TOPOLOGICAL SENSITIVITY ANALYSIS FOR THREE-DIMENSIONAL LINEAR ELASTICITY PROBLEM

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Abstract. In this work we use the Topological-Shape Sensitivity Method to obtain the topological derivative for three-dimensional linear elasticity problems, adopting the total potential energy as cost function and the equilibrium equation as constraint. This method, based on classical shape sensitivity analysis, leads to a systematic procedure to calculate the topological derivative. In particular, firstly we present the mechanical model, later we perform the shape derivative of the corresponding cost function and, finally, we calculate the final expression for the topological derivative using the Topological-Shape Sensitivity Method and results from classical asymptotic analysis around spherical cavities. In order to point out the applicability of the topological derivative in the context of topology optimization problems, we use this information as a descent direction to solve a three-dimensional topology design problem. Furthermore, through this example we also show that the topological derivative together with an appropriate mesh refinement strategy are able to capture high quality shapes even using a very simple topology algorithm.

1. Introduction

The topological derivative has been recognized as an alternative methodology and at the same time a promising tool to solve topology optimization problems (see [5, 6, 10, 30] and references therein). Moreover, this is a broad concept. In fact, the topological derivative may also be applied to analyze any kind of sensitivity problem in which discontinuous changes are allowable, for example, discontinuous changes on the shape of the boundary, on the boundary conditions, on the load system and/or on the parameters of the problem. The information given by the topological derivative is very useful in solving problems such as topology design, inverse problems (domain, boundary conditions and parameters characterization), image processing (enhancement and segmentation) and in the mechanical modeling of problems with changes on the configuration of the domain like fracture mechanics and damage. An extension of topological derivative in order to include arbitrary shaped holes and its applications to Laplace, Poisson, Helmoltz, Navier, Stokes and Navier-Stokes equations were developed by Masmoudi and Sokolowski and their respective co-workers (see, for instance, [2, 24] for applications of the topological derivative in the context of topology design and inverse problems).

Although the topological derivative is extremely general, this concept may become restrictive due to mathematical difficulties involved in its calculation. To overcame this difficult authors have put forward different approaches to calculate the topological derivative. In particular, we proposed an alternative method based on classical shape sensitivity analysis (see [3, 17, 18, 28, 31, 32, 34] and references therein). This approach, called Topological-Shape Sensitivity Method, has been applied for us in the following two-dimensional problems:

- Poisson: steady-state heat conduction problem taking into account both homogeneous and non-homogeneous Neumann and Dirichlet and also Robin boundary conditions on the hole [8, 26];
- Navier: plane stress and plane strain linear elasticity [9];
- Kirchhoff: thin plate bending problem [27];

Specifically, we considered respectively scalar second-order, vector second-order and scalar forth-order PDE two-dimensional problems. As a natural sequence of our research, in the present paper we apply the Topological-Shape Sensitivity Method to calculate the topological derivative in a vector second-order PDE three-dimensional problem. At this moment, we consider the three-dimensional linear elasticity problem taking the total potential energy as cost function and the state equation

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as constraint. Thus, for the sake of completeness, in Section 2 we present a brief description of the Topological-Shape Sensitivity Method. In Section 3 we use this approach to calculate the topological derivative for the problem under consideration: initially we present the mechanical model associated to three-dimensional linear elasticity, further we calculate the shape derivative for this problem adopting the total potential energy as cost function and the weak form of the state equation as constraint and then we obtain the expression for the topological derivative using classical asymptotic analysis around spherical cavities. Finally, in Section 4 we show that the topological derivative is a powerful tool to be applied in topology optimization context by using it as a descent direction to solve a three-dimensional topology design problem, whose result is improved with help of an appropriate adaptive mesh refinement strategy.

2. Topological-Shape Sensitivity Method

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. If the domain Ω is perturbed by introducing a small hole at an arbitrary point $\hat{\mathbf{x}} \in \Omega$, we have a new domain $\Omega_\varepsilon = \Omega - \overline{B}_\varepsilon$, whose boundary is denoted by $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$, where $\overline{B}_\varepsilon = B_\varepsilon \cup \partial B_\varepsilon$ is a ball of radius ε centered at point $\hat{\mathbf{x}} \in \Omega$. Therefore, we have the original domain without hole Ω and the new one Ω_ε with a small hole B_ε as shown in fig. (1).

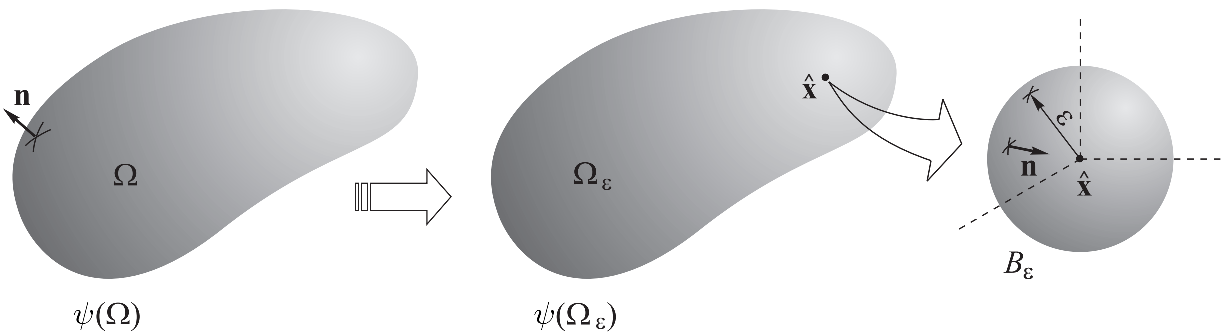


Figure 1. topological derivative concept

Thus, considering a cost function ψ defined in both domains, we have the following topological asymptotic expansion [10]

$$\psi\left(\Omega_\varepsilon\right) = \psi\left(\Omega\right) + f\left(\varepsilon\right) D_T\left(\hat{\mathbf{x}}\right) + \mathcal{R}(f(\varepsilon)) \,, \tag{2.1}$$

where $f(\varepsilon)$ is a negative function that decreases monotonically so that $f(\varepsilon) \to 0$ with $\varepsilon \to 0^+$ and $\mathcal{R}(f(\varepsilon))$ contains all higher order terms than $f(\varepsilon)$, that is, it satisfies

$$\mathcal{R}(f(\varepsilon)) : \lim_{\varepsilon\to 0} \frac{\mathcal{R}(f(\varepsilon))}{f(\varepsilon)} = 0 \,. \tag{2.2}$$

In addition, we can rewrite eq. (2.2) and, after taking the limit $\varepsilon \to 0$, $D_T(\hat{\mathbf{x}})$ may be recognized as the well-known topological derivative, that is

$$D_T\left(\hat{\mathbf{x}}\right) = \lim_{\varepsilon\to 0} \frac{\psi\left(\Omega_\varepsilon\right) - \psi\left(\Omega\right)}{f\left(\varepsilon\right)} \,. \tag{2.3}$$

Recently an alternative procedure to calculate the topological derivative, called Topological-Shape Sensitivity Method, was introduced by the authors [26]. This approach makes use of the whole mathematical framework (and results) developed for shape sensitivity analysis (see, for instance, the pioneering work of Murat & Simon [23]). The main result obtained in [26] may be briefly summarized in the following Theorem (see also [8, 25]):

Theorem 1. *Let $f(\varepsilon)$ be a function chosen in order to $0 < |D_T(\hat{\mathbf{x}})| < \infty$, then the topological derivative given by eq. (2.3) can be written as*

$$D_T\left(\hat{\mathbf{x}}\right) = \lim_{\varepsilon\to 0} \frac{1}{f'\left(\varepsilon\right)} \left.\frac{d}{d\tau}\psi\left(\Omega_\tau\right)\right|_{\tau=0} \,, \tag{2.4}$$

where $\tau \in \mathbb{R}^+$ is used to parameterize the domain. That is, for τ small enough, we have

$$\Omega_\tau := \left\{ \mathbf{x}_\tau \in \mathbb{R}^3 : \mathbf{x}_\tau = \mathbf{x} + \tau \mathbf{v},\ \mathbf{x} \in \Omega_\varepsilon \right\} \ . \tag{2.5}$$

