpha}.$$

Consequently, it follows that the function $w = J'_\sigma u$ satisfies (30). Hence the fact. □

A.2. **Duhamel's principle.** In this section, we establish Duhamel's principle in the space $H_\alpha(0,T)$ for time-fractional parabolic equations. To begin, we demonstrate a fundamental property of ∂_t^α in the context of convolving two functions, specifically when $\mathcal{D}(\partial_t^\alpha) = H_\alpha(0,T)$ (i.e., $\partial_t^\alpha = J^{-\alpha}$). To introduce this property, we define the convolution of two functions as follows

$$\phi * \psi(t) = \int_0^t \phi(t-\tau)\psi(\tau)\mathrm{d}\tau, \quad 0 < t < T \tag{33}$$

for $\phi \in L^2(0,\,T)$ and $\psi \in L^1(0,\,T)$. We then present the following lemma:

Lemma 15. *(see* [63, Theorem 4]*) For $\alpha \geq 0$, if $\phi \in H_\alpha(0,T)$ and $\psi \in L^1(0,T)$, then $\phi * \psi \in H_\alpha(0,T)$, and we have*

$$\partial_t^\alpha\big(\phi * \psi\big) = \partial_t^\alpha\big(\phi\big) * \psi.$$

Let's now focus on the following time-fractional diffusion problem:

$$\partial_t^\alpha \zeta + \mathcal{L}\zeta = \rho(t)\,\gamma(x) \quad \text{in} \quad L^2(0,T;L^2(\Omega)), \tag{34}$$

$$\zeta \in H_\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)). \tag{35}$$

In the system (34)-(35), we have $0 < \alpha < 1$, and the functions $\rho \in L^2(0,T)$ and $\gamma \in L^2(\Omega)$. Building upon the results established in [36], it is known that the problem (34)-(35) possesses a unique solution. We now prove the following lemma:

Lemma 16. *For $0 < \alpha < 1$, let $\varphi := J^{1-\alpha}\zeta$, where ζ represents the unique solution of* (34)-(35)*. Then $\varphi \in H_\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ satisfies*

$$\partial_t^\alpha \varphi + \mathcal{L}\varphi = \left(J^{1-\alpha}\rho\right)\gamma \ \textit{ in } \ L^2(0,T;L^2(\Omega)). \tag{36}$$

Proof. Since $\zeta \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, we can utilize Lemma 1 and the inclusion (2) to conclude

$$\varphi = J^{1-\alpha}\zeta \in H_{1-\alpha}(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \subset L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)). \tag{37}$$

Now, we establish that $\varphi \in H_\alpha(0,T;L^2(\Omega))$. Utilizing (4), we can express φ as

$$\varphi(\cdot,t) = J^{1-\alpha}\zeta(\cdot,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\zeta(\cdot,\tau)\mathrm{d}\tau, \ \ 0<t<T.$$

Define $\phi(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ and $\psi(t) = \zeta(\cdot,t)$. Then, the function φ can be expressed as the convolution of these two functions:

$$\varphi(\cdot,t) = \phi * \psi(t), \ \ 0<t<T.$$

Since $\phi \in L^1(0,T)$ and $\zeta \in H_\alpha\left(0,T;L^2(\Omega)\right)$, we can apply Lemma 15 to deduce

$$\varphi \in H_\alpha\left(0,T;L^2(\Omega)\right). \tag{38}$$

Therefore, from (37) and (38), we see $\zeta \in H_\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

Next, we will establish that ζ satisfies equation (36). By applying the operator $J^{1-\alpha}$ to the left-hand side of equation (34) and leveraging the injectivity property of $J^{1-\alpha}$ (as shown in Lemma 1), we can express it as

$$J^{1-\alpha}\left(\partial_t^\alpha \zeta\right) + \mathcal{L}\left(J^{1-\alpha}\zeta\right) = \left(J^{1-\alpha}\rho\right)\gamma. \tag{39}$$

Since $\zeta \in H_\alpha(0,T;L^2(\Omega))$, we can use Definition 2 to express $\partial_t^\alpha \zeta = J^{-\alpha}\zeta$. Furthermore, by referring to [36, Lemma 1.3(iv)], we obtain

$$J^\alpha J^{1-\alpha} = J^1. \tag{40}$$

Here, $(J^1 y)(t) = \displaystyle\int_0^t y(\tau)\,\mathrm{d}\tau$ for $0 \le t \le T$. To proceed, we apply $(J^\alpha)^{-1}$ to both sides of equation (40), yielding

$$J^\alpha J^{1-\alpha}\left(J^\alpha\right)^{-1} = J^{1-\alpha}. \tag{41}$$

Applying $(J^\alpha)^{-1}$ to the left-hand side of equation (41) and using the property $(J^\alpha)^{-1} J^\alpha z = z$ for z in $L^2(0,T)$, we get

$$J^{1-\alpha}\left(J^\alpha\right)^{-1} = \left(J^\alpha\right)^{-1} J^{1-\alpha}. \tag{42}$$

This implies that $J^{1-\alpha}\left(\partial_t^\alpha\right) = \partial_t^\alpha\left(J^{1-\alpha}\right)$. Consequently, we can conclude that

$$J^{1-\alpha}\left(\partial_t^\alpha\right) = \partial_t^\alpha\left(J^{1-\alpha}\right).$$

Consequently, we conclude that $\partial_t^\alpha\left(J^{1-\alpha}\zeta\right) + \mathcal{L}\left(J^{1-\alpha}\zeta\right) = \left(J^{1-\alpha}\rho\right)\gamma$. This completes the proof of Lemma 16. □

Building upon Lemma 16 and [41, Lemma 4], we can now derive Duhamel's principle for time-fractional parabolic equations in $H_\alpha(0,T)$-space.