Therefore, $\mathbf{x}_\tau|_{\tau=0} = \mathbf{x}$ and $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon$. In addition, considering that $\mathbf{n}$ is the outward normal unit vector (see fig. 1), then we can define the shape change velocity $\mathbf{v}$, *which is a smooth vector field in Ω_ε assuming the following values on the boundary $\partial\Omega_\varepsilon$*

$$\left\{ \begin{array}{ll} \mathbf{v} = -\mathbf{n} & \text{on } \partial B_\varepsilon \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega \end{array} \right. \tag{2.6}$$

and the shape sensitivity of the cost function in relation to the domain perturbation characterized by $\mathbf{v}$ is given by

$$\left. \frac{d}{d\tau} \psi\left(\Omega_\tau\right) \right|_{\tau=0} = \lim_{\tau \to 0} \frac{\psi\left(\Omega_\tau\right) - \psi(\Omega_\varepsilon)}{\tau} \ . \tag{2.7}$$

Proof. Let us take the derivative in relation to ε in both sides of eq. (2.1) to obtain

$$\frac{d}{d\varepsilon} \psi\left(\Omega_\varepsilon\right) = f'\left(\varepsilon\right) D_T\left(\hat{\mathbf{x}}\right) + \mathcal{R}'\left(f\left(\varepsilon\right)\right) f'\left(\varepsilon\right) \ , \tag{2.8}$$

where, from eq. (2.7), we observe, for $\tau \in \mathbb{R}^+$ small enough, that

$$\frac{d}{d\varepsilon} \psi\left(\Omega_\varepsilon\right) = \lim_{\tau \to 0} \frac{\psi\left(\Omega_\tau\right) - \psi(\Omega_\varepsilon)}{\tau} = \left. \frac{d}{d\tau} \psi\left(\Omega_\tau\right) \right|_{\tau=0} \ . \tag{2.9}$$

Considering the shape derivative of the cost function given by the above expression (eq. 2.9) and rearranging eq. (2.8) we obtain

$$\frac{1}{f'\left(\varepsilon\right)} \left. \frac{d}{d\tau} \psi\left(\Omega_\tau\right) \right|_{\tau=0} = D_T\left(\hat{\mathbf{x}}\right) + \mathcal{R}'\left(f\left(\varepsilon\right)\right) \ . \tag{2.10}$$

Finally, taking the limit $\varepsilon \to 0$ in eq. (2.10) and considering the definition of $\mathcal{R}(f(\varepsilon))$ given by eq. (2.2), we observe that

$$\lim_{\varepsilon \to 0} \mathcal{R}'\left(f\left(\varepsilon\right)\right) = 0 \quad \Rightarrow \quad D_T\left(\hat{\mathbf{x}}\right) = \lim_{\varepsilon \to 0} \frac{1}{f'\left(\varepsilon\right)} \left. \frac{d}{d\tau} \psi\left(\Omega_\tau\right) \right|_{\tau=0} \tag{2.11}$$

and we get the proof of the Theorem □

Note that the topological derivative given by eq. (2.3) can be seen as an extension of classical shape derivative, but with a mathematical difficulty concerning the lack of homeomorphism between Ω and Ω_ε. On the other hand, the above Theorem highlights that the topological derivative may be obtained by means of shape sensitivity analysis. Consequently, the Topological-Shape Sensitivity Method leads to a systematic approach to calculate the topological derivative of cost function ψ considering eq. (2.4). In fact, since the domains Ω_ε and Ω_τ have the same topology, we can to build an homeomorphic map between them. In addition, Ω_ε and Ω_τ may be respectively viewed as material and spatial configurations. Therefore, in order to calculate the shape derivative of the cost function (see eq. 2.7) we can use classical results from Continuum Mechanics like Reynolds' transport theorem and the concept of material derivatives of spatial fields [15].

3. The topological derivative in three-dimensional linear elasticity

To highlight the potentialities of the Topological-Shape Sensitivity Method, it will be applied to three-dimensional linear elasticity problems considering the total potential energy as cost function and the equilibrium equation in its weak form as constraint. Therefore, considering the above problem, initially we introduce the mechanical model, then we perform the shape sensitivity of the adopted cost function with respect to the shape change of the hole and finally we calculate the associated topological derivative.

3.1. **Mechanical model .** In this work, we consider a mechanical model restricted to small deformation and displacement and for the constitutive relation we adopt an isotropic linear elastic material. These assumptions lead to the classical three-dimensional linear elasticity theory [12]. In order to calculate the topological derivative associated to this problem, we need to state the equilibrium equations in the original domain Ω (without hole) and in the new one Ω_ε (with hole).

3.1.1. *Problem formulation in the original domain without hole.* The mechanical model associated to three-dimensional linear elasticity problem can be stated in its variational formulation as following: find the displacement vector field $\mathbf{u} \in \mathcal{U}$, such that

$$\int_\Omega \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\boldsymbol{\eta}) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta} \qquad \forall \boldsymbol{\eta} \in \mathcal{V} \,, \tag{3.1}$$

where Ω represents a deformable body with boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$, such that $\Gamma_N \cap \Gamma_D = \emptyset$, submitted to a set of surface forces $\bar{\mathbf{q}}$ on the Neumann boundary Γ_N and displacement constraints $\bar{\mathbf{u}}$ on the Dirichlet boundary Γ_D. Therefore, assuming that $\bar{\mathbf{q}} \in L^2(\Gamma_N)$, the admissible functions set $\mathcal{U}$ and the admissible variations space $\mathcal{V}$ are given, respectively, by

$$\mathcal{U} = \left\{ \mathbf{u} \in H^1(\Omega) : \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_D \right\} \,, \quad \mathcal{V} = \left\{ \boldsymbol{\eta} \in H^1(\Omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \Gamma_D \right\} \,. \tag{3.2}$$

In addition, the linearized Green deformation tensor $\mathbf{E}(\mathbf{u})$ and the Cauchy stress tensor $\mathbf{T}(\mathbf{u})$ are defined as

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) := \nabla^s\mathbf{u} \qquad \text{and} \qquad \mathbf{T}(\mathbf{u}) = \mathbf{C}\mathbf{E}(\mathbf{u}) = \mathbf{C}\nabla^s\mathbf{u} \,, \tag{3.3}$$

where $\mathbf{C} = \mathbf{C}^T$ is the elasticity tensor, that is, denoting by $\mathbf{I}$ and $\mathbb{I}$ respectively the second and fourth order identity tensors, and by E and ν the Young's modulus and the Poisson's ratio respectively, we have

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\,\mathbb{I} + \nu\,(\mathbf{I}\otimes\mathbf{I})\right] \quad \Rightarrow \quad \mathbf{C}^{-1} = \frac{1}{E}\left[(1+\nu)\,\mathbb{I} - \nu\,(\mathbf{I}\otimes\mathbf{I})\right] \,. \tag{3.4}$$

The Euler-Lagrange equation associated with the above variational problem, eq. (3.1), is given by the following boundary value problem: find $\mathbf{u}$ such that

$$\left\{ \begin{array}{lll} \operatorname{div}\mathbf{T}(\mathbf{u}) = \mathbf{0} & \text{in} & \Omega \\ \mathbf{u} = \bar{\mathbf{u}} & \text{on} & \Gamma_D \\ \mathbf{T}(\mathbf{u})\mathbf{n} = \bar{\mathbf{q}} & \text{on} & \Gamma_N \end{array} \right. . \tag{3.5}$$

3.1.2. *Problem formulation in the new domain with hole.* The problem stated in the original domain Ω must also be stated in the domain Ω_ε with a hole B_ε. Therefore, assuming null forces on the hole, we have the following variational problem: find the displacement vector field $\mathbf{u}_\varepsilon \in \mathcal{U}_\varepsilon$, such that

$$\int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\boldsymbol{\eta}_\varepsilon) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta}_\varepsilon \qquad \forall \boldsymbol{\eta}_\varepsilon \in \mathcal{V}_\varepsilon \,. \tag{3.6}$$

where set $\mathcal{U}_\varepsilon$ and space $\mathcal{V}_\varepsilon$ are respectively defined as

$$\mathcal{U}_\varepsilon = \left\{ \mathbf{u}_\varepsilon \in H^1(\Omega_\varepsilon) : \mathbf{u}_\varepsilon = \bar{\mathbf{u}} \text{ on } \Gamma_D \right\} \,, \quad \mathcal{V}_\varepsilon = \left\{ \boldsymbol{\eta}_\varepsilon \in H^1(\Omega_\varepsilon) : \boldsymbol{\eta}_\varepsilon = \mathbf{0} \text{ on } \Gamma_D \right\} \,. \tag{3.7}$$

As seen before, tensors $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)$ and $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)$ are respectively given by

$$\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \nabla^s\mathbf{u}_\varepsilon \qquad \text{and} \qquad \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{C}\nabla^s\mathbf{u}_\varepsilon \,, \tag{3.8}$$

where the elasticity tensor $\mathbf{C}$ is defined in eq. (3.4). In accordance with the variational problem given by eq. (3.6), the natural boundary condition on ∂B_ε is $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0}$ (homogeneous Neumann condition). Therefore, the Euler-Lagrange equation associated with this new variational problem is given by the following boundary value problem: find $\mathbf{u}_\varepsilon$ such that

$$\left\{ \begin{array}{lll} \operatorname{div}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{0} & \text{in} & \Omega_\varepsilon \\ \mathbf{u}_\varepsilon = \bar{\mathbf{u}} & \text{on} & \Gamma_D \\ \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \bar{\mathbf{q}} & \text{on} & \Gamma_N \\ \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0} & \text{on} & \partial B_\varepsilon \end{array} \right. . \tag{3.9}$$

3.2. **Shape sensitivity analysis .** Let us choose the total potential energy stored in the elastic solid under analysis as cost function. For simplicity, we assume that the external load remains fixed during the shape change. As it is well-known, different approaches can be applied to obtain the shape derivative of the cost function. However, in our particular case, as the cost function is associated to the potential of the state equation, the direct differentiation method will be adopted to calculate its shape derivative. Therefore, considering the total potential energy already written in the configuration Ω_τ, defined through eq. (2.5), then $\psi(\Omega_\tau) := \mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)$ can be expressed by

$$\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau) = \frac{1}{2}\int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau) - \int_{\Gamma_N} \bar{\mathbf{q}}\cdot\mathbf{u}_\tau \ , \tag{3.10}$$

where tensors $\mathbf{E}_\tau(\mathbf{u}_\tau)$ and $\mathbf{T}_\tau(\mathbf{u}_\tau)$ are respectively given by

$$\mathbf{E}_\tau(\mathbf{u}_\tau) = \nabla^s_\tau\mathbf{u}_\tau \qquad \text{and} \qquad \mathbf{T}_\tau(\mathbf{u}_\tau) = \mathbf{C}\nabla^s_\tau\mathbf{u}_\tau \ , \tag{3.11}$$

with $\nabla_\tau(\cdot)$ used to denote

$$\nabla_\tau(\cdot) := \frac{\partial}{\partial\mathbf{x}_\tau}(\cdot) \ . \tag{3.12}$$

In addition, $\mathbf{u}_\tau$ is the solution of the variational problem defined in the configuration Ω_τ, that is: find the displacement vector field $\mathbf{u}_\tau \in \mathcal{U}_\tau$ such that

$$\int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\boldsymbol{\eta}_\tau) = \int_{\Gamma_N} \bar{\mathbf{q}}\cdot\boldsymbol{\eta}_\tau \quad \forall\,\boldsymbol{\eta}_\tau\in\mathcal{V}_\tau \ , \tag{3.13}$$

where set $\mathcal{U}_\tau$ and space $\mathcal{V}_\tau$ are defined as

$$\mathcal{U}_\tau = \left\{\mathbf{u}_\tau \in H^1(\Omega_\tau) : \mathbf{u}_\tau = \bar{\mathbf{u}} \text{ on } \Gamma_D\right\} \ , \quad \mathcal{V}_\tau = \left\{\boldsymbol{\eta}_\tau \in H^1(\Omega_\tau) : \boldsymbol{\eta}_\tau = \mathbf{0} \text{ on } \Gamma_D\right\} \ . \tag{3.14}$$

Observe that from the well-known terminology of Continuum Mechanics, the domains $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon$ and Ω_τ can be interpreted as the material and the spatial configurations, respectively. Therefore, in order to calculate the shape derivative of cost function $\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)$, at $\tau = 0$, we may use Reynolds' transport theorem and the concept of material derivatives of spatial fields, that is [15]

$$\frac{d}{d\tau}\int_{\Omega_\tau}\varphi_\tau\bigg|_{\tau=0} = \int_{\Omega_\varepsilon}\left(\dot{\varphi}_\tau|_{\tau=0} + \varphi_\tau|_{\tau=0}\operatorname{div}\mathbf{v}\right) \ , \tag{3.15}$$

where φ_τ is a spatial scalar field and $\dot{(\cdot)}$ is used to denote

$$\dot{(\cdot)} := \frac{d(\cdot)}{d\tau} \ . \tag{3.16}$$

Taking into account the cost function defined through eq. (3.10) and assuming that parameters E, ν, $\bar{\mathbf{u}}$ and $\bar{\mathbf{q}}$ are constants in relation to the perturbation represented by τ, we have, from eq. (3.15) and following Theorem 1, eqs. (2.5,2.6), that

$$\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\bigg|_{\tau=0} = \frac{1}{2}\int_{\Omega_\varepsilon}\left[\frac{d}{d\tau}\left(\mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau)\right)\bigg|_{\tau=0} + \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)\operatorname{div}\mathbf{v}\right] - \int_{\Gamma_N}\bar{\mathbf{q}}\cdot\dot{\mathbf{u}}_\varepsilon \ , \tag{3.17}$$

where, according to the material derivatives of spatial fields [15], we have

$$\frac{d}{d\tau}\left(\mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau)\right)\bigg|_{\tau=0} = 2\left(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot(\nabla\mathbf{u}_\varepsilon\nabla\mathbf{v})^s\right) \ . \tag{3.18}$$

Substituting eq. (3.18) in eq. (3.17) we obtain

$$\begin{aligned}\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\bigg|_{\tau=0} &= \int_{\Omega_\varepsilon}\left[\frac{1}{2}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)\operatorname{div}\mathbf{v} - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot(\nabla\mathbf{u}_\varepsilon\nabla\mathbf{v})^s\right] \\ &\quad + \int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \int_{\Gamma_N}\bar{\mathbf{q}}\cdot\dot{\mathbf{u}}_\varepsilon \ .\end{aligned} \tag{3.19}$$

Since $\mathbf{u}_\varepsilon$ is the solution of the variational problem given by eq. (3.6) and considering that $\dot{\mathbf{u}}_\varepsilon \in \mathcal{V}_\varepsilon$, eq. (3.19) becomes

$$\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\bigg|_{\tau=0} = \int_{\Omega_\varepsilon}\boldsymbol{\Sigma}_\varepsilon\cdot\nabla\mathbf{v} \ , \tag{3.20}$$

where $\boldsymbol{\Sigma}_\varepsilon$ is the generalized Eshelby energy-momentum tensor (see, for instance, [7, 32]) given in this particular case by

$$\boldsymbol{\Sigma}_\varepsilon = \frac{1}{2}\left(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)\right)\mathbf{I} - (\nabla\mathbf{u}_\varepsilon)^T\,\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\ . \tag{3.21}$$

Remark 2. *It is interesting to observe that the Eshelby tensor $\boldsymbol{\Sigma}_\varepsilon$ appears as a duality pair with respect to $\nabla\mathbf{v}$, as can be seen in eq. (3.20). This fact allows us to interpret $\boldsymbol{\Sigma}_\varepsilon$ as the set of configurational forces* [16] *associated to the change in the configuration of Ω_ε characterized by $\nabla\mathbf{v}$.*