Lemma 17. *(Duhamel's principle). Given $\rho \in L^2(0,T)$ and $\gamma \in L^2(\Omega)$, let ζ be the solution of* (34)-(35)*. Then the weak solution $\varphi = J^{1-\alpha}\zeta$ to the problem* (36) *allows the representation*

$$\varphi(x,t) = \int_0^t \rho(\tau)Q(x,t-\tau)\mathrm{d}\tau, \quad (x,t)\in\Omega\times(0,T),$$

where Q satisfies the following system

$$\begin{cases} \partial_t^\alpha\Big(Q-\gamma\Big) + \mathcal{L}Q = 0 \ \ in \ \ L^2(0,T;L^2(\Omega)), \\ Q \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \\ Q-\gamma \in H_\alpha(0,T;L^2(\Omega)). \end{cases} \tag{43}$$

A.3. **Proof of Theorem 5.** To begin, we introduce a fundamental result in the domain of complex analysis: the Titchmarsh Convolution Theorem. The proof of this result can be found in [33].

Lemma 18. *(Titchmarsh Convolution Theorem). Let ϕ and ψ be integrable functions on the interval $(0,T)$ such that $\phi * \psi = 0$ a.e. in $(0,T)$ and $0 \in supp(\phi)$. Then, $\psi = 0$ a.e. in $(0,T)$. Here, $supp(\phi)$ represents the support of the function ϕ defined on $(0,T)$. It is defined as the complement of the largest open subset of $(0,T)$ where $\phi = 0$ a.e.*

Now, we proceed with the completion of the proof for Theorem 5. We define w as $w := J'_\sigma u$, where u is the solution of the system (7)-(9). By applying Proposition 14 and utilizing Duhamel's principle (refer to Lemma 17), we can express the function w as follows

$$J^{1-\alpha} w(\cdot,t) = \int_0^t \left(J'_\sigma \mu\right)(\tau) v(\cdot, t-\tau) \mathrm{d}\tau, \quad 0 < t < T. \tag{44}$$

Here, v satisfies the following system

$$\begin{cases} \partial_t^\alpha \Big(v - g\Big) + \mathcal{L} v = 0 \ \text{ in } \ L^2(0,T;L^2(\Omega)), \\ v - g \in H_\sigma(0,T;L^2(\Omega)), \\ v \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)). \end{cases} \tag{45}$$

Since $\frac{1}{2} < \sigma < 1$, the Sobolev embedding theorem ensures that $H_\sigma(0,T) \subset C[0,T]$. Consequently, from the second condition in (45), we have $(v-g) \in H_\sigma\left(0,T;L^2(\Omega)\right)$, which implies that $(v-g) \in C\left([0,T];L^2(\Omega)\right)$. In this scenario, we see that the initial condition $v(\cdot,0)$ can be expressed as (for more details, we refer the reader to [36])

$$v(\cdot,0) = g \ \text{ in } \ L^2\text{-sense}. \tag{46}$$

Proof of $5(i)$**.** Let's consider a point $x_0 \in \partial\Omega$, which is the same as the one mentioned in Theorem 5-(i). By using the assumption that $\partial_\nu u = 0$ at $\{x_0\}\times(0,T)$, we can derive the following:

$$\partial_\nu w(x_0,t) = 0, \quad 0 < t < T.$$

Differentiating two sides of equality (44) with respect to $x_0 \in \partial\Omega$, we obtain

$$\int_0^t \left(J'_\sigma \mu\right)(\tau) \partial_\nu v(x_0, t-\tau) \mathrm{d}\tau = J^{1-\alpha} \partial_\nu w(x_0,t) = 0, \quad 0 < t < T.$$

On the other hand, from (46), we observe that

$$\partial_\nu v(x_0,0) = \partial_\nu g(x_0) \neq 0.$$

Consequently, with the assistance of Lemma 18, we can deduce that $J'_\sigma \mu = 0$ in $L^2(0,T)$, and taking into account the injectivity property of J'_σ (see Lemma 3), it follows that $\mu = 0$ in ${}^{-\sigma}H\,(0,T)$.

Proof of $5(ii)$**.** Let's consider a point $x_0 \in \Omega$, which is the same as the one mentioned in Theorem 5-(ii), i.e.;

$$g(x_0) \neq 0 \quad \text{and} \quad u = 0 \ \text{ at } \ \{x_0\}\times(0,T). \tag{47}$$

Since $w = J'_\sigma u$, we can deduce that

$$w(x_0,t) = 0, \quad 0 < t < T.$$

From (44), we have

$$\int_0^t \left(J'_\sigma \mu\right)(\tau) v(x_0, t-\tau) \mathrm{d}\tau = J^{1-\alpha} w(x_0,t) = 0, \quad 0 < t < T.$$

Using that $v(x_0, 0) = g(x_0) \neq 0$, by invoking Lemma 18 we conclude that $J'_\sigma \mu = 0$ in $L^2(0, T)$. Further, considering the injectivity characteristic of J'_σ as described in Lemma 3,it consequently implies that $\mu = 0$ in ${}^{-\sigma}H\,(0, T)$. This completes the proof of Theorem 5.

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