Let us calculate again the shape derivative of cost function $\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)$ defined through eq. (3.10), at $\tau = 0$, using another version for Reynolds' transport theorem [15], that is,

$$\frac{d}{d\tau}\int_{\Omega_\tau}\varphi_\tau\bigg|_{\tau=0} = \int_{\Omega_\varepsilon}\varphi'_\tau|_{\tau=0} + \int_{\partial\Omega_\varepsilon}\varphi_\tau|_{\tau=0}\,(\mathbf{v}\cdot\mathbf{n})\ , \tag{3.22}$$

where φ_τ is a spatial scalar field and $(\cdot)'$ is used to denote

$$(\cdot)' := \frac{\partial(\cdot)}{\partial\tau} = \frac{d(\cdot)}{d\tau}\bigg|_{\mathbf{x}_\tau \text{ fixed}}\ . \tag{3.23}$$

Which results in

$$\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\bigg|_{\tau=0} = \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{v}\cdot\mathbf{n}) + \frac{1}{2}\int_{\Omega_\varepsilon}\frac{\partial}{\partial\tau}(\mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau))\bigg|_{\tau=0} - \int_{\Gamma_N}\bar{\mathbf{q}}\cdot\dot{\mathbf{u}}_\varepsilon\ , \tag{3.24}$$

where $\dot{\mathbf{u}}_\varepsilon$ can be written as [15]

$$\dot{\mathbf{u}}_\varepsilon = \mathbf{u}'_\varepsilon + (\nabla\mathbf{u}_\varepsilon)\mathbf{v} \quad\Rightarrow\quad \mathbf{u}'_\varepsilon = \dot{\mathbf{u}}_\varepsilon - (\nabla\mathbf{u}_\varepsilon)\mathbf{v}\ . \tag{3.25}$$

Taking into account the notation introduced through eq. (3.23) and from eq. (3.25), we have

$$\begin{aligned}\frac{\partial}{\partial\tau}(\mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau))\bigg|_{\tau=0} &= 2\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}'_\varepsilon)\\ &= 2\left(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\right)\ ,\end{aligned} \tag{3.26}$$

where

$$\boldsymbol{\varphi}_\varepsilon = (\nabla\mathbf{u}_\varepsilon)\mathbf{v} \quad\Rightarrow\quad \mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon) = \nabla^s\boldsymbol{\varphi}_\varepsilon\ . \tag{3.27}$$

Substituting eq. (3.26) in eq. (3.24) we obtain

$$\begin{aligned}\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\bigg|_{\tau=0} &= \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{v}\cdot\mathbf{n}) - \int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\\ &\quad + \int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \int_{\Gamma_N}\bar{\mathbf{q}}\cdot\dot{\mathbf{u}}_\varepsilon\\ &= \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{n}\cdot\mathbf{v}) - \int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\ ,\end{aligned} \tag{3.28}$$

since $\dot{\mathbf{u}}_\varepsilon \in \mathcal{V}_\varepsilon$ and $\mathbf{u}_\varepsilon$ is solution of eq. (3.6). In addition, we observe that

$$\int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon) = \int_{\partial\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\boldsymbol{\varphi}_\varepsilon\cdot\mathbf{n} - \int_{\Omega_\varepsilon}\operatorname{div}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))\cdot\boldsymbol{\varphi}_\varepsilon\ . \tag{3.29}$$

Considering this last result in eq. (3.28) and taking into account again that $\mathbf{u}_\varepsilon$ is solution of eq. (3.9), we have

$$\begin{aligned}\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\bigg|_{\tau=0} &= \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{v}\cdot\mathbf{n}) - \int_{\partial\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\boldsymbol{\varphi}_\varepsilon\cdot\mathbf{n}\\ &= \int_{\partial\Omega_\varepsilon}\left[\frac{1}{2}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))\mathbf{I} - (\nabla\mathbf{u}_\varepsilon)^T\,\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\right]\mathbf{n}\cdot\mathbf{v}\\ &= \int_{\partial\Omega_\varepsilon}\boldsymbol{\Sigma}_\varepsilon\mathbf{n}\cdot\mathbf{v}\ ,\end{aligned} \tag{3.30}$$

remembering that $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\varphi}_\varepsilon$ are respectively given by eq. (3.21) and eq. (3.27).

On the other hand, taking into account eq. (3.20) and considering the tensorial relation

$$\operatorname{div}(\mathbf{\Sigma}_\varepsilon^T \mathbf{v}) = \mathbf{\Sigma}_\varepsilon \cdot \nabla \mathbf{v} + \operatorname{div}\mathbf{\Sigma}_\varepsilon \cdot \mathbf{v} \ , \tag{3.31}$$

we can apply the divergence theorem to obtain

$$\left.\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\right|_{\tau=0} = \int_{\partial\Omega_\varepsilon} \mathbf{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{v} - \int_{\Omega_\varepsilon} \operatorname{div}\mathbf{\Sigma}_\varepsilon \cdot \mathbf{v} \ . \tag{3.32}$$

Thus, from eqs. (3.32,3.30) we observe that the Eshelby tensor has null divergence. In fact, since $\mathbf{v}$ is an arbitrary velocity field, then from the fundamental theorem of the calculus of variations it is straightforward to verify that

$$\int_{\Omega_\varepsilon} \operatorname{div}\mathbf{\Sigma}_\varepsilon \cdot \mathbf{v} = 0 \quad \forall\, \mathbf{v} \qquad \Leftrightarrow \qquad \operatorname{div}\mathbf{\Sigma}_\varepsilon = \mathbf{0} \tag{3.33}$$

and the shape derivative of cost function $\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)$ defined through eq. (3.10), at $\tau = 0$, becomes an integral defined on the boundary $\partial\Omega_\varepsilon$, that is,

$$\left.\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\right|_{\tau=0} = \int_{\partial\Omega_\varepsilon} \mathbf{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{v} \ . \tag{3.34}$$

In other words, the shape sensitivity of the problem only depends on the flux of the Eshelby tensor $\mathbf{\Sigma}_\varepsilon \mathbf{n}$ and on the velocity field $\mathbf{v}$ along the boundary $\partial\Omega_\varepsilon$.

3.3. **Topological sensitivity analysis .** In order to calculate the topological derivative using the Topological-Shape Sensitivity Method, we need to substitute eq. (3.34) in the result of Theorem 1 (eq. 2.4). Therefore, from the definition of the velocity field (eq. 2.6) and considering the shape derivative of the cost function (eq. 3.34), we have that

$$\left.\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\right|_{\tau=0} = -\int_{\partial B_\varepsilon} \mathbf{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{n} \ , \tag{3.35}$$

where

$$\mathbf{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{n} = \frac{1}{2}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} \cdot (\nabla \mathbf{u}_\varepsilon)\, \mathbf{n} \ . \tag{3.36}$$

In addition, taking into account homogeneous Neumann boundary condition on the hole, we have, from eq. (3.9), that $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0}$ on ∂B_ε, therefore

$$\left.\frac{d}{d\tau}\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)\right|_{\tau=0} = -\frac{1}{2}\int_{\partial B_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \ . \tag{3.37}$$

Finally, substituting eq. (3.37) in the result of the Theorem 1 (eq. 2.4), the topological derivative becomes

$$D_T(\hat{\mathbf{x}}) = -\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\int_{\partial B_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \ . \tag{3.38}$$

Considering the inverse of the constitutive relation $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{C}^{-1}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)$ (see eq. 3.4), then the integrand of eq. (3.38) may be expressed as a function of the stress tensor as following

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \frac{1}{E}\left[(1+\nu)\, \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) - \nu\, (\operatorname{tr}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))^2\right] \ . \tag{3.39}$$

Let us introduce a spherical coordinate system (r, θ, φ) centered at $\hat{\mathbf{x}}$ (see fig. 6), then the stress tensor $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))^T$, when defined on the boundary ∂B_ε, can be decomposed as

$$\begin{aligned} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)|_{\partial B_\varepsilon} &= T_\varepsilon^{rr}\,(\mathbf{e}_r \otimes \mathbf{e}_r) + T_\varepsilon^{r\theta}\,(\mathbf{e}_r \otimes \mathbf{e}_\theta) + T_\varepsilon^{r\varphi}\,(\mathbf{e}_r \otimes \mathbf{e}_\varphi) \\ &+ T_\varepsilon^{r\theta}\,(\mathbf{e}_\theta \otimes \mathbf{e}_r) + T_\varepsilon^{\theta\theta}\,(\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + T_\varepsilon^{\theta\varphi}\,(\mathbf{e}_\theta \otimes \mathbf{e}_\varphi) \\ &+ T_\varepsilon^{r\varphi}\,(\mathbf{e}_\varphi \otimes \mathbf{e}_r) + T_\varepsilon^{\theta\varphi}\,(\mathbf{e}_\varphi \otimes \mathbf{e}_\theta) + T_\varepsilon^{\varphi\varphi}\,(\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi) \ , \end{aligned} \tag{3.40}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are the spherical coordinate system basis such that

$$\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = \mathbf{e}_\varphi \cdot \mathbf{e}_\varphi = 1 \qquad \text{and} \qquad \mathbf{e}_r \cdot \mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{e}_\varphi = \mathbf{e}_\theta \cdot \mathbf{e}_\varphi = 0 \ . \tag{3.41}$$

Since we have homogeneous Neumann boundary condition on ∂B_ε, then

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{e}_r = \mathbf{0} \qquad \text{on} \qquad \partial B_\varepsilon \ . \tag{3.42}$$

From decomposition of the stress tensor shown in eq. (3.40) and taking into account eqs. (3.41,3.42), we observe that

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{e}_r = T_\varepsilon^{rr}\mathbf{e}_r + T_\varepsilon^{r\theta}\mathbf{e}_\theta + T_\varepsilon^{r\varphi}\mathbf{e}_\varphi = \mathbf{0} \qquad \Rightarrow \qquad T_\varepsilon^{rr} = T_\varepsilon^{r\theta} = T_\varepsilon^{r\varphi} = 0 \; . \tag{3.43}$$

Substituting eqs. (3.40,3.43) into eq. (3.39), the topological derivative given by eq. (3.38) may be written in terms of the stress tensor components in spherical coordinate, as following

$$\begin{aligned} D_T(\hat{\mathbf{x}}) &= -\frac{1}{2E}\lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\int_{\partial B_\varepsilon} d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi}) \\ &= -\frac{1}{2E}\lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\int_0^{2\pi}\left(\int_0^{\pi} d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi})\varepsilon^2 \sin\theta d\theta\right) d\varphi \; , \end{aligned} \tag{3.44}$$

where

$$d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi}) = (T_\varepsilon^{\theta\theta})^2 + (T_\varepsilon^{\varphi\varphi})^2 - 2\nu T_\varepsilon^{\theta\theta} T_\varepsilon^{\varphi\varphi} + 2(1+\nu)(T_\varepsilon^{\theta\varphi})^2 \; . \tag{3.45}$$

Now, it is sufficient to calculate the limit $\varepsilon \to 0$ in eq. (3.44) to obtain the final expression of the topological derivative. Thus, an asymptotic analysis [21] shall be performed in order to know the behavior of stress components $T_\varepsilon^{\theta\theta}$, $T_\varepsilon^{\theta\varphi}$ and $T_\varepsilon^{\varphi\varphi}$ when $\varepsilon \to 0$. This asymptotic expansion may be obtained from the analytical solution for a stress distribution around a spherical void in a three-dimensional elastic body [29], which is given, for any $\delta > 0$ and at $r = \varepsilon$, by (see Appendix A)

$$\begin{aligned} T_\varepsilon^{\theta\theta}\Big|_{\partial B_\varepsilon} = \frac{3}{4}\frac{1}{7-5\nu} &\{\sigma_1(\mathbf{u})\left[3 - 5(1-2\nu)\cos 2\varphi + 10\cos 2\theta \sin^2\varphi\right] \\ &+ \sigma_2(\mathbf{u})\left[3 + 5(1-2\nu)\cos 2\varphi + 10\cos 2\theta \cos^2\varphi\right] \\ &+ \sigma_3(\mathbf{u})\left[2(4-5\nu) - 10\cos 2\theta\right]\} + \mathcal{O}(\varepsilon^{1-\delta}) \; , \end{aligned} \tag{3.46}$$

$$T_\varepsilon^{\theta\varphi}\Big|_{\partial B_\varepsilon} = \frac{15}{2}\frac{1-\nu}{7-5\nu}(\sigma_1(\mathbf{u}) - \sigma_2(\mathbf{u}))\cos\theta \sin 2\varphi + \mathcal{O}(\varepsilon^{1-\delta}) \; , \tag{3.47}$$

$$\begin{aligned} T_\varepsilon^{\varphi\varphi}|_{\partial B_\varepsilon} = \frac{3}{4}\frac{1}{7-5\nu} &\{\sigma_1(\mathbf{u})\left[8 - 5\nu + 5(2-\nu)\cos 2\varphi + 10\nu\cos 2\theta \sin^2\varphi\right] \\ &+ \sigma_2(\mathbf{u})\left[8 - 5\nu - 5(2-\nu)\cos 2\varphi + 10\nu\cos 2\theta \cos^2\varphi\right] \\ &-2\sigma_3(\mathbf{u})(1 + 5\nu\cos 2\theta)\} + \mathcal{O}(\varepsilon^{1-\delta}) \; , \end{aligned} \tag{3.48}$$

where $\sigma_1(\mathbf{u})$, $\sigma_2(\mathbf{u})$ and $\sigma_3(\mathbf{u})$ are the principal stress values of tensor $\mathbf{T}(\mathbf{u})$, associated to the original domain without hole Ω (see eq. 3.1), evaluated at point $\hat{\mathbf{x}} \in \Omega$, that is $\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}}$.

Substituting the asymptotic expansion given by eqs. (3.46,3.47,3.48) in eq. (3.44) we observe that, in order to take the limit $\varepsilon \to 0$, function $f(\varepsilon)$ must be chosen such that

$$f'(\varepsilon) = -|\partial B_\varepsilon| = -4\pi\varepsilon^2 \quad \Rightarrow \quad f(\varepsilon) = -|B_\varepsilon| = -\frac{4}{3}\pi\varepsilon^3 \; . \tag{3.49}$$

Therefore, from this choice of function $f(\varepsilon)$ shown in eq. (3.49), the final expression for the topological derivative becomes a scalar function that depends on solution $\mathbf{u}$ associated to the original domain Ω (without hole), that is (see also [11, 22]):

- in terms of principal stress values $\sigma_1(\mathbf{u})$, $\sigma_2(\mathbf{u})$ and $\sigma_3(\mathbf{u})$ of tensor $\mathbf{T}(\mathbf{u})$

  $$D_T(\hat{\mathbf{x}}) = \frac{3}{4E}\frac{1-\nu}{7-5\nu}\left[10(1+\nu)S_1(\mathbf{u}) - (1+5\nu)S_2(\mathbf{u})\right] \; , \tag{3.50}$$

  where $S_1(\mathbf{u})$ and $S_2(\mathbf{u})$ are respectively given by

  $$S_1(\mathbf{u}) = \sigma_1(\mathbf{u})^2 + \sigma_2(\mathbf{u})^2 + \sigma_3(\mathbf{u})^2 \quad \text{and} \quad S_2(\mathbf{u}) = (\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u}) + \sigma_3(\mathbf{u}))^2 \; ; \tag{3.51}$$

- in terms of stress tensor $\mathbf{T}(\mathbf{u})$

  $$D_T(\hat{\mathbf{x}}) = \frac{3}{4E}\frac{1-\nu}{7-5\nu}\left[10(1+\nu)\mathbf{T}(\mathbf{u})\cdot\mathbf{T}(\mathbf{u}) - (1+5\nu)(\mathrm{tr}\mathbf{T}(\mathbf{u}))^2\right] \; ; \tag{3.52}$$

- in terms of stress $\mathbf{T}(\mathbf{u})$ and strain $\mathbf{E}(\mathbf{u})$ tensors

  $$D_T(\hat{\mathbf{x}}) = \frac{3}{4}\frac{1-\nu}{7-5\nu}\left[10\mathbf{T}(\mathbf{u})\cdot\mathbf{E}(\mathbf{u}) - \frac{1-5\nu}{1-2\nu}\mathrm{tr}\mathbf{T}(\mathbf{u})\,\mathrm{tr}\mathbf{E}(\mathbf{u})\right] \; ; \tag{3.53}$$

which was obtained from a simple manipulation considering the constitutive relation given by eq. (3.3). See also eq. (3.4).

Remark 3. *It is interesting to observe that if we take $\nu = 1/5$ in eq. (3.53), the final expression for the topological derivative in terms of* $\mathbf{T}(\mathbf{u})$ *and* $\mathbf{E}(\mathbf{u})$ *becomes*

$$D_T(\hat{\mathbf{x}}) = \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\mathbf{u}) \ . \tag{3.54}$$

4. A numerical experiment

As it was mentioned previously, the topological derivative furnishes the sensitivity of the cost functional when a small hole is created at an arbitrary point $\hat{\mathbf{x}}$ of the domain. Thus, from eq. (2.1), $D_T(\hat{\mathbf{x}})$ may be seen as a first order correction on $\psi(\Omega)$ to obtain $\psi(\Omega_\varepsilon)$, which allows us to naturally apply this derivative as a descent direction in topology optimization problems. In order to point out this feature, let us present a simple topology design algorithm based on the topological derivative given by eqs. (3.50,3.52, 3.53): considering the sequence $\{\Omega^i : |\Omega^i| \geq |\Omega^*|\}$, where i is the i-th iteration and $|\Omega^*|$ corresponds to the required final volume, then,

(1) **Provide** the initial domain Ω^0, the stop criterion $|\Omega^*|$ and the rate of material removal α.
(2) **While** $|\Omega^i| \geq |\Omega^*|$ **do**:
  : (a) compute $D_T(\hat{\mathbf{x}})$ $\forall \hat{\mathbf{x}} \in \Omega^i$;
  : (b) create holes at the points $\hat{\mathbf{x}}$ where the topological derivative assumes its smallest values, according to the volume of material to be removed at each iteration $\alpha |\Omega^i|$;
  : (c) define the new domain Ω^{i+1};
  : (d) make $i \leftarrow i + 1$.
(3) **Ensure** the desired final topology Ω^*.

According to eqs. (3.50,3.52,3.53), the topological derivative depends on the stress field $\mathbf{T}(\mathbf{u})$. In this research, $\mathbf{u}$ is computed via Finite Element Method (see, for instance, [20]) and the stress field $\mathbf{T}(\mathbf{u})$ is obtained by a post-processing [19, 33] of the approximated solution. Thus the topological derivative is evaluated at the nodal points of the finite elements mesh and the holes are created by elimination of the elements which share the nodes with smallest values of $D_T(\hat{\mathbf{x}})$, according to the volume of material to be removed at each iteration. Instead effectively create voids, we introduce a fictitious material 10^7 times softer than the real one. This procedure allows us to simplify the computational implementation of the above algorithm. Finally, in order to improve the results, we use an adaptive mesh refinement strategy based on the following idea: let h be the mesh parameter (size of the elements) associated to the original (uniform) mesh and let us consider h_s and h_v the mesh parameters respectively associated to solid (hard material) and voids (soft material), then we fix for example $h_v = 10\,h$ and compute h_s maintaining the total number of elements as constant as possible. In other words, we use a rough mesh for the voids, while a fine mesh is automatically generated in the remainder material. From this strategy, we obtain a more accurate evaluation of the stresses and a refined definition of the topology.

To point out the applicability of the above procedure, let us consider the design of a simply supported cube on the bottom under vertical load applied on the top [4], as shown in fig. (2). This cube has dimension $a \times a \times a$, where $a = 0.5[m]$. The load $\bar{q} = 10[KN]$ is distributed in a small centered circular region of radius equal to $0.03[m]$. The supports are circular of radius equal to $0.02[m]$, with center at $0.035[m]$ from the edges of the cube. The material properties are given by Young's modulus $E = 210 \times 10^3[MPa]$ and Poisson's ratio $\nu = 1/3$. Due to the problem symmetry, only a quarter of the cube was analyzed using four-nodes tetrahedron finite elements. The required final volume is $|\Omega^*| = 0.02\,|\Omega|$ and the rate of material removal is $\alpha = 0.05$ (5% of material is removed at each iteration).

Finally, the proposed adaptive remeshing algorithm was activated only four times during the topology design process. Details of the obtained results are shown in fig. (3) and in fig. (4), where we can also observe the shape of the transversal sections of the bars obtained at the end of the process. The corresponding finite element mesh associated to the final configuration is presented in fig. (5) showing a overwhelming concentration of elements in the regions of hard material, allowing an excellent identification of the topology.

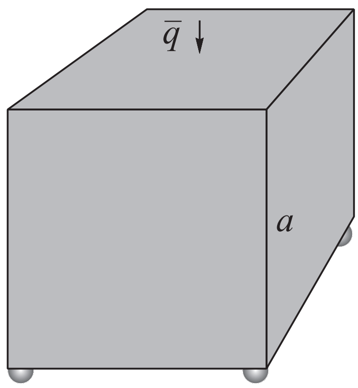
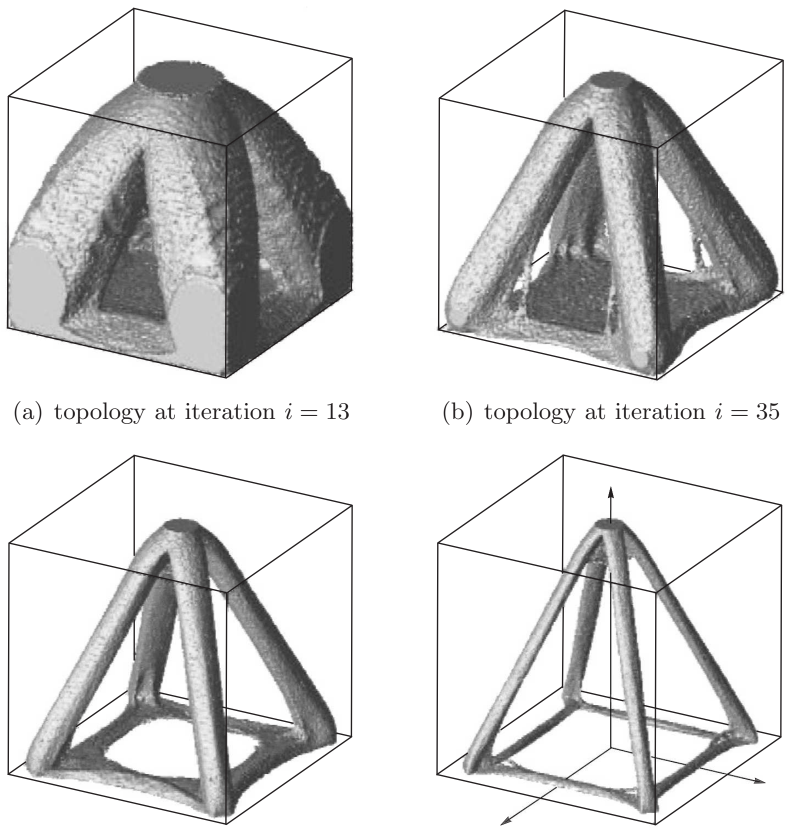


FIGURE 2. simply supported cube on the bottom under vertical load applied on the top [4].



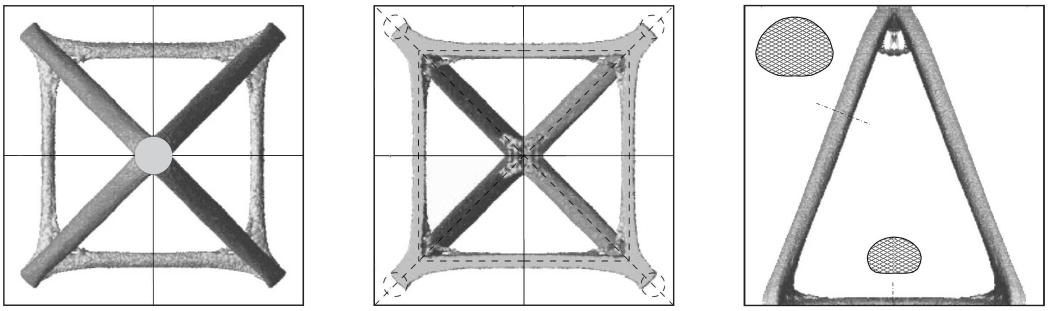
(a) topology at iteration $i = 13$

(b) topology at iteration $i = 35$

(c) topology at iteration $i = 52$

(d) topology at iteration $i = 76$

FIGURE 3. history of the topology design process.



(a) top (b) bottom (c) lateral

FIGURE 4. detail of the obtained final topology.

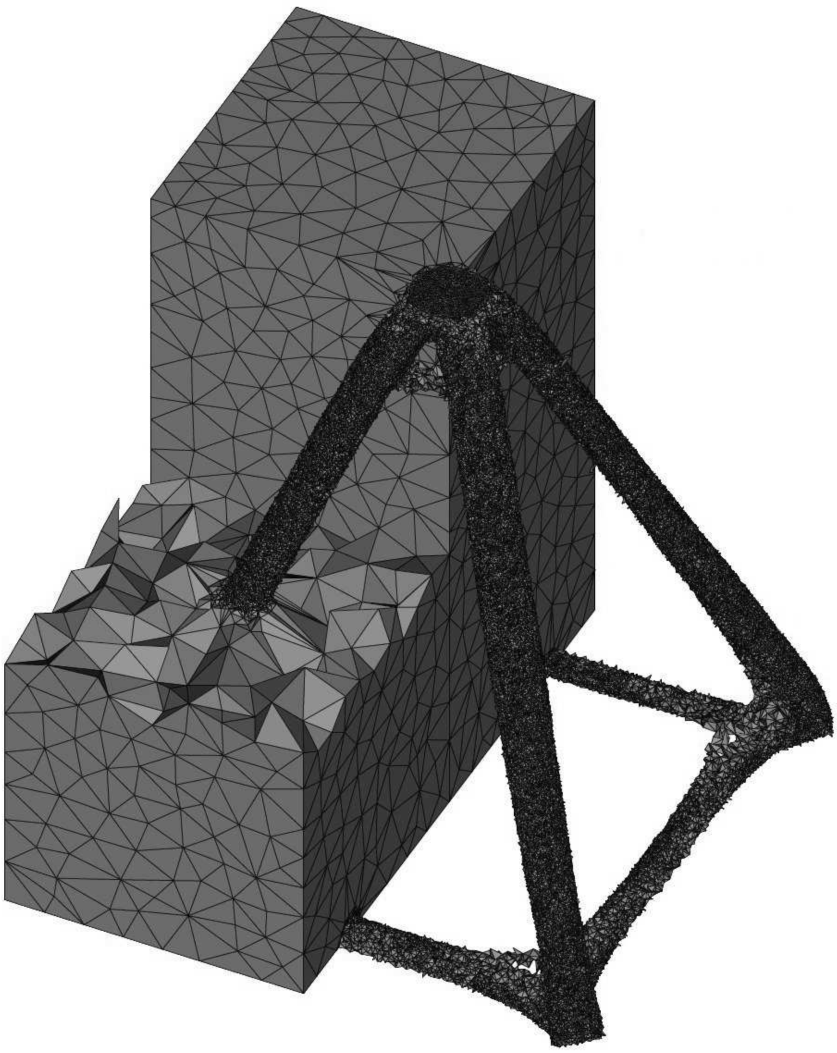


FIGURE 5. finite element mesh at the end of the topology design process.

5. Conclusions

In this study, we have calculated the topological derivative for three-dimensional linear elasticity taking the total potential energy as cost function and the state equation in its weak form as constraint. The relationship between shape and topological derivatives was formally established in Theorem 1, leading to the Topological-Shape Sensitivity Method. Therefore, results from classical shape sensitivity analysis could be applied to calculate the topological derivative as a systematic methodology.

In particular, we have obtained the explicit formula for the topological derivative for the problem under consideration given by eqs. (3.50,3.52,3.53), whose result can be applied in several engineering problems such as topology optimization of three-dimensional linear elastic structures. Indeed, the numerical experiment displays that the topological derivative, even when applied in conjunction with a very simple algorithm, allows us to obtain excellent results. However, others strategies may be explored, like the use of the topological derivative together with level-sets methods, as proposed in [1, 13, 14].

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Appendix A. Asymptotic Analysis

To perform the asymptotic analysis in relation to the parameter ε in Section 3.3 we present in this appendix the analytical solution for the stress distribution around a spherical cavity in a three-dimensional linear elastic body. Therefore, let us introduce a spherical coordinate system (r, θ, φ) centered at $\hat{\mathbf{x}}$, as shown in fig. 6.

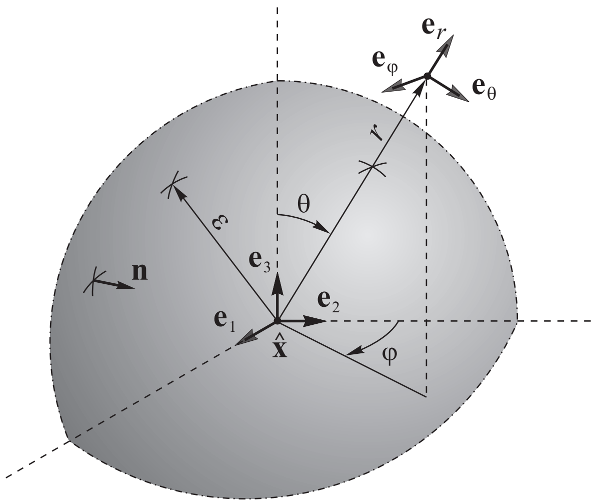

Figure 6. spherical coordinate system (r, θ, φ) positioned at the center $\hat{\mathbf{x}}$ of the ball B_ε.

Then, the stress distribution around the spherical cavity B_ε is given, for any $\delta > 0$, by

$$\begin{aligned}
T_\varepsilon^{rr} &= T_1^{rr} + T_2^{rr} + T_3^{rr} + \mathcal{O}(\varepsilon^{1-\delta}) \, , \\
T_\varepsilon^{r\theta} &= T_1^{r\theta} + T_2^{r\theta} + T_3^{r\theta} + \mathcal{O}(\varepsilon^{1-\delta}) \, , \\
T_\varepsilon^{r\varphi} &= T_1^{r\varphi} + T_2^{r\varphi} + T_3^{r\varphi} + \mathcal{O}(\varepsilon^{1-\delta}) \, , \\
T_\varepsilon^{\theta\theta} &= T_1^{\theta\theta} + T_2^{\theta\theta} + T_3^{\theta\theta} + \mathcal{O}(\varepsilon^{1-\delta}) \, , \\
T_\varepsilon^{\theta\varphi} &= T_1^{\theta\varphi} + T_2^{\theta\varphi} + T_3^{\theta\varphi} + \mathcal{O}(\varepsilon^{1-\delta}) \, , \\
T_\varepsilon^{\varphi\varphi} &= T_1^{\varphi\varphi} + T_2^{\varphi\varphi} + T_3^{\varphi\varphi} + \mathcal{O}(\varepsilon^{1-\delta}) \, ,
\end{aligned} \tag{A.1}$$

where T_i^{rr}, $T_i^{r\theta}$, $T_i^{r\varphi}$, $T_i^{\theta\theta}$, $T_i^{\theta\varphi}$ and $T_i^{\varphi\varphi}$, for $i = 1, 2, 3$, are written, as:

- for $i = 1$

$$T_1^{rr} = \frac{\sigma_1}{14-10\nu}\left[12\left(\frac{\varepsilon^3}{r^3}-\frac{\varepsilon^5}{r^5}\right)+\left(14-10\nu-10(5-\nu)\frac{\varepsilon^3}{r^3}+36\frac{\varepsilon^5}{r^5}\right)\sin^2\theta\sin^2\varphi\right] \ , \tag{A.2}$$

$$T_1^{r\theta} = \frac{\sigma_1}{14-10\nu}\left[7-5\nu+5(1+\nu)\frac{\varepsilon^3}{r^3}-12\frac{\varepsilon^5}{r^5}\right]\sin 2\theta\sin^2\varphi \ , \tag{A.3}$$

$$T_1^{r\varphi} = \frac{\sigma_1}{14-10\nu}\left[7-5\nu+5(1+\nu)\frac{\varepsilon^3}{r^3}-12\frac{\varepsilon^5}{r^5}\right]\sin\theta\sin 2\varphi \ , \tag{A.4}$$

$$\begin{aligned} T_1^{\theta\theta} = \frac{\sigma_1}{56-40\nu}\Bigg[&14-10\nu+(1+10\nu)\frac{\varepsilon^3}{r^3}+3\frac{\varepsilon^5}{r^5}-\left(14-10\nu+25(1-2\nu)\frac{\varepsilon^3}{r^3}-9\frac{\varepsilon^5}{r^5}\right)\cos 2\varphi \\ &+\left(28-20\nu-10(1-2\nu)\frac{\varepsilon^3}{r^3}+42\frac{\varepsilon^5}{r^5}\right)\cos 2\theta\sin^2\varphi\Bigg] \ , \end{aligned} \tag{A.5}$$

$$T_1^{\theta\varphi} = \frac{\sigma_1}{14-10\nu}\left[7-5\nu+5(1-2\nu)\frac{\varepsilon^3}{r^3}+3\frac{\varepsilon^5}{r^5}\right]\cos\theta\sin 2\varphi \ , \tag{A.6}$$

$$\begin{aligned} T_1^{\varphi\varphi} = \frac{\sigma_1}{56-40\nu}\Bigg[&28-20\nu+(11-10\nu)\frac{\varepsilon^3}{r^3}+9\frac{\varepsilon^5}{r^5}+\left(28-20\nu+5(1-2\nu)\frac{\varepsilon^3}{r^3}+27\frac{\varepsilon^5}{r^5}\right)\cos 2\varphi \\ &-30\left((1-2\nu)\frac{\varepsilon^3}{r^3}-\frac{\varepsilon^5}{r^5}\right)\cos 2\theta\sin^2\varphi\Bigg] \ , \end{aligned} \tag{A.7}$$

- for $i = 2$

$$T_2^{rr} = \frac{\sigma_2}{14-10\nu}\left[12\left(\frac{\varepsilon^3}{r^3}-\frac{\varepsilon^5}{r^5}\right)+\left(14-10\nu-10(5-\nu)\frac{\varepsilon^3}{r^3}+36\frac{\varepsilon^5}{r^5}\right)\sin^2\theta\cos^2\varphi\right] \ , \tag{A.8}$$

$$T_2^{r\theta} = \frac{\sigma_2}{14-10\nu}\left[7-5\nu+5(1+\nu)\frac{\varepsilon^3}{r^3}-12\frac{\varepsilon^5}{r^5}\right]\cos^2\varphi\sin 2\theta \ , \tag{A.9}$$

$$T_2^{r\varphi} = \frac{-\sigma_2}{14-10\nu}\left[7-5\nu+5(1+\nu)\frac{\varepsilon^3}{r^3}-12\frac{\varepsilon^5}{r^5}\right]\sin\theta\sin 2\varphi \ , \tag{A.10}$$

$$\begin{aligned} T_2^{\theta\theta} = \frac{\sigma_2}{56-40\nu}\Bigg[&14-10\nu+(1+10\nu)\frac{\varepsilon^3}{r^3}+3\frac{\varepsilon^5}{r^5}+\left(14-10\nu+25(1-2\nu)\frac{\varepsilon^3}{r^3}-9\frac{\varepsilon^5}{r^5}\right)\cos 2\varphi \\ &+\left(28-20\nu-10(1-2\nu)\frac{\varepsilon^3}{r^3}+42\frac{\varepsilon^5}{r^5}\right)\cos 2\theta\cos^2\varphi\Bigg] \ , \end{aligned} \tag{A.11}$$

$$T_2^{\theta\varphi} = \frac{-\sigma_2}{14-10\nu}\left[7-5\nu+5(1-2\nu)\frac{\varepsilon^3}{r^3}+3\frac{\varepsilon^5}{r^5}\right]\cos\theta\sin 2\varphi \ , \tag{A.12}$$

$$\begin{aligned} T_2^{\varphi\varphi} = \frac{\sigma_2}{56-40\nu}\Bigg[&28-20\nu+(11-10\nu)\frac{\varepsilon^3}{r^3}+9\frac{\varepsilon^5}{r^5}-\left(28-20\nu+5(1-2\nu)\frac{\varepsilon^3}{r^3}+27\frac{\varepsilon^5}{r^5}\right)\cos 2\varphi \\ &-30\left((1-2\nu)\frac{\varepsilon^3}{r^3}-\frac{\varepsilon^5}{r^5}\right)\cos 2\theta\cos^2\varphi\Bigg] \ , \end{aligned} \tag{A.13}$$

- for $i = 3$

$$T_3^{rr} = \frac{\sigma_3}{14-10\nu}\left[14-10\nu-(38-10\nu)\frac{\varepsilon^3}{r^3}+24\frac{\varepsilon^5}{r^5}\right.$$
$$\left.-\left(14-10\nu-10(5-\nu)\frac{\varepsilon^3}{r^3}+36\frac{\varepsilon^5}{r^5}\right)\sin^2\theta\right] , \tag{A.14}$$

$$T_3^{r\theta} = \frac{-\sigma_3}{14-10\nu}\left[14-10\nu+10(1+\nu)\frac{\varepsilon^3}{r^3}-24\frac{\varepsilon^5}{r^5}\right]\cos\theta\sin\theta , \tag{A.15}$$

$$T_3^{r\varphi} = 0 , \tag{A.16}$$

$$T_3^{\theta\theta} = \frac{\sigma_3}{14-10\nu}\left[(9-15\nu)\frac{\varepsilon^3}{r^3}-12\frac{\varepsilon^5}{r^5}+\left(14-10\nu-5(1-2\nu)\frac{\varepsilon^3}{r^3}+21\frac{\varepsilon^5}{r^5}\right)\sin^2\theta\right] , \tag{A.17}$$

$$T_3^{\theta\varphi} = 0 , \tag{A.18}$$

$$T_3^{\varphi\varphi} = \frac{\sigma_3}{14-10\nu}\left[(9-15\nu)\frac{\varepsilon^3}{r^3}-12\frac{\varepsilon^5}{r^5}-15\left((1-2\nu)\frac{\varepsilon^3}{r^3}-\frac{\varepsilon^5}{r^5}\right)\sin^2\theta\right] , \tag{A.19}$$

where σ_1, σ_2 and σ_3 are the principal stress values of tensor $\mathbf{T}(\mathbf{u})$, associated to the original domain without hole Ω, evaluated at point $\hat{\mathbf{x}} \in \Omega$, that is $\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}}$. In other words, tensor $\mathbf{T}(\mathbf{u})$ was diagonalized in the following way

$$\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}} = \sum_{i=1}^{3}\sigma_i(\mathbf{e}_i\otimes\mathbf{e}_i) , \tag{A.20}$$

where σ_i is the eigen-value associated to the $\mathbf{e}_i$ eigen-vector of tensor $\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}}$.

Remark 4. *It is important to mention that the stress distribution for $i = 1,2$ was obtained from a rotation of the stress distribution for $i = 3$. In addition, the derivation of this last result (for $i = 3$) can be found in* [29]*, for instance.*

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